Hamilton-Jacobi Equations and Two-Person Zero-Sum Differential Games with Unbounded Controls*

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Abstract

A two-person zero-sum differential game with unbounded controls is considered. Under proper coercivity conditions, the upper and lower value functions are characterized as the unique viscosity solutions to the corresponding upper and lower Hamilton–Jacobi–Isaacs equations, respectively. Consequently, when the Isaacs' condition is satisfied, the upper and lower value functions coincide, leading to the existence of the value function. Due to the unboundedness of the controls, the corresponding upper and lower Hamiltonians grow super linearly in the gradient of the upper and lower value functions, respectively. A uniqueness theorem of viscosity solution to Hamilton–Jacobi equations involving such kind of Hamiltonian is proved, without relying on the convexity/concavity of the Hamiltonian. Also, it is shown that the assumed coercivity conditions guaranteeing the finiteness of the upper and lower value functions are sharp in some sense.

Keywords. Two-person zero-sum differential games, unbounded control, Hamilton-Jacobi equation, viscosity solution.

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1 Introduction

Let us begin with the following control system:

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$$\begin{cases} \dot{y}(s) = f(s, y(s), u_1(s), u_2(s)), \qquad s \in [t, T], \\ y(t) = x. \end{cases} \tag{1.1}$$

where $f : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}^n$ is a given map. In the above, $y(\cdot)$ is the state trajectory taking values in $\mathbb{R}^n$, and $(u_1(\cdot), u_2(\cdot))$ is the control pair taken from the set $\mathcal{U}_1^{\sigma_1}[t, T] \times \mathcal{U}_2^{\sigma_2}[t, T]$ of *admissible controls*, defined by the following:

$$\begin{cases} \mathcal{U}_1^{\sigma_1}[t, T] = \Big\{ u_1 : [t, T] \to U_1 \ \Big| \ \int_t^T |u_1(s)|^{\sigma_1} ds < \infty \Big\}, \\ \mathcal{U}_2^{\sigma_2}[t, T] = \Big\{ u_2 : [t, T] \to U_2 \ \Big| \ \int_t^T |u_2(s)|^{\sigma_2} ds < \infty \Big\}, \end{cases}$$

with U_1 and U_2 being closed subsets of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively, and with some $\sigma_1, \sigma_2 \geq 1$. Note that U_1 and U_2 are allowed to be unbounded, and they could even be $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively. The *performance functional* associated with (1.1) is the following:

$$J(t, x; u_1(\cdot), u_2(\cdot)) = \int_t^T g(s, y(s), u_1(s), u_2(s)) ds + h(y(T)), \tag{1.2}$$

with $g : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ being some given maps.

The above setting can be used to describe a two-person zero-sum differential game: Player 1 wants to select a control $u_1(\cdot) \in \mathcal{U}_1^{\sigma_1}[t, T]$ so that the functional (1.2) is minimized and Player 2 wants to select a control $u_2(\cdot) \in \mathcal{U}_2^{\sigma_2}[t, T]$ so that the functional (1.2) is maximized. Therefore, $J(t, x; u_1(\cdot), u_2(\cdot))$ is a *cost functional* for Player 1 and a *payoff functional* for Player 2. Also, if U_2 is a singleton, the above is reduced to a standard optimal control problem.

Under some mild conditions, for any *initial pair* $(t, x) \in [0, T] \times \mathbb{R}^n$ and control pair $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1^{\sigma_1}[t, T] \times \mathcal{U}_2^{\sigma_2}[t, T]$, the state equation (1.1) admits a unique solution $y(\cdot) \equiv y(\cdot\,; t, x, u_1(\cdot), u_2(\cdot))$, and the performance functional $J(t, x; u_1(\cdot), u_2(\cdot))$ is well-defined. By adopting the notion of Elliott–Kalton strategies ([7]), we can define the *upper* and *lower value functions* $V^{\pm} : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ (see Section 3 for details). Further, when $V^{\pm}(\cdot\,,\cdot)$ are differentiable, they should satisfy the following upper and lower Hamilton-Jacobi-Isaacs (HJI, for short) equations, respectively:

$$\begin{cases} V_t^{\pm}(t, x) + H^{\pm}(t, x, V_x^{\pm}(t, x)) = 0, \qquad (t, x) \in [0, T] \times \mathbb{R}^n, \\ V^{\pm}(T, x) = h(x), \qquad x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $H^{\pm}(t, x, p)$ are *upper* and *lower Hamiltonians* defined by the following, respectively:

$$\begin{cases} H^+(t, x, p) = \inf\limits_{u_1 \in U_1} \sup\limits_{u_2 \in U_2} \Big[\langle\, p, f(t, x, u_1, u_2)\, \rangle + g(t, x, u_1, u_2) \Big], \\ H^-(t, x, p) = \sup\limits_{u_2 \in U_2} \inf\limits_{u_1 \in U_1} \Big[\langle\, p, f(t, x, u_1, u_2)\, \rangle + g(t, x, u_1, u_2) \Big], \\ \qquad\qquad\qquad\qquad (t, x, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n. \end{cases} \tag{1.4}$$

When the sets U_1 and U_2 are bounded, the above differential game is well-understood ([8]): Under reasonable conditions, the upper and lower value functions $V^{\pm}(\cdot\,,\cdot)$ are unique viscosity solutions to the corresponding upper and lower HJI equations, respectively. Consequently, in the case that the following *Isaacs condition*:

$$H^+(t,x,p)=H^-(t,x,p),\qquad \forall(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n, \tag{1.5}$$

holds, the upper and lower values coincide and the two-person zero-sum differential game has the value function

$$V(t,x)=V^+(t,x)=V^-(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^n. \tag{1.6}$$

For comparison purposes, let us now take a closer look at the properties that the upper and lower value functions $V^{\pm}(\cdot\,,\cdot)$ and the upper and lower Hamiltonians $H^{\pm}(\cdot\,,\cdot\,,\cdot)$ have, under classical assumptions. To this end, let us recall the following classical assumption:

(B) Functions $f:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}^n$, $g:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}$, and $h:\mathbb{R}^n\to\mathbb{R}$ are continuous. There exists a constant $L>0$ and a continuous function $\omega:[0,\infty)\times[0,\infty)\to[0,\infty)$, increasing in each of its arguments and $\omega(r,0)=0$ for all $r\ge 0$, such that for all $t,s\in[0,T],\ x,y\in\mathbb{R}^n,\ (u_1,u_2)\in U_1\times U_2$,

$$\begin{cases} |f(t,x,u_1,u_2)-f(s,y,u_1,u_2)|\le L|x-y|+\omega(|x|\vee|y|,|t-s|),\\ |g(t,x,u_1,u_2)-g(s,y,u_1,u_2)|\le \omega(|x|\vee|y|,|x-y|+|t-s|),\\ |h(x)-h(y)|\le\omega(|x|\vee|y|,|x-y|),\\ |f(t,0,u_1,u_2)|+|g(t,0,u_1,u_2)|+|h(0)|\le L,\end{cases} \tag{1.7}$$

where $|x|\vee|y|=\max\{|x|,|y|\}$.

Note that condition (1.7) implies that the continuity and growth of $(t,x)\mapsto(f(t,x,u_1,u_2),$ $g(t,x,u_1,u_2))$ are uniform in $(u_1,u_2)\in U_1\times U_2$. This essentially will be the case if U_1 and U_2 are bounded (or compact metric spaces). Let us state the following proposition.

Proposition 1.1. *Under assumption* (B), *one has the following:*

(i) *The upper and lower value functions $V^{\pm}(\cdot\,,\cdot)$ are well-defined continuous functions. Moreover, they are the unique viscosity solutions to the upper and lower HJI equations* (1.3), *respectively. In particular, if Isaacs' condition* (1.5) *holds, the upper and lower value functions coincide.*

(ii) *The upper and lower Hamiltonians $H^{\pm}(\cdot\,,\cdot\,,\cdot)$ satisfy the following: For all $t,s\in[0,T],\ x,y,p,q\in\mathbb{R}^n$,*

$$\begin{cases} |H^{\pm}(t,x,p)-H^{\pm}(s,y,q)|\le L(1+|x|)|p-q|\\ \qquad\qquad +(1+|p|\wedge|q|)\Big(L|x-y|+\omega(|x|\vee|y|,|t-s|)\Big),\\ |H^{\pm}(t,x,p)|\le L(1+|x|)|p|+L+\omega(|x|,|x|),\end{cases} \tag{1.8}$$

where $|p| \wedge |q| = \min\{|p|, |q|\}$, *and* $\omega : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is nondecreasing in each argument, and* $\omega(r, 0) = 0$ *for all* $r \geq 0$.

Condition (1.8) plays an important role in the proof of the uniqueness of viscosity solution to HJI equations ([4, 11]). Note that, in particular, (1.8) implies that $p \mapsto H^{\pm}(t, x, p)$ is at most of linear growth.

Unfortunately, the above property (1.8) fails, in general, when the control domains U_1 and/or U_2 is unbounded. To make this more convincing, we look at a one-dimensional linear-quadratic (LQ, for short) optimal control problem (which amounts to saying that $U_1 = \mathbb{R}$ and $U_2 = \{0\}$). Let the state equation be

$$\dot{y}(s) = y(s) + u(s), \qquad s \in [t, T],$$

with a quadratic cost functional

$$J(t, x; u(\cdot)) = \frac{1}{2}\Big[\int_t^T \Big(|y(s)|^2 + |u(s)|^2\Big)ds + |y(T)|^2\Big].$$

Then the Hamiltonian is

$$H(t, x, p) = \inf_{u \in \mathbb{R}} \Big[p(x + u) + \frac{|x|^2 + |u|^2}{2}\Big] = xp + \frac{x^2}{2} - \frac{p^2}{2}.$$

Thus, $p \mapsto H(t, x, p)$ is of quadratic growth and (1.8) fails.

Optimal control problems with unbounded control domains were studied in [2, 6]. Uniqueness of viscosity solution to the corresponding Hamilton-Jacobi-Bellman equation was proved by some arguments relying on the convexity/concavity of the corresponding Hamiltonian with respect to p. Therefore, the results of [2, 6] do not cover the general case of two-person zero-sum differential games, for which the upper and lower Hamiltonians are not necessarily convex or concave, even if the Isaacs' condition holds (see a typical case of such in the next section).

The main purpose of this paper is to study two-person zero-sum differential games with unbounded controls. A main motivation comes from the problem of what we call the affine-quadratic (AQ, for short) two-person zero-sum differential games, by which we mean that the right hand side of the state equation is affine in the controls, and the integrand of the performance functional is quadratic in the controls (see Section 2). This is a natural generalization of the classical LQ problems.

For general two-person zero-sum differential games with unbounded controls, under some very mild coercivity conditions, the upper and lower Hamiltonians $H^{\pm}(t, x, p)$ are proved to be well-defined, continuous, and locally Lipschitz in p. Therefore, the upper and lower HJI equations can be formulated. Then we will establish the uniqueness of viscosity solutions to a general first order Hamilton-Jacobi equation which includes our upper and lower HJI equations of the differential game. By assuming a little stronger coercivity conditions, together with some additional conditions (guaranteeing the well-posedness of the state equation, etc.), we show that the upper and lower value functions can be well-defined and are continuous. Combining the above results, one obtains a characterization of the upper and lower value functions of the differential game as the unique viscosity solutions to the corresponding upper and lower HJI equations. Then if in addition, the Isaacs' condition holds, the upper and lower value functions coincide which yields the existence of the value function of the differential game.

As we have seen above, due to the unboundedness of the controls, the upper and lower Hamiltonians $H^{\pm}(t,x,p)$ will grow super linearly in p. Our approach to the uniqueness of viscosity solution to such HJ equations is a careful modification of the original proofs found in [4, 11]. We would also like to mention here that due to the unboundedness of the control, the continuity of the upper and lower value functions $V^{\pm}(t,x)$ in t is quite subtle. To prove that, we need to establish a modified principle of optimality and fully use the coercivity conditions. It is interesting to indicate that the assumed coercivity conditions that ensuring the finiteness of the upper and lower value functions are actually sharp in some sense, which was illustrated by a one-dimensional LQ situation.

For some literature, more or less relevant to the current paper, we refer the readers to [13, 10, 9, 1, 16, 14], and references cited therein.

The rest of the paper is organized as follows. In Section 2, we make some brief observations on an AQ two-person differential game, for which we have a situation that the Isaacs' condition holds and the upper and lower Hamiltonians $H^{\pm}(t,x,p)$ are quadratic in p but may be neither convex nor concave. Section 3 is devoted to a study of upper and lower Hamiltonians. The uniqueness of viscosity solutions to a class of HJ equations will be proved in Section 4. In Section 5, we will show that under certain conditions, the upper and lower value functions are well-defined and continuous. Finally, in Section 6, we show that the assumed coercivity conditions ensuring the upper and lower value functions to be well-defined are sharp in some sense.

2 An Affine-Quadratic Two-Person Differential Game

To better understand two-person zero-sum differential games with unbounded controls, in this section, we look at a nontrivial special case which is a main motivation of this paper. Consider the following state equation:

$$\begin{cases} \dot y(s) = A(s,y(s)) + B_1(s,y(s))u_1(s) + B_2(s,y(s))u_2(s), \qquad s\in[t,T], \\ y(t) = x, \end{cases} \tag{2.1}$$

for some suitable matrix valued functions $A(\cdot\,,\cdot)$, $B_1(\cdot\,,\cdot)$, and $B_2(\cdot\,,\cdot)$. The state $y(\cdot)$ takes values in $\mathbb{R}^n$ and the control $u_i(\cdot)$ takes values in $U_i = \mathbb{R}^{m_i}$ $(i=1,2)$. The performance functional is given by

$$
\begin{aligned}
J(t,x;u_1(\cdot),u_2(\cdot)) = \int_t^T \Big[& Q(s,y(s)) + \frac{1}{2}\langle R_1(s,y(s))u_1(s), u_1(s)\rangle \\
& + \langle S(s,y(s))u_1(s), u_2(s)\rangle - \frac{1}{2}\langle R_2(s,y(s))u_2(s), u_2(s)\rangle \\
& + \langle \theta_1(s,y(s)), u_1(s)\rangle + \langle \theta_2(s,y(s)), u_2(s)\rangle \Big] ds + G(y(T)),
\end{aligned} \tag{2.2}
$$

for some scalar functions $Q(\cdot\,,\cdot)$ and $G(\cdot)$, some vector valued functions $\theta_1(\cdot\,,\cdot)$ and $\theta_2(\cdot\,,\cdot)$, and some matrix valued functions $R_1(\cdot\,,\cdot)$, $R_2(\cdot\,,\cdot)$, and $S(\cdot\,,\cdot)$. Note that the right hand side of the state equation is affine in the controls $u_1(\cdot)$ and $u_2(\cdot)$, and the integrand in the performance functional is up to quadratic in $u_1(\cdot)$ and $u_2(\cdot)$. Therefore, we refer to such a problem as an *affine-quadratic* (AQ, for short) *two-person zero-sum differential game*. We also note that due to the presence of the term $\langle S(s,y(s))u_1(s), u_2(s)\rangle$, controls $u_1(\cdot)$ and $u_2(\cdot)$ cannot be completely separated. Let us now introduce the following basic hypotheses concerning the above AQ two-person zero-sum differential game.

(AQ1) The maps

$$
A:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n, \quad B_1:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times m_1}, \quad B_2:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times m_2},
$$

are continuous.

(AQ2) The maps

$$
\begin{aligned}
& Q:[0,T]\times\mathbb{R}^n\to\mathbb{R}, \quad G:\mathbb{R}^n\to\mathbb{R}, \\
& R_1:[0,T]\times\mathbb{R}^n\to\mathcal{S}^{m_1}, \quad R_2:[0,T]\times\mathbb{R}^n\to\mathcal{S}^{m_2}, \quad S:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{m_2\times m_1}, \\
& \theta_1:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{m_1}, \quad \theta_2:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{m_2}
\end{aligned}
$$

are continuous (where $\mathcal{S}^m$ stands for the set of all $(m\times m)$ symmetric matrices), and $R_1(t,x)$ and $R_2(t,x)$ are positive definite for all $(t,x)\in[0,T]\times\mathbb{R}^n$.

With the above hypotheses, we let

$$
\begin{aligned}
\mathbb{H}(t,x,p,u_1,u_2) = & \langle p, A(t,x) + B_1(t,x)u_1 + B_2(t,x)u_2\rangle + Q(t,x) \\
& + \frac{1}{2}\langle R_1(t,x)u_1, u_1\rangle + \langle S(t,x)u_1, u_2\rangle - \frac{1}{2}\langle R_2(t,x)u_2, u_2\rangle \\
& + \langle \theta_1(t,x), u_1\rangle + \langle \theta_2(t,x), u_2\rangle .
\end{aligned} \tag{2.3}
$$

Our result concerning the above-defined function is the following proposition.

Proposition 2.1. Let (AQ1)–(AQ2) hold. Then the matrix $\begin{pmatrix} R_1(t,x) & S(t,x)^T \\ S(t,x) & -R_2(t,x)\end{pmatrix}$ is invertible, and

$$
\begin{aligned}
\mathbb{H}(t,x,p,u_1,u_2) = & \frac{1}{2}\langle R_1(t,x)(u_1-\bar u_1), u_1-\bar u_1\rangle + \langle S(t,x)(u_1-\bar u_1), u_2-\bar u_2\rangle \\
& - \frac{1}{2}\langle R_2(t,x)(u_2-\bar u_2), u_2-\bar u_2\rangle + Q_0(t,x,p),
\end{aligned} \tag{2.4}
$$

where

$$\begin{pmatrix} \bar u_1 \\ \bar u_2 \end{pmatrix} = -\begin{pmatrix} R_1(t,x) & S(t,x)^T \\ S(t,x) & -R_2(t,x) \end{pmatrix}^{-1} \begin{pmatrix} B_1(t,x)^T p + \theta_1(t,x) \\ B_2(t,x)^T p + \theta_2(t,x) \end{pmatrix}, \tag{2.5}$$

and

$$\begin{aligned} &Q_0(t,x,p) = Q(t,x) + \langle\, p, A(t,x)\,\rangle \\ &\quad -\frac{1}{2}\begin{pmatrix} B_1(t,x)^T p + \theta_1(t,x) \\ B_2(t,x)^T p + \theta_2(t,x) \end{pmatrix}^T \begin{pmatrix} R_1(t,x) & S(t,x)^T \\ S(t,x) & -R_2(t,x) \end{pmatrix}^{-1} \begin{pmatrix} B_1(t,x)^T p + \theta_1(t,x) \\ B_2(t,x)^T p + \theta_2(t,x) \end{pmatrix}, \end{aligned} \tag{2.6}$$

Further, $(\bar u_1, \bar u_2)$ given by (2.5) is the saddle point of $(u_1,u_2) \mapsto \mathbb{H}(t,x,p,u_1,u_2)$, namely,

$$\mathbb{H}(t,x,p,\bar u_1,u_2) \le \mathbb{H}(t,x,p,\bar u_1,\bar u_2) \le \mathbb{H}(t,x,p,u_1,\bar u_2), \qquad \forall (u_1,u_2) \in U_1 \times U_2, \tag{2.7}$$

and consequently, the Isaacs' condition is satisfied:

$$\begin{aligned} &H^+(t,x,p) \equiv \inf_{u_1\in U_1}\sup_{u_2\in U_2} \mathbb{H}(t,x,p,u_1,u_2) \\ &= \sup_{u_2\in U_2}\inf_{u_1\in U_1} \mathbb{H}(t,x,p,u_1,u_2) \equiv H^-(t,x,p) = Q_0(t,x,p), \\ &\qquad\qquad\qquad \forall (t,x,p) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n. \end{aligned} \tag{2.8}$$

Proof. For simplicity of notation, we suppress (t,x). We write

$$\begin{aligned} \mathbb{H}(p,u_1,u_2) = &\frac{1}{2}\langle\, R_1(u_1-\bar u_1), u_1-\bar u_1\,\rangle + \langle\, S(u_1-\bar u_1), u_2-\bar u_2\,\rangle \\ &-\frac{1}{2}\langle\, R_2(u_2-\bar u_2), u_2-\bar u_2\,\rangle + Q_0, \end{aligned}$$

with $\bar u_1$, $\bar u_2$, and Q_0 undetermined. Then

$$\begin{aligned} &\langle\, p, A\,\rangle + Q + \langle\, B_1^T p + \theta_1, u_1\,\rangle + \langle\, B_2^T p + \theta_2, u_2\,\rangle \\ &+\frac{1}{2}\langle\, R_1u_1, u_1\,\rangle + \langle\, Su_1, u_2\,\rangle - \frac{1}{2}\langle\, R_2u_2, u_2\,\rangle = \mathbb{H}(p,u_1,u_2) \\ &= \frac{1}{2}\langle\, R_1u_1, u_1\,\rangle + \langle\, Su_1, u_2\,\rangle - \frac{1}{2}\langle\, R_2u_2, u_2\,\rangle \\ &\quad -\langle\, R_1\bar u_1, u_1\,\rangle - \langle\, S^T\bar u_2, u_1\,\rangle - \langle\, S\bar u_1, u_2\,\rangle + \langle\, R_2\bar u_2, u_2\,\rangle \\ &\quad +\frac{1}{2}\langle\, R_1\bar u_1, \bar u_1\,\rangle + \langle\, S\bar u_1, \bar u_2\,\rangle - \frac{1}{2}\langle\, R_2\bar u_2, \bar u_2\,\rangle + Q_0. \end{aligned}$$

Hence, we must have

$$\begin{cases} B_1^T p + \theta_1 = -R_1\bar u_1 - S^T\bar u_2, \\ B_2^T p + \theta_2 = -S\bar u_1 + R_2\bar u_2, \\ \langle\, p, A\,\rangle + Q = \frac{1}{2}\langle\, R_1\bar u_1, \bar u_1\,\rangle + \langle\, S\bar u_1, \bar u_2\,\rangle - \frac{1}{2}\langle\, R_2\bar u_2, \bar u_2\,\rangle + Q_0. \end{cases}$$

Consequently,

$$\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix}\begin{pmatrix} \bar u_1 \\ \bar u_2 \end{pmatrix} = -\begin{pmatrix} B_1^T p+\theta_1 \\ B_2^T p+\theta_2 \end{pmatrix}.$$

Note that

$$\det\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix} = \det\begin{pmatrix} R_1 & 0 \\ 0 & -(R_2+SR_1^{-1}S^T) \end{pmatrix}$$
$$= (-1)^{m_2}\det(R_1)\det(R_2+SR_1^{-1}S^T)\neq 0.$$

Thus, $\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix}$ is invertible, which yields

$$\begin{pmatrix} \bar u_1 \\ \bar u_2 \end{pmatrix} = -\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix}^{-1}\begin{pmatrix} B_1^T p+\theta_1 \\ B_2^T p+\theta_2 \end{pmatrix},$$

and

$$Q_0 = \langle\, p, A\,\rangle + Q - \frac{1}{2}\begin{pmatrix} \bar u_1 \\ \bar u_2 \end{pmatrix}^T\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix}\begin{pmatrix} \bar u_1 \\ \bar u_2 \end{pmatrix}$$
$$= \langle\, p, A\,\rangle + Q - \frac{1}{2}\begin{pmatrix} B_1^T p+\theta_1 \\ B_2^T p+\theta_2 \end{pmatrix}^T\begin{pmatrix} R_1 & S^T \\ S & -R_2 \end{pmatrix}^{-1}\begin{pmatrix} B_1^T p+\theta_1 \\ B_2^T p+\theta_2 \end{pmatrix},$$

proving (3.5). Now, we see that

$$\mathbb{H}(p,\bar u_1,u_2) = -\frac{1}{2}\langle\, R_2(u_2-\bar u_2), u_2-\bar u_2\,\rangle + Q_0 \le Q_0 = \mathbb{H}(p,\bar u_1,\bar u_2)$$
$$\le \frac{1}{2}\langle\, R_1(u_1-\bar u_1), u_1-\bar u_1\,\rangle + Q_0(t,x,p) = \mathbb{H}(p,u_1,\bar u_2),$$

which means that $(\bar u_1,\bar u_2)$ is a saddle point of $\mathbb{H}(t,x,p,u_1,u_2)$. Then the Isaacs condition (2.8) follows easily. Finally, since R_1 and R_2 are positive definite, the saddle point must be unique. □

We see that in the current case, $p\mapsto H^\pm(t,x,p)$ is quadratic, and is neither convex nor concave in general. As a matter of fact, the Hessian $H^\pm_{pp}(t,x,p)$ of $H^\pm(t,x,p)$ is given by the following:

$$H^\pm_{pp}(t,x,p) = -\frac{1}{2}\begin{pmatrix} B_1(t,x)^T \\ B_2(t,x)^T \end{pmatrix}^T\begin{pmatrix} R_1(t,x) & S(t,x)^T \\ S(t,x) & -R_2(t,x) \end{pmatrix}^{-1}\begin{pmatrix} B_1(t,x)^T \\ B_2(t,x)^T \end{pmatrix}.$$

which is indefinite in general.

We have seen from the above that in order the upper and lower Hamiltonians to be well-defined, the only crucial assumption that we made is the positive definiteness of the matrix-valued maps $R_1(\cdot\,,\cdot)$ and $R_2(\cdot\,,\cdot)$. Whereas, in order to study the AQ two-person zero-sum differential games, we need a little stronger hypotheses. For the state equation to be well-posed for reasonable controls, we need the following assumption.

(AQ1)′ The maps

$$A:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n,\quad B_1:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times m_1},\quad B_2:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times m_2},$$

are continuous and for some constant $L>0$,

$$\begin{cases}|A(t,x)-A(t,y)|+|B_1(t,x)-B_1(t,y)|+|B_2(t,x)-B_2(t,y)|\le L|x-y|,\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t\in[0,T],\ x,y\in\mathbb{R}^n,\\ |A(t,0)|+|B_1(t,0)|+|B_2(t,0)|\le L,\qquad \forall t\in[0,T].\end{cases}\tag{2.9}$$

Under (AQ1)′, for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $(u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1^1[t,T]\times\mathcal{U}_2^1[t,T]$, state equation (2.1) admits a unique solution $y(\cdot)\equiv y(\cdot\,;t,x,u_1(\cdot),u_2(\cdot))$. Moreover, we have

$$\begin{aligned}|y(s)|&\le|x|+\int_t^s\Big[|A(r,y(r))+B_1(r,y(r))u_1(r)+B_2(r,y(r))u_2(r)|\Big]dr\\&\le|x|+L\int_t^s\Big(1+|y(r)|\Big)\Big(1+|u_1(r)|+|u_2(r)|\Big)dr,\end{aligned}$$

which leads to the following estimate:

$$|y(s)|\le C\Big[1+|x|+\int_t^T\Big(|u_1(r)|+|u_2(r)|\Big)dr\Big]e^{L\int_t^T(|u_1(r)|+|u_2(r)|)dr},\qquad s\in[t,T].\tag{2.10}$$

Hereafter, C stands for a generic constant which can be different from line to line. Now, for the performance functional to be well-defined, we need to assume the following:

(AQ2)′ The maps

$$\begin{aligned}&Q:[0,T]\times\mathbb{R}^n\to\mathbb{R},\quad G:\mathbb{R}^n\to\mathbb{R},\\&R_1:[0,T]\times\mathbb{R}^n\to\mathcal{S}^{m_1},\quad R_2:[0,T]\times\mathbb{R}^n\to\mathcal{S}^{m_2},\quad S:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{m_2\times m_1},\\&\theta_1:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{m_1},\quad \theta_2:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{m_2}\end{aligned}$$

are continuous and there are constants $L,c,\mu>0$ such that

$$\begin{aligned}&|Q(t,x)|+|G(x)|+|R_1(t,x)|+|R_2(t,x)|+|\theta_1(t,x)|+|\theta_2(t,x)|\le L(1+|x|^\mu),\\&R_1(t,x)\ge cI_{m_1},\quad R_2(t,x)\ge cI_{m_2},\qquad \forall(t,x)\in[0,T]\times\mathbb{R}^n.\end{aligned}\tag{2.11}$$

Under (AQ1)′–(AQ2)′, we have

$$\begin{aligned}&\Big|Q(s,y(s))+\frac12\langle R_1(s,y(s))u_1(s),u_1(s)\rangle+\langle S(s,y(s))u_1(s),u_2(s)\rangle\\&\qquad-\frac12\langle R_2(s,y(s))u_2(s),u_2(s)\rangle+\langle\theta_1(s,y(s)),u_1(s)\rangle+\langle\theta_2(s,y(s)),u_2(s)\rangle\Big|\\&\le C\Big(1+|y(s)|^\mu\Big)\Big(1+|u_1(s)|^2+|u_2(s)|^2\Big),\end{aligned}$$

and

$$|G(y(T))|\le L(1+|y(T)|^\mu).$$

Hence, for any $(u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1^2[t,T]\times\mathcal{U}_2^2[t,T]$, noting (2.10), we see that the performance functional $J(t,x;u_1(\cdot),u_2(\cdot))$ is well-defined. Also, we note that

$$\begin{aligned}&Q(t,x)+\frac{1}{2}\langle\, R_1(t,x)u_1,u_1\,\rangle+\langle\, S(t,x)u_1,u_2\,\rangle-\frac{1}{2}\langle\, R_2(t,x)u_2,u_2\,\rangle\\&\qquad\qquad\qquad+\langle\,\theta_1(t,x),u_1\,\rangle+\langle\,\theta_2(t,x),u_2\,\rangle\\&\le C(1+|x|^\mu)(1+|u_1|^2)-\frac{c}{4}|u_2|^2,\end{aligned}$$

and

$$\begin{aligned}&Q(t,x)+\frac{1}{2}\langle\, R_1(t,x)u_1,u_1\,\rangle+\langle\, S(t,x)u_1,u_2\,\rangle-\frac{1}{2}\langle\, R_2(t,x)u_2,u_2\,\rangle\\&\qquad\qquad\qquad+\langle\,\theta_1(t,x),u_1\,\rangle+\langle\,\theta_2(t,x),u_2\,\rangle\\&\ge -C(1+|x|^\mu)(1+|u_2|^2)+\frac{c}{4}|u_1|^2,\end{aligned}$$

Then, it is possible to define the upper and lower value functions.

3 Upper and Lower Hamiltonians

In this section, we will carefully look at the upper and lower Hamiltonians associated with general two-person zero-sum differential games with unbounded controls. First of all, we introduce the following standing assumption which will be assumed throughout of the rest of the paper without further mentioning.

(H0) For $i=1,2$, the set $U_i\subseteq\mathbb{R}^{m_i}$ is closed and

$$0\in U_i,\qquad i=1,2. \tag{3.1}$$

The time horizon $T>0$ is fixed.

Note that U_i could be unbounded and may even be equal to $\mathbb{R}^{m_i}$. Condition (3.1) is for convenience. We may make a translation of the control domains and make corresponding changes in the control systems and performance functional to achieve this.

Inspired by the AQ two-person zero-sum differential games, let us now introduce the following assumptions for the involved functions f and g in the state equation (1.1) and the performance functional (1.2). We denote $\mathbb{R}_+=[0,\infty)$.

(H1) Map $f:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}^n$ is continuous and there are constants $\sigma_1,\sigma_2\ge0$ and a nondecreasing function $N:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\begin{aligned}&|f(t,x,u_1,u_2)|\le N(|x|)(1+|u_1|^{\sigma_1}+|u_2|^{\sigma_2}),\\&\qquad\qquad\qquad\forall(t,x,u_1,u_2)\in[0,T]\times\mathbb{R}^n\times U_1\times U_2.\end{aligned}\tag{3.2}$$

(H2) Map $g:[0,T]\times\mathbb{R}^n\times U_1\times U_2\to\mathbb{R}$ is continuous and there exist constants $c,\rho_1,\rho_2>0$, and a nondecreasing function $N:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\begin{aligned}&c|u_1|^{\rho_1}-N(|x|)(1+|u_2|^{\rho_2})\le g(t,x,u_1,u_2)\le N(|x|)(1+|u_1|^{\rho_1})-c|u_2|^{\rho_2},\\&\qquad\qquad\qquad\qquad\forall(t,x,u_1,u_2)\in[0,T]\times\mathbb{R}^n\times U_1\times U_2.\end{aligned}\tag{3.3}$$

Note that in the above, we let the function $N(\cdot)$ be the same in (H1) and (H2). Replacing the smaller by the larger, we can always achieve that. We will also need the following compatibility hypothesis which is crucial below.

(H3) The constants $\sigma_1, \sigma_2, \rho_1, \rho_2$ in (H1)–(H2) satisfy the following:

$$\sigma_1 < \rho_1, \qquad \sigma_2 < \rho_2. \tag{3.4}$$

It is not hard to see that the above (H1)–(H3) includes the AQ two-person zero-sum differential game described in the previous section as a special case. Now, we let

$$\begin{aligned}\mathbb{H}(t,x,p,u_1,u_2) = \langle\, p, f(t,x,u_1,u_2)\,\rangle + g(t,x,u_1,u_2),&\\ (t,x,u_1,u_2)\in[0,T]\times\mathbb{R}^n\times U_1\times U_2.&\end{aligned} \tag{3.5}$$

Then the *upper* and *lower Hamiltonians* are defined as follows:

$$\begin{cases} H^+(t,x,p) = \inf\limits_{u_1\in U_1}\sup\limits_{u_2\in U_2}\mathbb{H}(t,x,p,u_1,u_2), \\ H^-(t,x,p) = \sup\limits_{u_2\in U_2}\inf\limits_{u_1\in U_1}\mathbb{H}(t,x,p,u_1,u_2),\end{cases} \qquad (t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n. \tag{3.6}$$

Note that the upper and lower Hamiltonians are nothing to do with the function $h(\cdot)$ (appears as the terminal cost/payoff in (1.2)). The main result of this section is the following.

Proposition 3.1. *Under* (H1)–(H3), *the upper and lower Hamitonians* $H^\pm(\cdot\,,\cdot\,,\cdot)$ *are well-defined. Moreover, they are continuous and there exists a nondecreasing continuous function* $N:\mathbb{R}_+\to\mathbb{R}_+$ *such that*

$$|H^\pm(t,x,p)|\le N(|x|+|p|), \qquad \forall (t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n, \tag{3.7}$$

and

$$\begin{aligned}|H^\pm(t,x,p)-H^\pm(t,x,q)|\le N(|x|+|p|+|q|)|p-q|,&\\ \forall (t,x)\in[0,T]\times\mathbb{R}^n,\ \ p,q\in\mathbb{R}^n.&\end{aligned} \tag{3.8}$$

Proof. In what follows, $N(\cdot)$ will represent a generic nondecreasing continuous function from $\mathbb{R}_+$ to itself, and it could be different from line to line, and C will be a generic constant which could be different from line to line. Let us look at $H^+(t,x,p)$ carefully ($H^-(t,x,p)$ can be treated similarly). To this end, we first observe the following:

$$\begin{aligned}&\mathbb{H}(t,x,p,u_1,u_2)\le |p|\,|f(t,x,u_1,u_2)| + g(t,x,u_1,u_2)\\ &\le |p|N(|x|)(1+|u_1|^{\sigma_1}+|u_2|^{\sigma_2}) + N(|x|)(1+|u_1|^{\rho_1}) - c|u_2|^{\rho_2}\\ &= N(|x|)\Big[|p|(1+|u_1|^{\sigma_1}) + (1+|u_1|^{\rho_1})\Big] + |p|N(|x|)|u_2|^{\sigma_2} - c|u_2|^{\rho_2}\\ &\le N(|x|)\Big[1+|p|+|p||u_1|^{\sigma_1}+|u_1|^{\rho_1}\Big] + C\Big(|p|N(|x|)\Big)^{\frac{\rho_2}{\rho_2-\sigma_2}} - \frac{c}{2}|u_2|^{\rho_2}\\ &\le N(|x|+|p|)(1+|u_1|^{\rho_1}) - \frac{c}{2}|u_2|^{\rho_2}.\end{aligned} \tag{3.9}$$

Thus, $u_2 \mapsto \mathbb{H}(t,x,p,u_1,u_2)$ is coercive from above. Consequently, since U_2 is closed, for any given $(t,x,p,u_1) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U_1$, there exists a $\bar u_2 \equiv \bar u_2(t,x,p,u_1) \in U_2$ such that

$$\begin{aligned}&\mathcal{H}^+(t,x,p,u_1) \equiv \mathbb{H}(t,x,p,u_1,\bar u_2) = \sup_{u_2\in U_2} \mathbb{H}(t,x,p,u_1,u_2)\\ &= \sup_{u_2\in U_2,\, |u_2|\le|\bar u_2|} \mathbb{H}(t,x,p,u_1,u_2) \le N(|x|+|p|)(1+|u_1|^{\rho_1}) - \frac{c}{2}|\bar u_2|^{\rho_2}\\ &\le N(|x|+|p|)(1+|u_1|^{\rho_1}).\end{aligned} \tag{3.10}$$

Hence,

$$\begin{aligned}H^+(t,x,p) &= \inf_{u_1\in U_1}\sup_{u_2\in U_2} \mathbb{H}(t,x,p,u_1,u_2) \le \sup_{u_2\in U_2} \mathbb{H}(t,x,p,0,u_2)\\ &\equiv \mathcal{H}^+(t,x,p,0) \le N(|x|+|p|).\end{aligned} \tag{3.11}$$

which means that $H^+(t,x,p)$ is well-defined and locally bounded from above. On the other hand, similar to the above, we have

$$\begin{aligned}&\mathbb{H}(t,x,p,u_1,u_2) \ge -|p|\,|f(t,x,u_1,u_2)| - g(t,x,u_1,u_2)\\ &\ge -N(|x|+|p|)(1+|u_2|^{\rho_2}) + \frac{c}{2}|u_1|^{\rho_1}.\end{aligned} \tag{3.12}$$

Therefore,

$$\begin{aligned}&\mathcal{H}^+(t,x,p,u_1) = \sup_{u_2\in U_2} \mathbb{H}(t,x,p,u_1,u_2) \ge \mathbb{H}(t,x,p,u_1,0)\\ &\ge -N(|x|+|p|) + \frac{c}{2}|u_1|^{\rho_1} \ge -N(|x|+|p|).\end{aligned} \tag{3.13}$$

Combining the above, we obtain (3.7) for $H^+(\cdot\,,\cdot\,,\cdot)$.

Next, we want to get the local Lipschitz continuity of $p \mapsto H^+(t,x,p)$. To this end, we note that (3.10) and (3.13) imply

$$\begin{aligned}\frac{c}{2}|\bar u_2|^{\rho_2} &\le N(|x|+|p|)(1+|u_1|^{\rho_1}) - \mathcal{H}^+(t,x,p,u_1)\\ &\le N(|x|+|p|)(1+|u_1|^{\rho_1}) + N(|x|+|p|)\\ &\le N(|x|+|p|)|u_1|^{\rho_1} + 2N(|x|+|p|).\end{aligned} \tag{3.14}$$

For the above $N(|x|+|p|)$ (which is taken to be the largest among those in (3.11), (3.13) and (3.14)), let

$$U_1(x,p) = \Big\{u_1 \in U_1 \;\Big|\; \frac{c}{2}|u_1|^{\rho_1} \le 2N(|x|+|p|) + 1\Big\},$$

which is a compact set. Then for any $u_1 \in U_1 \setminus U_1(x,p)$, we have (by (3.13) and (3.11))

$$\begin{aligned}&\mathcal{H}^+(t,x,p,u_1) \ge -N(|x|+|p|) + \frac{c}{2}|u_1|^{\rho_1}\\ &> N(|x|+|p|) + 1 \ge H^+(t,x,p) + 1 = \inf_{u_1\in U_1} \mathcal{H}^+(t,x,p,u_1) + 1.\end{aligned}$$

Hence,

$$\inf_{u_1\in U_1} \mathcal{H}^+(t,x,p,u_1) = \inf_{u_1\in U_1(x,p)} \mathcal{H}^+(t,x,p,u_1).$$

Next, by (3.14), we see that for $u_1 \in U_1(x,p)$,

$$\frac{c}{2}|\bar u_2|^{\rho_2} \le N(|x|+|p|)\Big[\frac{4}{c}N(|x|+|p|)+\frac{2}{c}\Big]+2N(|x|+|p|)\le N(|x|+|p|),$$

with a different $N(\cdot)$. Combining the above, we see that there exists a nondecreasing continuous function $N:\mathbb{R}_+\to\mathbb{R}_+$ such that for any $(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$,

$$H^+(t,x,p)=\inf_{u_1\in U_1(N(|x|+|p|))}\ \sup_{u_2\in U_2(N(|x|+|p|))}\mathbb{H}(x,p,u_1,u_2), \tag{3.15}$$

where

$$U_i(r)=\Big\{u_i\in U_i\ |\ |u_i|\le r\Big\},\qquad i=1,2.$$

Next, we observe the following: For any $u_i\in U_i(N(|x|+|p|+|q|+1))$ $(i=1,2)$

$$\begin{aligned}&|\mathbb{H}(t,x,p,u_1,u_2)-\mathbb{H}(t,x,q,u_1,u_2)|\\&\le|p-q|\,|f(t,y,u_1,u_2)|\le N(|x|)(1+|u_1|^{\sigma_1}+|u_2|^{\sigma_2})|p-q|\\&\le N(|x|)\Big[1+N(|x|+|p|+|q|+1)^{\sigma_1}+N(|x|+|p|+|q|+1)^{\sigma_2}\Big]|p-q|\\&\le N(|x|+|p|+|q|)|p-q|,\end{aligned}$$

with a possible different $N(\cdot)$. Hence, (3.8) holds for $H^+(\cdot\,,\cdot\,,\cdot)$. Likewise, due to the fact that the infimum and supremum in (3.15) can be taken on compact sets, we can prove the continuity of $(t,x)\mapsto H^+(t,x,p)$. □

4 Uniqueness of Viscosity Solution

Consider the following HJ equation:

$$\begin{cases}V_t+H(t,x,V_x)=0, & (t,x)\in[0,T]\times\mathbb{R}^n,\\ V(T,x)=h(x), & x\in\mathbb{R}^n.\end{cases} \tag{4.1}$$

We recall the following definition.

Definition 4.1. (i) A continuous function $V(\cdot\,,\cdot)$ is called a *viscosity sub-solution* of (4.1) if

$$V(T,x)\le h(x),\qquad \forall x\in\mathbb{R}^n,$$

and for any continuous differentiable function $\varphi(\cdot\,,\cdot)$, if $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$ is a local maximum of $(t,x)\mapsto V(t,x)-\varphi(t,x)$, then

$$\varphi_t(t_0,x_0)+H(t_0,x_0,\varphi_x(t_0,x_0))\ge 0.$$

(ii) A continuous function $V(\cdot\,,\cdot)$ is called a *viscosity super-solution* of (4.1) if

$$V(T,x)\ge h(x),\qquad \forall x\in\mathbb{R}^n,$$

and for any continuous differentiable function $\varphi(\cdot\,,\cdot)$, if $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$ is a local minimum of $(t,x)\mapsto V(t,x)-\varphi(t,x)$, then

$$\varphi_t(t_0,x_0)+H(t_0,x_0,\varphi_x(t_0,x_0))\le 0.$$

(iii) If continuous function $V(\cdot\,,\cdot)$ is viscosity sub-solution and super-solution of (4.1), it is called a *viscosity solution* of (4.1).

We have the following theorem.

Theorem 4.2. Let $H:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and $h:\mathbb{R}^n\to\mathbb{R}$ be continuous and $p\mapsto H(t,x,p)$ is local Lipschitz. Then the HJ equation (4.1) has at most one viscosity solution.

Proof. First of all, by the continuity of $H(\cdot\,,\cdot\,,\cdot)$ and the local Lipschitz continuity of $p\mapsto H(t,x,p)$, we can suppose that there are continuous functions $M:\mathbb{R}_+\to\mathbb{R}_+$ and $\omega:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ with $M(\cdot)$ being nondecreasing, and $\omega(\cdot\,,\cdot)$ being nondecreasing in each of its arguments, and $\omega(r,0)=0$, for all $r\ge 0$ such that

$$\begin{cases}|H(t,x,p)-H(t,x,q)|\le M\Big(|x|+|p|+|q|\Big)|p-q|,\\ |H(t,x,p)-H(t,y,p)|\le \omega\Big(|x|+|y|+|p|,|t-s|+|x-y|\Big),\\ \qquad\qquad\qquad\qquad\forall t,s\in\Big[0,T\Big],\ x,y,p,q\in\mathbb{R}^n.\end{cases}\tag{4.2}$$

We split the rest of the proof into several steps.

Step 1. Construction of the domain $\mathcal{N}_{\delta,\varepsilon}$.

Suppose $V(\cdot\,,\cdot)$ and $\widehat V(\cdot\,,\cdot)$ are two viscosity solutions of the HJ equation (4.1). To prove the uniqueness it suffices to prove

$$V(t,x)\le \hat V(t,x),\qquad\qquad \forall (t,x)\in[0,T]\times\mathbb{R}^n.\tag{4.3}$$

Suppose the above fails. Then, without loss of generality, we may assume that

$$\sup_{t\in[T-\tau,T]}\Big[V(t,0)-\widehat V(t,0)\Big]>0,\qquad \forall\tau\in(0,T).\tag{4.4}$$

Now, let us fix such a $\tau>0$, small enough with

$$\tau<\min\Big\{T,\frac{1}{L}\Big\}.$$

Then let $L_0>0$ be undetermined such that

$$L_0\ge\frac{L}{1-L\tau},\tag{4.5}$$

and define

$$\mathcal{N}=\Big\{(t,x)\in(T-\tau,T)\times\mathbb{R}^n\;\Big|\;|x|<L_0(t-T+\tau)\Big\}.$$

This is a cylindrical domain: The vertex is at $(T-\tau,0)$, the axis is the t-axis, opening in the positive t-direction, and the base is a disk centered at $x=0$ with radius $L_0\tau$. Clearly,

$$(t,0)\in\overline{\mathcal{N}},\qquad \forall t\in[T-\tau,T]. \tag{4.6}$$

Next, we take a constant $K>0$ satisfying (note (4.4) and (4.6))

$$K>\sup_{(t,x,s,y)\in\mathcal{N}\times\mathcal{N}}\Big[V(t,x)-\widehat{V}(s,y)\Big]\geq\sup_{(t,x)\in\mathcal{N}}\Big[V(t,x)-\widehat{V}(t,x)\Big]>0. \tag{4.7}$$

Then let $\delta>0$ such that

$$\frac{(2K)^\mu}{\delta^\mu}+2\delta<L_0\tau. \tag{4.8}$$

This and (4.5) can be simultaneously achieved by enlarging L_0 if necessary. We keep in mind that L_0 can further be enlarged, which will be done towards the end of the proof. Now, let $\varepsilon\in(0,\delta)$ and define

$$\mathcal{N}_{\delta,\varepsilon}=\Big\{(t,x)\in(T-\tau,T)\times\mathbb{R}^n \,|\, \langle\, x\,\rangle_{\varepsilon}\leq L_0(t-T)+\delta\Big\}, \tag{4.9}$$

where $\langle\, x\,\rangle_{\varepsilon}=\sqrt{|x|^2+\varepsilon^2}$. Then for any $(t,x)\in\mathcal{N}_{\delta,\varepsilon}$, we have (noting (4.8))

$$|x|\leq\sqrt{|x|^2+\varepsilon^2}=\langle\, x\,\rangle_{\varepsilon}\leq L_0(t-T)+\delta\leq L_0(t-T+\tau).$$

This means

$$\mathcal{N}_{\delta,\varepsilon}\subseteq\mathcal{N}. \tag{4.10}$$

Also, we have (since $\varepsilon<\delta$)

$$\langle\, 0\,\rangle_{\varepsilon}=\varepsilon<\delta\leq L_0(t-T+\tau),$$

provided (making use of (4.8))

$$t>T-\tau+\frac{\delta}{L_0}=T-\frac{L_0\tau-\delta}{L_0}>T-\frac{\tau}{2}.$$

Hence, we have

$$(0,t)\in\mathcal{N}_{\delta,\varepsilon},\qquad\forall t\in[T-\frac{\tau}{2},T).$$

Consequently, by (4.4), we obtain

$$\sup_{(t,x,s,y)\in\mathcal{N}_{\delta,\varepsilon}\times\mathcal{N}_{\delta,\varepsilon}}\Big[V(t,x)-\widehat{V}(s,y)\Big]\geq\sup_{t\in[T-\frac{\tau}{2},T]}\Big[V(t,0)-\widehat{V}(t,0)\Big]=\sigma>0. \tag{4.11}$$

We observe that for any $(t,x)\in\mathcal{N}_{\delta,\varepsilon}$ and $p,q\in\mathbb{R}^n$, the following holds

$$\begin{aligned}&|H(t,x,p)-H(t,x,q)|\leq M\Big(|x|+|p|+|q|\Big)|p-q|\\&\leq M\Big(\langle\, x\,\rangle_{\varepsilon}+|p|+|q|\Big)|p-q|\\&\leq M\Big(L_0(t-T)+\delta+|p|+|q|\Big)|p-q|\leq M\Big(\delta+|p|+|q|\Big)|p-q|.\end{aligned} \tag{4.12}$$

Step 2. Construction and analysis of the auxiliary function.

We take $\zeta(\cdot) \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} \zeta(r) = \begin{cases} 0, & r \leq -\delta, \\ -K, & r \geq 0, \end{cases} \\ \zeta'(r) \leq 0, \qquad |\zeta'(r)| \leq \dfrac{2K}{\delta}, \qquad \forall r \in \mathbb{R}. \end{cases} \tag{4.13}$$

For any $\alpha, \beta > 0$ small enough, and $0 < \gamma < \frac{\sigma}{4\tau}$, we define

$$\begin{aligned} \Phi(t,x,s,y) &= V(t,x) - \widehat{V}(s,y) - \frac{1}{\alpha}|t-s|^2 - \frac{1}{\beta}|x-y|^2 \\ &\quad + \zeta\Big(\langle\, x \,\rangle_{\varepsilon} - L_0(t-T) - 2\delta \Big) + \zeta\Big(\langle\, y \,\rangle_{\varepsilon} - L_0(s-T) - 2\delta \Big) \\ &\quad + \gamma(t+s) - 2\gamma T, \qquad\qquad \forall (t.x,s,y) \in \mathcal{N} \times \mathcal{N}. \end{aligned} \tag{4.14}$$

Since Φ is continuous, and $\overline{\mathcal{N} \times \mathcal{N}}$ is compact, we may assume that Φ attains its maximum over $\overline{\mathcal{N} \times \mathcal{N}}$ at $(t_0, x_0, s_0, y_0) \in \overline{\mathcal{N} \times \mathcal{N}}$. Note that the point (t_0, x_0, s_0, y_0) depends on the choice of (α, β) (with $\delta, \varepsilon, \gamma$ fixed). Our next goal is to show that there exists an $r_0 > 0$ such that for any $0 < \alpha, \beta < r_0$,

$$(t_0, x_0, s_0, y_0) \in \mathcal{N}_{\delta,\varepsilon} \times \mathcal{N}_{\delta,\varepsilon}. \tag{4.15}$$

We first claim that

$$\begin{cases} \langle\, x_0 \,\rangle_{\varepsilon} < L_0(t_0 - T) + 2\delta, \\ \langle\, y_0 \,\rangle_{\varepsilon} < L_0(s_0 - T) + 2\delta, \end{cases} \tag{4.16}$$

Indeed, if (4.16) is not true, then either

$$\langle\, x_0 \,\rangle_{\varepsilon} - L_0(t_0 - T) + 2\delta > 0,$$

or

$$\langle\, y_0 \,\rangle_{\varepsilon} - L_0(s_0 - T) + 2\delta > 0.$$

Thus, by the optimality of (t_0, x_0, s_0, y_0), we have

$$\begin{aligned} 0 &= V(T,0) - \widehat{V}(T,0) + 2\zeta(\varepsilon - 2\delta) = \Phi(T,0,T,0) \leq \Phi(t_0,x_0,s_0,y_0) \\ &\leq V(t_0,x_0) - \widehat{V}(s_0,y_0) - \frac{1}{\alpha}|t_0-s_0|^2 - \frac{1}{\beta}|x_0-y_0|^2 \\ &\quad + \zeta\Big(\langle\, x_0 \,\rangle_{\varepsilon} - L_0(t_0-T) - 2\delta \Big) + \zeta\Big(\langle\, y_0 \,\rangle_{\varepsilon} - L_0(s_0-T) - 2\delta \Big) \\ &\quad + \gamma(t_0+s_0) - 2\gamma T \\ &< K + \zeta\Big(\langle\, x_0 \,\rangle_{\varepsilon} - L_0(t_0-T) - 2\delta \Big) + \zeta\Big(\langle\, y_0 \,\rangle_{\varepsilon} - L_0(s_0-T) - 2\delta \Big) \\ &\quad + \gamma(t_0+s_0) - 2\gamma T \\ &\leq K - K + \gamma(t_0+s_0) - 2\gamma T \leq 0, \end{aligned} \tag{4.17}$$

which is a contradiction. Hence (4.16) holds.

Next, we claim that for small enough α, β, the following cannot happen:

$$t_0 = T, \quad \text{or} \quad s_0 = T. \tag{4.18}$$

To show the above, let us first look the following consequence of the optimality of (t_0, x_0, s_0, y_0):

$$\begin{aligned}
&V(t_0,x_0) - \widehat V(t_0,x_0) + 2\zeta\Big(\langle\, x_0\,\rangle_{\,\varepsilon} - L_0(t_0-T) - 2\delta\Big)\\
&\qquad + V(s_0,y_0) - \widehat V(s_0,y_0) + 2\zeta\Big(\langle\, y_0\,\rangle_{\,\varepsilon} - L_0(s_0-T) - 2\delta\Big) + 2\gamma(t_0+s_0) - 4\gamma T\\
&= \Phi(t_0,x_0,t_0,x_0) + \Phi(s_0,y_0,s_0,y_0) \le 2\Phi(t_0,x_0,s_0,y_0)\\
&\le 2V(t_0,x_0) - 2\widehat V(s_0,y_0) - \frac{2}{\alpha}|t_0-s_0|^2 - \frac{2}{\beta}|x_0-y_0|^2\\
&\qquad + 2\zeta\Big(\langle\, x_0\,\rangle_{\,\varepsilon} - L_0(t_0-T) - 2\delta\Big)\\
&\qquad + 2\zeta\Big(\langle\, y_0\,\rangle_{\,\varepsilon} - L_0(s_0-T) - 2\delta\Big) + 2\gamma(t_0+s_0) - 4\gamma T,
\end{aligned} \tag{4.19}$$

which yields

$$\begin{aligned}
\frac{2}{\alpha}|t_0-s_0|^2 + \frac{2}{\beta}|x_0-y_0|^2 &\le V(t_0,x_0) - V(s_0,y_0) + \widehat V(t_0,x_0) - \widehat V(s_0,y_0)\\
&\le 2\eta(|t_0-s_0| + |x_0-y_0|),
\end{aligned} \tag{4.20}$$

where

$$\eta(r) = \frac{1}{2}\sup_{\substack{|t-s|+|x-y|\le r\\ (t,x,s,y)\in\mathcal N\times\mathcal N}} \Big\{|V(t_0,x_0) - V(s_0,y_0)| + |\widehat V(t_0,x_0) - \widehat V(s_0,y_0)|\Big\}. \tag{4.21}$$

Clearly, by the continuity of $V(\cdot\,,\cdot)$ and $\widehat V(\cdot\,,\cdot)$, together with the boundedness of $\mathcal N$, we have

$$0 = \lim_{r\to 0}\eta(r) \le \sup_{r>0}\eta(r) \equiv \eta_0 < \infty. \tag{4.22}$$

Then it follows from (4.20) that

$$|t_0 - s_0| \le \sqrt{\alpha\eta_0}, \qquad |x_0 - y_0| \le \sqrt{\beta\eta_0}. \tag{4.23}$$

Combining (4.20) with (4.21), we further have

$$\frac{1}{\alpha}|t_0-s_0|^2 + \frac{1}{\beta}|x_0-y_0|^2 \le \eta(\sqrt{\alpha\eta_0} + \sqrt{\beta\eta_0}). \tag{4.24}$$

Now, let us show that (4.18) is not possible when $\alpha, \beta > 0$ are small enough by contradiction. Suppose for a sequence $(\alpha_m, \beta_m) \to (0,0)$, the corresponding maximum point, denoted by, (t_m, x_m, s_m, y_m) of Φ over $\overline{\mathcal N_{\delta,\varepsilon}\times\mathcal N_{\delta,\varepsilon}}$ satisfies

$$t_m = T, \quad \text{or} \quad s_m = T, \quad \forall m \ge 1. \tag{4.25}$$

By (4.23), we obtain

$$|x_m - y_m| \to 0, \quad t_m, s_m \to T, \quad \text{as} \quad m \to \infty. \tag{4.26}$$

Hence, noting $2\gamma\tau < \frac{\sigma}{2}$, we have

$$\begin{aligned}
0 < \frac{\sigma}{2} &\le \sup_{(t,x)\in\mathcal{N}_{\delta,\varepsilon}} \Big[V(t,x) - \widehat{V}(t,x) + 2\gamma(t-T)\Big] = \sup_{(t,x)\in\mathcal{N}_{\delta,\varepsilon}} \Phi(t,x,t,x) \\
&\le \sup_{(t,x,s,y)\in\mathcal{N}_{\varepsilon,\delta}\times\mathcal{N}_{\delta,\varepsilon}} \Phi(t,x,s,y) = \Phi(t_m,x_m,s_m,y_m) \\
&\le V(t_m,x_m) - \widehat{V}(s_m,y_m) \to 0, \qquad \text{as} \quad m\to\infty,
\end{aligned} \tag{4.27}$$

leading to a contradiction. Hence, when $\alpha,\beta>0$ small, (4.18) is not possible.

Combining what we have proved, we have (4.15).

Step 3. Completion of the proof by the definition of viscosity solutions.

Since for $0<\alpha,\beta<r_0$, the point (t_0,x_0,s_0,y_0) is in the interior of $\mathcal{N}_{\delta,\varepsilon}\times\mathcal{N}_{\delta,\varepsilon}$, we are ready to use the definition of viscosity solutions. First, we see that the function

$$\begin{aligned}
(t,x)\mapsto V(t,x) - \Big\{\widehat{V}(s_0,y_0) + \frac{1}{\alpha}|t-s_0|^2 + \frac{1}{\beta}|x-y_0|^2 \\
-\zeta\Big(\langle x\rangle_\varepsilon - L_0(t-T) - 2\delta\Big) \\
-\zeta\Big(\langle y_0\rangle_\varepsilon - L_0(s_0-T) - 2\delta\Big) - \gamma(t+s_0) + 2\gamma T\Big\},
\end{aligned} \tag{4.28}$$

attain its maximum at $(t_0,x_0)\in\mathcal{N}_{\delta,\varepsilon}$. Hence we have

$$\frac{2}{\alpha}(t_0-s_0) + \zeta'(X)L_0 - \gamma + H\Big(t_0,x_0,\frac{2}{\beta}(x_0-y_0) - \zeta'(X)\langle x_0\rangle_\varepsilon^{-1}x_0\Big) \ge 0, \tag{4.29}$$

where $X = \langle x_0\rangle_\varepsilon - L_0(t_0-T) - 2\delta$. Similarly, the function

$$\begin{aligned}
(s,y)\mapsto \widehat{V}(s,y) - \Big\{V(t_0,x_0) - \frac{1}{\alpha}|t_0-s|^2 - \frac{1}{\beta}|x_0-y|^2 \\
+\zeta\Big(\langle x_0\rangle_\varepsilon - L_0(t_0-T) - 2\delta\Big) \\
+\zeta\Big(\langle y\rangle_\varepsilon - L_0(s-T) - 2\delta\Big) + \gamma(t_0+s) - 2\gamma T\Big\},
\end{aligned} \tag{4.30}$$

attain its minimum at $(s_0,y_0)\in\mathcal{N}_{\delta,\varepsilon}$. Hence we have

$$-\frac{2}{\alpha}(s_0-t_0) - \zeta'(Y)L_0 + \gamma + H\Big(s_0,y_0,-\frac{2}{\beta}(y_0-x_0) + \zeta'(Y)\langle y_0\rangle_\varepsilon^{-1}y_0\Big) \le 0, \tag{4.31}$$

where $Y = \langle y_0\rangle_\varepsilon - L_0(s_0-T) - 2\delta$. Combining (1.19) – (1.24), we obtain

$$\begin{aligned}
2\gamma \le L_0\Big(\zeta'(X)+\zeta'(Y)\Big) + H(t_0,x_0,\frac{2}{\beta}(x_0-y_0) - \zeta'(X)\langle x_0\rangle_\varepsilon^{-1}x_0\Big) \\
-H(s_0,y_0,-\frac{2}{\beta}(y_0-x_0) + \zeta'(Y)\langle y_0\rangle_\varepsilon^{-1}y_0\Big).
\end{aligned} \tag{4.32}$$

Choose some sequence $\alpha\downarrow 0$ such that (t_0, x_0, s_0, y_0) converges. For notational simplicity, we still denote the limit by (t_0, x_0, s_0, y_0) itself. Note that for this limit, by (4.19), we must have $s_0 = t_0$, and (4.21) becomes

$$\frac{2}{\beta}|y_0 - x_0|^2 \le \eta(\sqrt{\beta\eta_0}). \tag{4.33}$$

Then (4.32) implies

$$\begin{aligned}
2\gamma &\le L_0\Big(\zeta'(X)+\zeta'(Y)\Big) + H(t_0, x_0, \frac{2}{\beta}(x_0-y_0) - \zeta'(X)\,\langle\, x_0\,\rangle_{\varepsilon}^{-1} x_0\Big) \\
&\quad - H(t_0, y_0, -\frac{2}{\beta}(y_0-x_0) + \zeta'(Y)\,\langle\, y_0\,\rangle_{\varepsilon}^{-1} y_0\Big) \\
&\le L_0\Big(\zeta'(X)+\zeta'(Y)\Big) + M\Big(\delta + \Big|\frac{2}{\beta}(x_0-y_0) - \zeta'(X)\,\langle\, x_0\,\rangle_{\varepsilon}^{-1} x_0\Big| \\
&\quad + \Big|\frac{2}{\beta}(y_0-x_0) - \zeta'(Y)\,\langle\, y_0\,\rangle_{\varepsilon}^{-1} y_0\Big|\Big)\Big|\zeta'(X)\,\langle\, x_0\,\rangle_{\varepsilon}^{-1} x_0 + \zeta'(Y)\,\langle\, y_0\,\rangle_{\varepsilon}^{-1} y_0\Big| \\
&\quad + \omega\Big(|x_0| + |y_0| + \Big|\frac{2}{\beta}(x_0-y_0) - \zeta'(X)\,\langle\, x_0\,\rangle_{\varepsilon}^{-1} x_0\Big|, |x_0-y_0|\Big).
\end{aligned} \tag{4.34}$$

Note that

$$\Big|\frac{2}{\beta}(x_0-y_0) - \zeta'(X)\,\langle\, x_0\,\rangle_{\varepsilon}^{-1} x_0\Big| \le 2\Big|\frac{x_0-y_0}{\beta}\Big| + \frac{2K}{\delta}.$$

Likewise,

$$\Big|\frac{2}{\beta}(y_0-x_0) - \zeta'(Y)\,\langle\, y_0\,\rangle_{\varepsilon}^{-1} y_0\Big| \le 2\Big|\frac{x_0-y_0}{\beta}\Big| + \frac{2K}{\delta}.$$

Hence, we obtain, taking into account that $\zeta'(r) \le 0$ for all $r \in \mathbb{R}$,

$$\begin{aligned}
2\gamma &\le L_0\Big(\zeta'(X)+\zeta'(Y)\Big) \\
&\quad + M\Big(\delta + 4\Big|\frac{x_0-y_0}{\beta}\Big| + \frac{4K}{\delta}\Big)\Big(|\zeta'(X)| + |\zeta'(Y)|\Big) \\
&\quad + \omega(|x_0| + |y_0| + \frac{|x_0-y_0|}{\beta} + \frac{2K}{\delta}, |x_0-y_0|\Big) \\
&\le -\Big\{L_0 - M\Big(\delta + 4\Big|\frac{x_0-y_0}{\beta}\Big| + \frac{4K}{\delta}\Big)\Big\}\Big(|\zeta'(X)| + |\zeta'(Y)|\Big) \\
&\quad + \omega(|x_0| + |y_0| + \frac{|x_0-y_0|}{\beta} + \frac{2K}{\delta}, |x_0-y_0|\Big).
\end{aligned}$$

By further enlarge L_0, if necessary, we have

$$0 < 2\gamma \le \omega\Big(|x_0| + |y_0| + \frac{|x_0-y_0|}{\beta} + \frac{2K}{\delta}, |x_0-y_0|\Big).$$

Let $\beta \to 0$ and using (4.33), we obtain a contradiction. Therefore (4.3) holds, and our conclusion follows. □

Remark 4.3. The essential idea of modification on the original proof of Ishii ([11]) is that we realize the flexibility of L_0 which can be enlarged whenever we need that. This enable us to handle the case that $p \mapsto H(t, x, p)$ is only local Lipschitz.

The following corollary is clear.

Corollary 4.4. *Let* (H1)–(H3) *hold, and let* $h : \mathbb{R}^n \to \mathbb{R}$ *be continuous. Then each of the following upper and lower HJI equations*

$$\begin{cases} V_t^{\pm}(t,x) + H^{\pm}(t,x,V^{\pm}(t,x)) = 0, \qquad (t,x) \in [0,T] \times \mathbb{R}^n, \\ V^{\pm}(T,x) = h(x), \qquad x \in \mathbb{R}^n \end{cases} \tag{4.35}$$

has at most one viscosity solution, where $H^{\pm}(\cdot\,,\cdot\,,\cdot)$ *are upper and lower Hamiltonians defined by* (3.6).

5 Upper and Lower Value Functions

In this section, we are going to define the upper and lower value functions via the so-called Elliott–Kalton strategies. Some basic properties of upper and lower value functions will be established carefully.

5.1 State trajectories and Elliott–Kalton strategies

Let us introduce the following hypotheses which are strengthened versions of (H1)–(H3).

(H1)′ Map $f : [0,T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}^n$ is continuous and there exist constants $L > 0$, $\sigma_1, \sigma_2 \geq 0$ such that

$$\begin{aligned} &|f(t,x,u_1,u_2) - f(t,y,u_1,u_2)| \leq L(1 + |u_1|^{\sigma_1} + |u_2|^{\sigma_2})|x - y|, \\ &\qquad\qquad\qquad\qquad \forall (t,u_1,u_2) \in [0,T] \times U_1 \times U_2,\ x,y \in \mathbb{R}^n, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &|f(t,x,u_1,u_2)| \leq L(1 + |x| + |u_1|^{\sigma_1} + |u_2|^{\sigma_2}), \\ &\qquad\qquad\qquad \forall (t,x,u_1,u_2) \in [0,T] \times \mathbb{R}^n \times U_1 \times U_2. \end{aligned} \tag{5.2}$$

(H2)′ Map $g : [0,T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}$ satisfies the following:

$$\begin{aligned} &|g(t,x,u_1,u_2) - g(t,y,u_1,u_2)| \leq L(1 + |x|^{\mu} + |y|^{\mu} + |u_1|^{\rho_1} + |u_2|^{\rho_2})|x - y|, \\ &\qquad\qquad\qquad\qquad \forall (t,u_1,u_2) \in [0,T] \times U_1 \times U_2,\ x,y \in \mathbb{R}^n, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} &c|u_1|^{\rho_1} - L(1 + |x|^{\mu} + |u_2|^{\rho_2}) \leq g(t,x,u_1,u_2) \leq L(1 + |x|^{\mu} + |u_1|^{\rho_1}) - c|u_2|^{\rho_2}, \\ &\qquad\qquad\qquad\qquad\qquad \forall (t,x,u_1,u_2) \in [0,T] \times \mathbb{R}^n \times U_1 \times U_2. \end{aligned} \tag{5.4}$$

Also, map $h : \mathbb{R}^n \to \mathbb{R}$ is continuous and

$$\begin{cases} |h(x) - h(y)| \leq L(1 + |x|^{\mu} + |y|^{\mu})|x - y|, \qquad \forall x,y \in \mathbb{R}^n, \\ |h(0)| \leq L. \end{cases} \tag{5.5}$$

Further, the compatibility hypothesis (H3) is now replaced by the following:

(H3)$'$ The constants $\sigma_1, \sigma_2, \rho_1, \rho_2, \mu$ appear in (H1)$'$–(H2)$'$ satisfy the following:

$$\sigma_1(1 \vee \mu) < \rho_1, \qquad \sigma_2(1 \vee \mu) < \rho_2. \tag{5.6}$$

Let us first present the following result concerning the state trajectories.

Proposition 5.1. *Let* (H1)$'$ *hold. Then, for any* $(t,x) \in [0,T) \times \mathbb{R}^n$, $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1^{\sigma_1}[t,T] \times \mathcal{U}_2^{\sigma_2}[t,T]$, *state equation* (1.1) *admits a unique solution* $y(\cdot) \equiv y(\cdot\,; t, x, u_1(\cdot), u_2(\cdot)) \equiv y_{t,x}(\cdot)$. *Moreover, the following estimate holds:*

$$|y_{t,x}(s)| \le C_0\Big[1 + |x| + \int_t^s \Big(|u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\Big)dr\Big], \qquad s \in [t,T], \tag{5.7}$$

with $C_0 = e^{LT}(1 + L + LT)$, *and*

$$|y(s) - x| \le Le^{LT}\Big[(1 + |x|)(s - t) + \int_t^s \Big(|u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\Big)dr\Big], \quad s \in [t,T]. \tag{5.8}$$

Further, if $(\bar t, \bar x) \in [0,T] \times \mathbb{R}^n$ *with* $\bar t \in [t,T]$, *and* $y_{\bar t,\bar x}(\cdot) \equiv y(\cdot\,; \bar t, \bar x, u_1(\cdot), u_2(\cdot))$, *then*

$$\begin{aligned} |y_{t,x}(s) - y_{\bar t,\bar x}(s)| \le Ce^{L\int_t^T(|u_1(r)|^{\sigma_1}+|u_2(r)|^{\sigma_2})dr}\Big\{|x-\bar x| \\ +\Big[1+|x|+\int_t^T\Big(|u_1(r)|^{\sigma_1}+|u_2(r)|^{\sigma_2}\Big)dr\Big]|\bar t-t|+\int_t^{\bar t}\Big(|u_1(s)|^{\sigma_1}+|u_2(s)|^{\sigma_2}\Big)ds\Big\}. \end{aligned} \tag{5.9}$$

Proof. First, under (H1)$'$, for any $(t,x) \in [0,T) \times \mathbb{R}^n$, and any $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1^{\sigma_1}[t,T] \times \mathcal{U}_2^{\sigma_2}[t,T]$, we have

$$\begin{aligned} &|f(s, y_1, u_1(s), u_2(s)) - f(s, y_2, u_1(s), u_2(s))| \\ &\quad \le L\Big(1 + |u_1(s)|^{\sigma_1} + |u_2(s)|^{\sigma_2}\Big)|y_1 - y_2|, \qquad s \in [t,T],\ y_1, y_2 \in \mathbb{R}^n, \end{aligned}$$

and

$$|f(s, y, u_1(t), u_2(t))| \le L\Big(1 + |y| + |u_1(s)|^{\sigma_1} + |u_2(s)|^{\sigma_2}\Big), \quad \forall (s,y) \in [t,T] \times \mathbb{R}^n.$$

Hence, by a standard argument, the state equation (1.1) admits a unique solution $y(\cdot) \equiv y(\cdot\,; t, x, u_1(\cdot), u_2(\cdot))$. Moreover,

$$\begin{aligned} |y(s)| &\le |x| + \int_t^s |f(r, y(r), u_1(r), u_2(r))|dr \\ &\le |x| + \int_t^s L\Big(1 + |y(r)| + |u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\Big)dr \equiv \Theta(s). \end{aligned}$$

Then

$$\dot\Theta(s) = L\Big(1 + |y(s)| + |u_1(s)|^{\sigma_1} + |u_2(s)|^{\sigma_2}\Big) \le L\Theta(s) + L\Big(1 + |u_1(s)|^{\sigma_1} + |u_2(s)|^{\sigma_2}\Big).$$

Thus, making use of Gronwall's inequality, we have

$$\begin{aligned}|y(s)| &\le \Theta(s) \le e^{L(s-t)}\Theta(t) + \int_t^s e^{L(s-r)} L\Big(1 + |u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\Big)dr \\ &\le e^{L(T-t)}|x| + Le^{L(T-t)}(T-t) + Le^{L(T-t)} \int_t^s \Big(|u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\Big)dr \\ &\le e^{LT}\Big[LT + |x| + L\int_t^s \Big(|u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\Big)dr\Big].\end{aligned}$$

Hence, (5.7) follows. Also, we have

$$\begin{aligned}|y(s) - x| &\le \int_t^s |f(r, y(r), u_1(r), u_2(r))|dr \\ &\le \int_t^s L\Big(1 + |x| + |y(s) - x| + |u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\Big)dr.\end{aligned}$$

Thus, by Gwonwall's inequality, one obtains (5.8).

Now, for any $(t,x), (\bar t, \bar x) \in [0,T] \times \mathbb{R}^n$, with $0 \le t \le \bar t \le T$, by denoting $y_{t,x}(\cdot) = y(\cdot\,; t, x, u_1(\cdot), u_2(\cdot))$, and $y_{\bar t,\bar x}(\cdot) = y(\cdot\,; \bar t, \bar x, u_1(\cdot), u_2(\cdot))$, we have

$$\begin{aligned}|y_{t,x}(s) - y_{\bar t,\bar x}(s)| &\le |x - \bar x| + \int_t^{\bar t} |f(\tau, y_{t,x}(\tau), u_1(\tau), u_2(\tau))|d\tau \\ &\qquad + \int_{\bar t}^s |f(\tau, y_{t,x}(\tau), u_1(\tau), u_2(\tau)) - f(\tau, y_{\bar t,\bar x}(\tau), u_1(\tau), u_2(\tau))|d\tau \\ &\le |x - \bar x| + \int_t^{\bar t} L\Big(1 + |y_{t,x}(\tau)| + |u_1(\tau)|^{\sigma_1} + |u_2(\tau)|^{\sigma_2}\Big)d\tau \\ &\qquad + \int_{\bar t}^s L\Big(1 + |u_1(\tau)|^{\sigma_1} + |u_2(\tau)|^{\sigma_2}\Big)|y_{t,x}(\tau) - y_{\bar t,\bar x}(\tau)|d\tau \\ &\le |x - \bar x| + \int_t^{\bar t} C\Big[1 + |x| + \int_t^T \Big(|u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\Big)dr + |u_1(\tau)|^{\sigma_1} + |u_2(\tau)|^{\sigma_2}\Big]d\tau \\ &\qquad + \int_{\bar t}^s L\Big(1 + |u_1(\tau)|^{\sigma_1} + |u_2(\tau)|^{\sigma_2}\Big)|y_{t,x}(\tau) - y_{\bar t,\bar x}(\tau)|d\tau \\ &\le |x - \bar x| + C\Big[1 + |x| + \int_t^T \Big(|u_1(r)|^{\sigma_1} + |u_2(r)|^{\sigma_2}\Big)dr\Big]|\bar t - t| + \int_t^{\bar t}\Big(|u_1(\tau)|^{\sigma_1} + |u_2(\tau)|^{\sigma_2}\Big)d\tau \\ &\qquad + \int_{\bar t}^s L\Big(1 + |u_1(\tau)|^{\sigma_1} + |u_2(\tau)|^{\sigma_2}\Big)|y_{t,x}(\tau) - y_{\bar t,\bar x}(\tau)|d\tau.\end{aligned}$$

Then by Gronwall's inequality, we obtain (5.9). □

From the above proposition, together with (H2)′, we see that for any $u_i(\cdot) \in \mathcal{U}_i^{\rho_i}[t,T]$ (which is smaller than $\mathcal{U}_i^{\sigma_i}[t,T]$), $i = 1,2$, the performance functional $J(t,x;u_1(\cdot),u_2(\cdot))$ is well-defined. Let us now introduce the following definition which is a modification of the notion introduced in [7].

Definition 5.2. A map $\alpha_1 : \mathcal{U}_2^1[t,T] \to \mathcal{U}_1^\infty[t,T]$ is called an Elliott–Kalton (E-K, for short) strategy for Player 1 if it is *non-anticipating*, namely, for any $u_2(\cdot), \bar u_2(\cdot) \in \mathcal{U}_2^1[t,T]$, and any $\hat t \in [t,T]$,

$$\alpha_1[u_2(\cdot)](s) = \alpha_1[\bar u_2(\cdot)](s), \qquad \text{a.e. } s \in [t, \hat t],$$

provided

$$u_1(s) = \bar{u}_1(s), \qquad \text{a.e. } s \in [t, \hat{t}].$$

The set of all E-K strategies for Player 1 is denoted by $\mathcal{A}_1[t,T]$. An E-K strategy $\alpha_2 : \mathcal{U}_2^1[t,T] \to \mathcal{U}_1^\infty[t,T]$ for Player 2 can be defined similarly. The set of all E-K strategies for Player 2 is denoted by $\mathcal{A}_2[t,T]$.

Note that as far as the state equation is concerned, one could define an E-K strategy α_1 for Player I as a map $\alpha_1 : \mathcal{U}_2^{\sigma_2}[t,T] \to \mathcal{U}_1^{\sigma_1}[t,T]$. Whereas, as far as the performance functional is concerned, one might have to restrictively define $\alpha_1 : \mathcal{U}_2^{\rho_2}[t,T] \to \mathcal{U}_1^{\rho_1}[t,T]$. We note that the numbers $\sigma_1, \sigma_2, \rho_1, \rho_2$ appeared in (H1)$'$–(H2)$'$ might not be the "optimal" ones, in some sense (for example, σ_1 and σ_2 might be larger than necessary, and ρ_1 and ρ_2 could be smaller than they should be, and so on). Our above definition is somehow "universal". The domain $\mathcal{U}_2^1[t,T]$ of α_1 is large enough to cover possible $u_2(\cdot)$ in some larger space than $\mathcal{U}_2^{\sigma_2}[t,T]$, and the co-domain $\mathcal{U}_1^\infty[t,T]$ is large enough so that the integrability of $\alpha_1[u_2(\cdot)]$ is ensured and the supremum will remain the same due to the density of $\mathcal{U}_1^\infty[t,T]$ in $\mathcal{U}_1^{\rho_1}[t,T]$. In what follows, we simply denote

$$\mathcal{U}_i[t,T] = \mathcal{U}_i^\infty[t,T], \qquad i = 1, 2.$$

Recall that $0 \in U_i$ $(i = 1, 2)$. For later convenience, we hereafter let $u_1^0(\cdot) \in \mathcal{U}_1[t,T]$ and $u_2^0(\cdot) \in \mathcal{U}_2[t,T]$ be defined by

$$u_1^0(s) = 0, \quad u_2^0(s) = 0, \qquad \forall s \in [t,T],$$

and let $\alpha_1^0 \in \mathcal{A}_1[t,T]$ be the E-K strategy that

$$\alpha_1^0[u_2(\cdot)](s) = 0, \qquad \forall s \in [t,T], \quad u_2(\cdot) \in \mathcal{U}_2^1[t,T].$$

We call such an α_1^0 the *zero E-K strategy* for Player 1. Similarly, we define zero E-K strategy $\alpha_2^0 \in \mathcal{A}_2[t,T]$ for Player 2.

Now, we define

$$\begin{cases} V^+(t,x) = \displaystyle\sup_{\alpha_2 \in \mathcal{A}_2[t,T]} \inf_{u_1(\cdot) \in \mathcal{U}_1[t,T]} J(t,x;u_1(\cdot), \alpha_2[u_1(\cdot)]), \\ V^-(t,x) = \displaystyle\inf_{\alpha_1 \in \mathcal{A}_1[t,T]} \sup_{u_2(\cdot) \in \mathcal{U}_2[t,T]} J(t,x;\alpha_1[u_2(\cdot)], u_2(\cdot)). \end{cases} \qquad (t,x) \in [0,T] \times \mathbb{R}^n, \quad (5.10)$$

which are called *upper* and *lower value functions* of our two-person zero-sum differential game.

5.2 Upper and lower value functions, and principle of optimality

Although the upper and lower value functions are formally defined in (5.10), there seems to be no guarantee that they are well-defined. The following result states that under suitable conditions, $V^\pm(\cdot\,,\cdot)$ are indeed well-defined.

Theorem 5.3. *Let* (H1)$'$–(H3)$'$ hold. Then the upper and lower value functions $V^{\pm}(\cdot\,,\cdot)$ are well-defined and there exists a constant $C>0$ such that

$$|V^{\pm}(t,x)|\le C(1+|x|^{\mu}),\qquad (t,x)\in[0,T]\times\mathbb{R}^n. \tag{5.11}$$

Moreover,

$$\begin{cases} V^+(t,x)=\sup\limits_{\alpha_2\in\mathcal{A}_2[t,T;N(|x|)]}\ \inf\limits_{u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]} J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]),\\ V^-(t,x)=\inf\limits_{\alpha_1\in\mathcal{A}_1[t,T;N(|x|)]}\ \sup\limits_{u_2(\cdot)\in\mathcal{U}_2[t,T;N(|x|)]} J(t,x;\alpha_1[u_2(\cdot)],u_2(\cdot)),\end{cases} \tag{5.12}$$

where $N:[0,\infty)\to\big[0,\infty)$ is some nondecreasing continuous function,

$$\mathcal{U}_i[t,T;r]=\Big\{u_i\in\mathcal{U}_i[t,T]\;\Big|\;\int_t^T|u_i(s)|^{\rho_i}ds\le r\Big\},\qquad i=1,2,$$

and

$$\begin{cases}\mathcal{A}_1[t,T;r]=\Big\{\alpha_1:\mathcal{U}_2^1[t,T]\to\mathcal{U}_1[t,T;r]\mid\alpha_1\in\mathcal{A}_1[t,T]\Big\},\\ \mathcal{A}_2[t,T;r]=\Big\{\alpha_2:\mathcal{U}_1^1[t,T]\to\mathcal{U}_2[t,T;r]\mid\alpha_2\in\mathcal{A}_2[t,T]\Big\}.\end{cases}$$

Proof. For any $(t,x)\in[0,T]\times\mathbb{R}^n$ and $(u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$, we have

$$\begin{aligned}&J(t,x;u_1(\cdot),u_2(\cdot))=\int_t^T g(s,y(s),u_1(s),u_2(s))ds+h(y(T))\\ &\ge\int_t^T\Big[c|u_1(s)|^{\rho_1}-L\Big(1+|y(s)|^{\mu}+|u_2(s)|^{\rho_2}\Big)\Big]ds-L\Big(1+|y(T)|^{\mu}\Big)\\ &\ge\int_t^T\Big\{c|u_1(s)|^{\rho_1}-2L\Big[1+C_0^{\mu}\Big(1+|x|+\int_t^T\{|u_1(r)|^{\sigma_1}+|u_2(r)|^{\sigma_2}\}dr\Big)^{\mu}+L|u_2(s)|^{\rho_2}\Big]ds.\end{aligned}$$

Thus, in the case $\mu>1$, we have

$$\begin{aligned}J(t,x;u_1(\cdot),u_2(\cdot))&\ge-C(1+|x|^{\mu})+\int_t^T\Big\{c|u_1(s)|^{\rho_1}-C\Big(\int_t^T|u_1(r)|^{\sigma_1}dr\Big)^{\mu}\\ &\qquad-C\Big(\int_t^T|u_2(r)|^{\sigma_2}dr\Big)^{\mu}\Big]-L|u_2(s)|^{\rho_2}\Big\}ds\\ &\ge-C(1+|x|^{\mu})-C\int_t^T|u_2(s)|^{\rho_2}ds+\int_t^T\Big[c|u_1(s)|^{\rho_1}-C|u_1(s)|^{\sigma_1\mu}\Big]ds,\end{aligned}$$

and in the case $\mu\in[0,1]$,

$$\begin{aligned}J(t,x;u_1(\cdot),u_2(\cdot))&\ge-C(1+|x|^{\mu})+\int_t^T\Big\{c|u_1(s)|^{\rho_1}-C\Big(\int_t^T|u_1(r)|^{\sigma_1}dr\Big)^{\mu}\\ &\qquad-C\Big(\int_t^T|u_2(r)|^{\sigma_2}dr\Big)^{\mu}\Big]-L|u_2(s)|^{\rho_2}\Big\}ds\\ &\ge-C(1+|x|^{\mu})-C\int_t^T|u_s(s)|^{\rho_2}ds+\int_t^T\Big[c|u_1(s)|^{\rho_1}-C|u_1(s)|^{\sigma_1}\Big]ds.\end{aligned}$$

Here, we have used the compatibility condition (5.6). From the above, we see that

$$\begin{aligned} J(t,x;u_1(\cdot),u_2(\cdot)) &\geq -C(1+|x|^\mu) - C\int_t^T |u_2(s)|^{\sigma_2}ds \\ &\qquad + \int_t^T \Big[c|u_1(s)|^{\rho_1} - C|u_1(s)|^{\sigma_1(\mu\vee 1)}\Big]ds \\ &\geq -C(1+|x|^\mu) - C\int_t^T |u_2(s)|^{\sigma_2}ds + \frac{c}{2}\int_t^T |u_1(s)|^{\rho_1}ds \\ &\geq -C(1+|x|^\mu) - C\int_t^T |u_2(s)|^{\sigma_2}ds. \end{aligned} \tag{5.13}$$

Consequently,

$$\begin{aligned} V^+(t,x) &= \sup_{\alpha_2\in\mathcal{A}_2[t,T]} \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T]} J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]) \\ &\geq \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T]} J(t,x;u_1(\cdot),\alpha_2^0[u_1(\cdot)]) \geq -C(1+|x|^\mu). \end{aligned}$$

Likewise, for any $(u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$, we have

$$\begin{aligned} &J(t,x;u_1(\cdot),u_2(\cdot)) = \int_t^T g(s,y(s),u_1(s),u_2(s))ds + h(y(T)) \\ &\leq C(1+|x|^\mu) + C\int_t^T |u_1(s)|^{\rho_1}ds - \frac{c}{2}\int_t^T |u_2(s)|^{\rho_2}ds \\ &\leq C(1+|x|^\mu) + C\int_t^T |u_1(s)|^{\rho_1}ds. \end{aligned} \tag{5.14}$$

Thus,

$$\begin{aligned} V^+(t,x) &= \sup_{\alpha_2\in\mathcal{A}_2[0,T]} \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T]} J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]) \\ &\leq \sup_{\alpha_2\in\mathcal{A}_2[t,T]} J(t,x;u_1^0(\cdot),\alpha_2[u_1^0(\cdot)]) \leq C\Big(1+|x|^\mu\Big). \end{aligned}$$

Similar results also hold for the lower value function $V^-(\cdot\,,\cdot)$. Therefore, we obtain that $V^\pm(t,x)$ are well-defined for all $(t,x)\in[0,T]\times\mathbb{R}^n$ and (5.11) holds.

Next, for the constant $C>0$ appearing in (5.11), we set

$$N(r) = \frac{4C}{c}(1+r^\mu).$$

Then for any $u_1(\cdot)\in\mathcal{U}_1[t,T]\setminus\mathcal{U}_1[t,T;N(|x|)]$, from (5.13), we see that

$$\begin{aligned} &J(t,x;u_1(\cdot),\alpha_2^0[u_1(\cdot)]) \geq -C(1+|x|^\mu) + \frac{c}{2}\int_t^T |u_1(s)|^{\rho_1}ds > C(1+|x|^\mu) \\ &\geq V^+(t,x) = \sup_{\alpha_2\in\mathcal{A}_2[t,T]} \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T]} J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]). \end{aligned}$$

Thus,

$$V^+(t,x) = \sup_{\alpha_2\in\mathcal{A}_2[t,T]} \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]} J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]). \tag{5.15}$$

Consequently, from (5.14), for any $u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]$, we have

$$\begin{aligned}-C(1+|x|^\mu)\le V^+(t,x)&\le\sup_{\alpha_2\in\mathcal{A}_2[t,T]}J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)])\\&\le C(1+|x|^\mu)+C\int_t^T|u_1(s)|^{\rho_1}ds-\frac{c}{2}\int_t^T|\alpha_2[u_1(\cdot)](s)|^{\rho_2}ds\\&\le C(1+|x|^\mu)+2C^2(1+|x|^\mu)-\frac{c}{2}\int_t^T|\alpha_2[u_1(\cdot)](s)|^{\rho_2}ds.\end{aligned}$$

This implies that

$$\frac{c}{2}\int_t^T|\alpha_2[u_1(\cdot)](s)|^{\rho_2}ds\le\widetilde{C}(1+|x|^\mu),\qquad\forall u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)],\tag{5.16}$$

with $\widetilde{C}=2C(C+1)>0$ being another absolute constant. Hence, if we replace the original $N(r)$ by the following:

$$N(r)=\frac{4\widetilde{C}}{c}(1+r^\mu),$$

and let

$$\mathcal{A}_2[t,T;r]=\Big\{\alpha_2\in\mathcal{A}_2[t,T]\;\Big|\;\int_t^T|\alpha_2[u_1(\cdot)](s)|^{\rho_2}ds\le N(|x|)\Big\},$$

then the first relation in (5.12) holds.

The second relation in (5.12) can be proved similarly. □

Next, we want to establish a modified Bellman's principle of optimality. To this end, we introduce some sets. For any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $\bar t\in(t,T]$, let

$$\mathcal{U}_i[t,\bar t;r]=\Big\{u_i(\cdot)\in\mathcal{U}_i[t,T]\mid\int_t^{\bar t}|u_i(s)|^{\rho_i}ds\le r\Big\},\qquad i=1,2,$$

and

$$\begin{cases}\mathcal{A}_1[t,\bar t;r]=\Big\{\alpha_1:\mathcal{U}_2^1[t,T]\to\mathcal{U}_1[t,\bar t;r]\mid\alpha_1\in\mathcal{A}_1[t,T]\Big\},\\\mathcal{A}_2[t,\bar t;r]=\Big\{\alpha_2:\mathcal{U}_1^1[t,T]\to\mathcal{U}_2[t,\bar t;r]\mid\alpha_2\in\mathcal{A}_2[t,T]\Big\}.\end{cases}$$

It is clear that

$$\begin{cases}\mathcal{U}_i[t,T;r]\subseteq\mathcal{U}_i[t,\bar t;r]\subseteq\mathcal{U}_i[t,T],\\\mathcal{A}_i[t,T;r]\subseteq\mathcal{A}_i[t,\bar t;r]\subseteq\mathcal{A}_i[t,T],\end{cases}\qquad i=1,2.$$

Thus, from the proof of Theorem 5.3, we see that for a suitable choice of $N(\cdot)$, say, $N(r)=C(1+r^\mu)$ for some large $C>0$, the following holds:

$$\begin{cases}V^+(t,x)=\displaystyle\sup_{\alpha_2\in\mathcal{A}_2[t,\bar t;N(|x|)]}\inf_{u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]}J(t,x;u_1(\cdot),\alpha_2[u_1(\cdot)]),\\V^-(t,x)=\displaystyle\inf_{\alpha_1\in\mathcal{A}_1[t,\bar t;N(|x|)]}\sup_{u_2(\cdot)\in\mathcal{U}_2[t,\bar t;N(|x|)]}J(t,x;\alpha_1[u_2(\cdot)],u_2(\cdot)).\end{cases}\tag{5.17}$$

We now state the following modified Bellman's principle of optimality.

Theorem 5.4. *Let* (H1)′–(H3)′ *hold. Let* $(t,x)\in[0,T)\times\mathbb{R}^n$ *and* $\bar t\in(t,T]$. *Let* $N:[0,\infty)\to[0,\infty)$ *be a nondecreasing continuous function such that* (5.17) *holds. Then*

$$\begin{aligned}V^+(t,x)=\sup_{\alpha_2\in\mathcal{A}_2[t,\bar t;N(|x|)]}\ \inf_{u_1(\cdot)\in\mathcal{U}_1[t,\bar t;N(|x|)]}\Big\{\int_t^{\bar t}g(s,y(s),u_1(s),\alpha_2[u_1(\cdot)](s))ds\\+V^+(\bar t,y(\bar t))\Big\},\end{aligned}\tag{5.18}$$

and

$$\begin{aligned}V^-(t,x)=\inf_{\alpha_1\in\mathcal{A}_1[t,\bar t;N(|x|)]}\ \sup_{u_2(\cdot)\in\mathcal{U}_2[t,\bar t;N(|x|)]}\Big\{\int_t^{\bar t}g(s,y(s),\alpha_1[u_2(\cdot)](s),u_2(s))ds\\+V^-(\bar t,y(\bar t))\Big\}.\end{aligned}\tag{5.19}$$

We note that if in (5.18) and (5.19), $\mathcal{A}_i[t,\bar t;N(|x|)]$ and $\mathcal{U}_i[t,\bar t;N(|x|)]$ are replaced by $\mathcal{A}_i[t,T]$ and $\mathcal{U}_i[t,T]$, respectively, the result is standard and the proof is routine. However, in the above case, some careful modification is necessary. For readers' convenience, we provide a proof in the appendix.

We point out that our modified principle of optimality will play an essential role in the next subsection.

5.3 Continuity of upper and lower value functions

In this subsection, we are going to establish the continuity of the upper and lower value functions. Let us state the main results now.

Theorem 5.5. *Let* (H1)′–(H3)′ *hold. Then* $V^\pm(\cdot\,,\cdot)$ *are continuous. Moreover, there exists a nondecreasing continuous function* $N:[0,\infty)\to[0,\infty)$ *such that the following estimates hold:*

$$|V^\pm(t,x)-V^\pm(t,\bar x)|\le N(|x|\vee|\bar x|)|x-\bar x|,\qquad t\in[0,T],\ x,\bar x\in\mathbb{R}^n,\tag{5.20}$$

and

$$|V^\pm(t,x)-V^\pm(\bar t,x)|\le N(|x|)|t-\bar t|^{\frac{\rho_1-\sigma_1}{\rho_1}\wedge\frac{\rho_2-\sigma_2}{\rho_2}},\qquad\forall t,\bar t\in[0,T],\ x\in\mathbb{R}^n.\tag{5.21}$$

Proof. We will only prove the conclusions for $V^+(\cdot\,,\cdot)$. The conclusions for $V^-(\cdot\,,\cdot)$ can be proved similarly.

First, let $0\le t\le T$, $x,\bar x\in\mathbb{R}^n$, and let $N:[0,\infty)\to[0,\infty)$ be nondecreasing and continuous such that (5.12) holds. Take

$$u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|\vee|\bar x|)],\quad\alpha_2\in\widetilde{\mathcal{A}}_2[t,T;N(|x|\vee|\bar x|)].\tag{5.22}$$

Denote $u_2(\cdot)=\alpha_2[u_1(\cdot)]$. For the simplicity of notations, in what follows, we will let $N(\cdot)$ be a generic nondecreasing function which can be different line by line. Making use of Proposition 5.1, we have

$$\begin{aligned}|y_{t,x}(s)|,|y_{t,\bar x}(s)|&\le C_0\Big[1+|x|\vee|\bar x|+\int_t^T\Big(|u_1(r)|^{\sigma_1}+|u_2(r)|^{\sigma_2}\Big)dr\Big]\\&\le N(|x|\vee|\bar x|),\qquad s\in[t,T],\end{aligned}$$

and

$$\begin{aligned}|y_{t,x}(s)-y_{t,\bar x}(s)|&\le Ce^{L\int_t^s(|u_1(t)|^{\sigma_1}+|u_2(t)|^{\sigma_2})dt}|x-\bar x|\\&\le N(|x|\vee|\bar x|)|x-\bar x|,\qquad s\in[t,T].\end{aligned}$$

Consequently,

$$\begin{aligned}&|J(t,x;u_1(\cdot),u_2(\cdot))-J(t,\bar x;u_1(\cdot),u_2(\cdot))|\\&\le\int_t^T|g(s,y_{t,x}(s),u_1(s),u_2(s))-g(s,y_{t,\bar x}(s),u(s))|ds+|h(y_{t,x}(T))-h(y_{t,\bar x}(T))|\\&\le\int_t^T L(1+|u_1(s)|^{\rho_1}+|u_2(s)|^{\rho_2}+|y_{t,x}(s)|^{\mu}+|y_{t,\bar x}(s)|^{\mu})|y_{t,x}(s)-y_{t,\bar x}(s)|ds\\&\quad+L\Big(1+|y_{t,x}(T)|^{\mu}+|y_{t,\bar x}(T)|^{\mu}\Big)|y_{t,x}(T)-y_{t,\bar x}(T)|\\&\le C\Big\{1+\int_t^T|u_1(s)|^{\rho_1}ds+\int_t^T|u_2(s)|^{\rho_2}ds+\Big(|x|\vee|\bar x|\Big)^{(\mu\vee1)\mu}\Big\}\sup_{s\in[t,T]}|y_{t,x}(s)-y_{t,\bar x}(s)|\\&\le N(|x|\vee|\bar x|)|x-\bar x|.\end{aligned}$$

Since the above estimate is uniform in $(u_1(\cdot),\alpha_2)$ satisfying (5.22), we obtain (5.20) for $V^+(\cdot\,,\cdot)$.

We now prove the continuity in t. From the modified principle of optimality, we see that for any $\varepsilon>0$, there exists an $\alpha_2^\varepsilon\in\mathcal{A}_2[t,\bar t;N(|x|)]$ such that

$$\begin{aligned}&V^+(t,x)-\varepsilon\le\inf_{u_1(\cdot)\in\mathcal{U}_1[t,\bar t;N(|x|)]}\Big\{\int_t^{\bar t}g(s,y(s),u_1(\cdot),\alpha_2^\varepsilon[u_1(\cdot)](s))ds+V^+(\bar t,y(\bar t))\Big\}\\&\le\int_t^{\bar t}g(s,y(s),0,\alpha_2^\varepsilon[u_1^0(\cdot)](s))ds+V^+(\bar t,y(\bar t))\\&\le\int_t^{\bar t}L\Big(1+|y(s)|^{\mu}-c|\alpha_2[u_1^0(\cdot)](s))|^{\rho_2}\Big)ds+V^+(\bar t,x)+|V^+(\bar t,y(\bar t))-V^+(\bar t,x)|\\&\le\int_t^{\bar t}L\Big(1+|y(s)|^{\mu}\Big)ds+V^+(\bar t,x)+|V^+(\bar t,y(\bar t))-V^+(\bar t,x)|.\end{aligned}$$

By Proposition 5.1, we have (denote $u_2^\varepsilon(\cdot)=\alpha_2^\varepsilon[u_1^0(\cdot)]$)

$$\begin{aligned}&|y(\bar t)-x|\le C\Big[(1+|x|)(\bar t-t)+\int_t^{\bar t}|u_2^\varepsilon(s)|^{\sigma_2}ds\Big]\\&\le C\Big[(1+|x|)(\bar t-t)+\Big(\int_t^{\bar t}|u_2^\varepsilon(s)|^{\rho_2}ds\Big)^{\frac{\sigma_2}{\rho_2}}(\bar t-t)^{\frac{\rho_2-\sigma_2}{\rho_2}}\Big]\\&\le C\Big[(1+|x|)(\bar t-t)+N(|x|)(\bar t-t)^{\frac{\rho_2-\sigma_2}{\rho_2}}\Big].\end{aligned}$$

Also,

$$|y(s)| \leq C_0\Big[1+|x|+\int_t^{\bar t}|u_2^\varepsilon(s)|^{\sigma_2}ds\Big] \leq N(|x|), \qquad s\in[t,\bar t].$$

Hence, by the proved (5.20), we obtain

$$|V^+(\bar t,y(\bar t))-V^+(\bar t,x)| \leq N(|x|\vee|y(\bar t)|)|y(\bar t)-x| \leq N(|x|)(\bar t-t)^{\frac{\rho_2-\sigma_2}{\rho_2}}.$$

Consequently,

$$V^+(t,x)-V^+(\bar t,x) \leq N(|x|)(\bar t-t)^{\frac{\rho_2-\sigma_2}{\rho_2}}+\varepsilon,$$

which yields

$$V^+(t,x)-V^+(\bar t,x) \leq N(|x|)(\bar t-t)^{\frac{\rho_2-\sigma_2}{\rho_2}}.$$

On the other hand,

$$V^+(t,x) \geq \inf_{u_1(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]}\Big\{\int_t^{\bar t} g(s,y(s),u_1(s),0)ds+V^+(\bar t,y(\bar t))\Big\}.$$

Hence, for any $\varepsilon>0$, there exists a $u_1^\varepsilon(\cdot)\in\mathcal{U}_1[t,T;N(|x|)]$ such that

$$\begin{aligned}
&V^+(t,x)+\varepsilon \geq \int_t^{\bar t} g(s,y(s),u_1^\varepsilon(s),0)ds+V^+(\bar t,y(\bar t))\\
&\geq -\int_t^{\bar t} L\Big(1+|y(s)|^\mu\Big)ds+c\int_t^{\bar t}|u_1^\varepsilon(s)|^{\rho_1}ds+V^+(\bar t,x)-|V^+(\bar t,y(\bar t))-V^+(\bar t,x)|\\
&\geq -\int_t^{\bar t} L\Big(1+|y(s)|^\mu\Big)ds+V^+(\bar t,x)-|V^+(\bar t,y(\bar t))-V^+(\bar t,x)|.
\end{aligned}$$

Now, in the current case, we have

$$\begin{aligned}
|y(\bar t)-x| &\leq C\Big[(1+|x|)(\bar t-t)+\int_t^{\bar t}|u_1^\varepsilon(s)|^{\sigma_1}ds\Big]\\
&\leq C\Big[(1+|x|)(\bar t-t)+\Big(\int_t^{\bar t}|u_1^\varepsilon(s)|^{\rho_1}ds\Big)^{\frac{\sigma_1}{\rho_1}}(\bar t-t)^{\frac{\rho_1-\sigma_1}{\rho_1}}\Big]\\
&\leq C\Big[(1+|x|)(\bar t-t)+N(|x|)(\bar t-t)^{\frac{\rho_1-\sigma_1}{\rho_1}}\Big].
\end{aligned}$$

Also,

$$|y(s)| \leq C_0\Big[1+|x|+\int_t^{\bar t}|u_1^\varepsilon(s)|^{\sigma_1}ds\Big] \leq N(|x|), \qquad s\in[t,\bar t].$$

Hence, by the proved (5.20), we obtain

$$|V^+(\bar t,y(\bar t))-V^+(\bar t,x)| \leq N(|x|\vee|y(\bar t)|)|y(\bar t)-x| \leq N(|x|)(\bar t-t)^{\frac{\rho_1-\sigma_1}{\rho_1}}.$$

Consequently,

$$V^+(t,x)-V^+(\bar t,x) \geq -N(|x|)(\bar t-t)^{\frac{\rho_1-\sigma_1}{\rho_1}}-\varepsilon,$$

which yields

$$V^+(t,x)-V^+(\bar t,x) \leq -N(|x|)(\bar t-t)^{\frac{\rho_1-\sigma_1}{\rho_1}}.$$

Hence, we obtain the estimate (5.21) for $V^+(\cdot\,,\cdot)$. $\square$

Once we have the continuity, we are able to routinely prove the following result.

Theorem 5.6. *Let* (H1)$'$–(H3)$'$ *hold. Then* $V^{\pm}(\cdot\,,\cdot)$ *are the unique viscosity solution to the upper and lower HJI equations* (4.35), *respectively. Further, if the Isaacs' condition holds:*

$$H^+(t,x,p) = H^-(t,x,p), \qquad \forall (t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \tag{5.23}$$

then

$$V^+(t,x) = V^-(t,x), \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n.$$

6 Remarks on the Existence of Viscosity Solutions to HJ Equations.

We have seen that (H1)–(H3) enable us to defined the upper and lower Hamiltonians so that the upper and lower HJI equations can be well-formulated. Moreover, under some even weaker conditions, we can proved the uniqueness of the viscosity solutions to the upper and lower HJI equations. On the other hand, we have assumed much stronger hypotheses (H1)$'$–(H3)$'$ to obtain the upper and lower value functions $V^{\pm}(\cdot\,,\cdot)$ being well-defined so that the corresponding upper and lower HJI equations have viscosity solutions. In another word, weaker conditions ensure the uniqueness of viscosity solutions to the upper and lower HJI equations, and stronger conditions seem to be needed for the existence. There are some general existence results of viscosity solutions for the first order HJ equations in the literature, see [12, 3, 15, 10, 5]. A natural question is whether the conditions that we assumed for the existence of viscosity solutions are sharp (or close to be necessary). In this section, we present a simple situation which tells us that our conditions are sharp in some sense.

We consider the following one-dimensional controlled linear system:

$$\begin{cases} \dot y(s) = Ay(s) + B_1 u_1(s) + B_2 u_2(s), \qquad s \in [t,T], \\ y(t) = x, \end{cases} \tag{6.1}$$

with the performance functional:

$$J(t,x;u_1(\cdot),u_2(\cdot)) = \int_t^T \Big[Qy(s)^2 + R_1 u_1(s)^2 - R_2 u_2(s)^2 \Big] ds + Gy(T)^2, \tag{6.2}$$

where $A, B_1, B_2, A, R_1, R_2, G \in \mathbb{R}$. We assume that

$$R_1, R_2 > 0. \tag{6.3}$$

Note that in the current case,

$$\sigma_1 = \sigma_2 = 1, \quad \mu = \rho_1 = \rho_2 = 2.$$

Thus,

$$\sigma_i(1\vee\mu)=\rho_i,\qquad i=1,2,$$

which violates (5.6). In the current case, we have

$$\begin{aligned}&H^{\pm}(t,x,p)=H(t,x,p)=\inf_{u_1}\sup_{u_2}\Big[pf(t,x,u_1,u_2)+g(t,x,u_1,u_2)\Big]\\&=Apx+Qx^2+\inf_{u_1}\Big[R_1u_1^2+pB_1u_1\Big]-\inf_{u_2}\Big[R_2u_2^2-pB_2u_2\Big]\\&=Apx+Qx^2+\Big(\frac{B_2^2}{4R_2}-\frac{B_1^2}{4R_1}\Big)p^2.\end{aligned}\tag{6.4}$$

Consequently, the upper and lower HJI equation have the same form:

$$\begin{cases}V_t(t,x)+AxV_x(t,x)+Qx^2+\Big(\dfrac{B_2^2}{4R_2}-\dfrac{B_1^2}{4R_1}\Big)V_x(t,x)^2=0,\quad (t,x)\in[0,T]\times\mathbb{R},\\ V(T,x)=Gx^2,\qquad x\in\mathbb{R}.\end{cases}\tag{6.5}$$

If the above HJI equation has a viscosity solution, by the uniqueness, the solution has to be of the following form:

$$V(t,x)=p(t)x^2,\qquad (t,x)\in[0,T]\times\mathbb{R},\tag{6.6}$$

where $p(\cdot)$ is the solution to the following Riccati equation:

$$\begin{cases}\dot p(t)+2Ap(t)+Q+\Big(\dfrac{B_2^2}{R_2}-\dfrac{B_1^2}{R_1}\Big)p(t)^2=0,\qquad t\in[0,T],\\ p(T)=G.\end{cases}\tag{6.7}$$

In another word, the solvability of (6.5) is equivalent to that of (6.7).

Our claim is that Riccati equation (6.7) is not always solvable for any $T>0$. To state our result in a relatively neat way, let us rewrite equation (6.7) as follows:

$$\begin{cases}\dot p+\alpha p+\beta p^2+\gamma=0,\\ p(T)=g,\end{cases}\tag{6.8}$$

with

$$\alpha=2A,\qquad \beta=\frac{B_2^2}{R_2^2}-\frac{B_1^2}{R_1^2},\qquad \gamma=Q,\qquad g=G.$$

Note that β could be positive, negative, or zero. We have the following result.

Proposition 6.1. *Riccati equation* (6.8) *admits a solution on* $[0,T]$ *for any* $T>0$ *if and only if one of the following holds:*

$$\beta=0;\tag{6.9}$$

or

$$\beta<0,\quad \alpha^2-4\beta\gamma<0;\tag{6.10}$$

or

$$\beta \neq 0, \quad \alpha^2 - 4\beta\gamma \geq 0, \quad 2\beta g - \alpha - \frac{\beta\sqrt{\alpha^2 - 4\beta\gamma}}{|\beta|} \leq 0. \tag{6.11}$$

The proof is elementary and straightforward. For reader's convenience, we provide a proof in the appendix.

It is clear that there are a lot of cases for which the Riccati equation is not solvable. For example,

$$\alpha = \beta = \gamma = 1,$$

which violates (6.10). Also, the case

$$\alpha = -2, \quad \beta = \gamma = g = 1,$$

which violates (6.11). For the above two cases, Riccati equation (6.8) does not have a global solution on $[0, T]$ for some $T > 0$. Correspondingly we have some two-person zero-sum differential game with unbounded controls for which the coercivity condition (5.6) fails and the upper and lower value functions could not be defined on the whole time interval $[0, T]$, or equivalently, the corresponding upper/lower HJI equation have no viscosity solutions on $[0, T]$.

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Appendix

Proof of Theorem 5.4. We only prove (5.18). The other can be proved similarly. Since $N(|x|)$ and $\bar t$ are fixed, for notational simplicity, we denote below that

$$\widetilde{\mathcal{U}}_1=\mathcal{U}_1[t,\bar t;N(|x|)],\qquad \widetilde{\mathcal{A}}_2=\mathcal{A}_2[t,\bar t;N(|x|)].$$

Denote the right hand side of (5.18) by $\widehat V^+(t,x)$. For any $\varepsilon>0$, there exists an $\alpha_2^\varepsilon\in\widetilde{\mathcal{A}}_2$ such that

$$\widehat V^+(t,x)-\varepsilon<\inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1}\Big\{\int_t^{\bar t}g(s,y(s),u_1(s),\alpha_2^\varepsilon[u_1(\cdot)](s))ds+V^+(\bar t,y(\bar t))\Big\}.$$

By the definition of $V^+(\bar t,y(\bar t))$, there exists an $\bar\alpha_2^\varepsilon\in\mathcal{A}_2[\bar t,T]$ such that

$$V^+(\bar t,y(\bar t))-\varepsilon<\inf_{\bar u_1(\cdot)\in\mathcal{U}_1[\bar t,T]}J(\bar t,y(\bar t);\bar u_1(\cdot),\bar\alpha_2^\varepsilon[\bar u_1(\cdot)]).$$

Now, we define an extension $\widehat{\alpha}_2^\varepsilon \in \mathcal{A}_2[t,T]$ of $\alpha_2^\varepsilon \in \mathcal{A}_2[\bar{t},T]$ as follows: For any $u_1(\cdot) \in \mathcal{U}_1[t,T]$,

$$\widehat{\alpha}_2^\varepsilon[u_1(\cdot)](s) = \begin{cases} \alpha_2^\varepsilon[u_1(\cdot)](s), & s \in [t,\bar{t}), \\ \bar{\alpha}_2^\varepsilon[u_1(\cdot)\big|_{[\bar{t},T]}](s), & s \in [\bar{t},T]. \end{cases}$$

Since $\alpha_2^\varepsilon \in \widetilde{\mathcal{A}}_2$, we have

$$\int_t^{\bar{t}} |\widehat{\alpha}^\varepsilon[u_1(\cdot)](s)|^{\rho_2} ds = \int_t^{\bar{t}} |\alpha_2^\varepsilon[u_1(\cdot)](s)|^{\rho_2} ds \le N(|x|).$$

This means that $\widehat{\alpha}_2^\varepsilon \in \widetilde{\mathcal{A}}_2$. Consequently,

$$\begin{aligned}
&V^+(t,x) \ge \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1} J(t,x;u_1(\cdot),\widehat{\alpha}_2^\varepsilon[u_1(\cdot)]) \\
&= \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1} \Big\{ \int_t^{\bar{t}} g(s,y(s),u_1(s),\alpha_2^\varepsilon[u_1(\cdot)](s))ds + J(\bar{t},y(\bar{t});u_1(\cdot)\big|_{[\bar{t},T]},\bar{\alpha}_2^\varepsilon[u_1(\cdot)\big|_{[\bar{t},T]})\Big\} \\
&\ge \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1} \Big\{ \int_t^{\bar{t}} g(s,y(s),u_1(s),\alpha_2^\varepsilon[u_1(\cdot)](s))ds + \inf_{\bar{u}_1(\cdot)\in\mathcal{U}_1[\bar{t},T]} J(\bar{t},y(\bar{t});\bar{u}_1(\cdot),\bar{\alpha}_2^\varepsilon[\bar{u}_1(\cdot))\Big\} \\
&\ge \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1} \Big\{ \int_t^{\bar{t}} g(s,y(s),u_1(s),\alpha_2^\varepsilon[u_1(\cdot)](s))ds + V^+(\bar{t},y(\bar{t}))\Big\} - \varepsilon \\
&\ge \widehat{V}^+(t,x) - 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\widehat{V}^+(t,x) \le V^+(t,x).$$

On the other hand, for any $\varepsilon > 0$, there exists an $\alpha_2^\varepsilon \in \widetilde{\mathcal{A}}_2$ such that

$$V^+(t,x) - \varepsilon < \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1} J(t,x;u_1(\cdot),\alpha_2^\varepsilon[u_1(\cdot)]).$$

Also, by definition of $\widehat{V}^+(t,x)$,

$$\widehat{V}^+(t,x) \ge \inf_{u_1(\cdot)\in\widetilde{\mathcal{U}}_1} \Big\{ \int_t^{\bar{t}} g(s,y(s),u_1(s),\alpha_2^\varepsilon[u_1(\cdot)](s))ds + V^+(\bar{t},y(\bar{t}))\Big\}.$$

Thus, there exists a $u_1^\varepsilon(\cdot) \in \widetilde{\mathcal{U}}_1$ such that

$$\widehat{V}^+(t,x) + \varepsilon \ge \int_t^{\bar{t}} g(s,y(s),u_1^\varepsilon(s),\alpha_2^\varepsilon[u_1^\varepsilon(\cdot)](s))ds + V^+(\bar{t},y(\bar{t})).$$

Now, for any $\bar{u}_1(\cdot) \in \mathcal{U}_1[\bar{t},T]$, define a particular extension $\widetilde{u}_1(\cdot) \in \mathcal{U}_1[t,T]$ by the following:

$$\widetilde{u}_1(s) = \begin{cases} u_1^\varepsilon(s), & s \in [t,\bar{t}), \\ \bar{u}_1(s), & s \in [\bar{t},T]. \end{cases}$$

Namely, we patch $u_1^\varepsilon(\cdot)$ to $\bar u_1(\cdot)$ on $[t,\bar t)$. Since

$$\int_t^{\bar t}|\widetilde u_1(s)|^{\rho_1}ds=\int_t^{\bar t}|u_1^\varepsilon(s)|^{\rho_1}ds\le N(|x|),$$

we see that $\widetilde u_1(\cdot)\in\widetilde{\mathcal U}_1$. Next, we define a restriction $\bar\alpha_2^\varepsilon\in\mathcal A[\bar t,T]$ of $\alpha_2^\varepsilon\in\widetilde{\mathcal A}_2$, as follows:

$$\bar\alpha_2^\varepsilon[\bar u_1(\cdot)]=\alpha_2^\varepsilon[\widetilde u_1(\cdot)].$$

For such an $\bar\alpha_2^\varepsilon$, we have

$$V^+(\bar t,y(\bar t))\ge\inf_{\bar u_1(\cdot)\in\mathcal U_1[\bar t,T]}J(\bar t,y(\bar t),\bar u_1(\cdot),\bar\alpha_2^\varepsilon[\bar u_1(\cdot)]).$$

Hence, there exists a $\bar u_1^\varepsilon(\cdot)\in\mathcal U_1[\bar t,T]$ such that

$$V^+(\bar t,y(\bar t))+\varepsilon>J(\bar t,y(\bar t),\bar u_1^\varepsilon(\cdot),\bar\alpha_2^\varepsilon[\bar u_1^\varepsilon(\cdot)]).$$

Then we further let

$$\widetilde u_1^\varepsilon(s)=\begin{cases}u_1^\varepsilon(s), & s\in[t,\bar t),\\ \bar u_1^\varepsilon(s), & s\in[\bar t,T].\end{cases}$$

Again, $\widetilde u_1^\varepsilon(\cdot)\in\widetilde{\mathcal U}_1$, and therefore,

$$\begin{aligned}&\widehat V^+(t,x)+\varepsilon\ge\int_t^{\bar t}g(s,y(s),u_1^\varepsilon(s),\alpha_2^\varepsilon[u_1^\varepsilon(\cdot)](s))ds+V^+(\bar t,y(\bar t))\\&\ge\int_t^{\bar t}g(s,y(s),u_1^\varepsilon(s),\alpha_2^\varepsilon[u_1^\varepsilon(\cdot)](s))ds+J(\bar t,y(\bar t),\bar u_1^\varepsilon(\cdot),\bar\alpha_2^\varepsilon[\bar u_1^\varepsilon(\cdot)])-\varepsilon\\&=J(t,x;\widetilde u_1^\varepsilon(\cdot),\alpha_2^\varepsilon[\widetilde u_1^\varepsilon(\cdot)])-\varepsilon\\&\ge\inf_{u_1(\cdot)\in\widetilde{\mathcal U}_1[t,T]}J(t,x;u_1(\cdot),\alpha_2^\varepsilon[u_1(\cdot)])-\varepsilon\ge V^+(t,x)-2\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we obtain

$$\widehat V^+(t,x)\ge V^+(t,x).$$

This completes the proof. □

Proof of Proposition 6.1. Recall that we are considering the following Riccati equation:

$$\begin{cases}\dot p+\alpha p+\beta p^2+\gamma=0,\\ p(T)=g,\end{cases}$$

Case 1. $\beta=0$. The Riccati equation reads

$$\begin{cases}\dot p+\alpha p+\gamma=0,\\ p(T)=g.\end{cases}$$

This is linear equation. Thus, a global solution $p(\cdot)$ uniquely exists on $[0,T]$.

Case 2. $\beta \neq 0$. Then Riccati equation reads

$$\begin{cases} \dot{p} + \beta\Big[\Big(p - \dfrac{\alpha}{2\beta}\Big)^2 + \dfrac{4\beta\gamma - \alpha^2}{4\beta^2}\Big] = 0, \\ p(T) = g. \end{cases}$$

Let

$$\kappa = \frac{\sqrt{|\alpha^2 - 4\beta\gamma|}}{2|\beta|} \geq 0.$$

There are three subcases.

Subscase 1. Suppose

$$\alpha^2 - 4\beta\gamma = 0.$$

Then the Riccati equation becomes

$$\begin{cases} \dot{p} + \beta\Big(p - \dfrac{\alpha}{2\beta}\Big)^2 = 0, \\ p(T) = g. \end{cases}$$

Therefore, in the case

$$2\beta g - \alpha = 0,$$

we have that $p(t) \equiv \frac{\alpha}{2\beta}$ is the (unique) global solution on $[0,T]$. Now, let

$$2\beta g - \alpha \neq 0.$$

Then we have

$$\frac{dp}{(p - \frac{\alpha}{2\beta})^2} = -\beta dt,$$

which leads to

$$\frac{1}{p(t) - \frac{\alpha}{2\beta}} = \frac{1}{g - \frac{\alpha}{2\beta}} - \beta(T-t) = \frac{2\beta - \beta(2\beta g - \alpha)(T-t)}{2\beta g - \alpha}.$$

Thus,

$$p(t) = \frac{\alpha}{2\beta} + \frac{2\beta g - \alpha}{2\beta - \beta(2\beta g - \alpha)(T-t)},$$

which is well-defined on $[0,T]$ if and only if

$$2 - (2\beta g - \alpha)(T-t) \neq 0, \qquad t \in [0,T].$$

This is equivalent to the following:

$$(2\beta g - \alpha)T < 2.$$

The above is true for all $T > 0$ if and only if

$$2\beta g - \alpha \leq 0,$$

Subcase 2. Suppose

$$\alpha^2 - 4\beta\gamma < 0.$$

Then the Riccati equation is

$$\dot{p} + \beta\Big[\Big(p - \frac{\alpha}{2\beta}\Big)^2 + \kappa^2\Big] = 0.$$

Hence,

$$\frac{dp}{(p - \frac{\alpha}{2\beta})^2 + \kappa^2} = -\beta dt,$$

which results in

$$\frac{1}{\kappa}\tan^{-1}\Big[\frac{1}{\kappa}\Big(p(t) - \frac{\alpha}{2\beta}\Big)\Big] = -\beta t + C.$$

By the terminal condition,

$$C = \beta T + \frac{1}{\kappa}\tan^{-1}\Big[\frac{1}{\kappa}\Big(g - \frac{\alpha}{2\beta}\Big)\Big]$$

Consequently,

$$\tan^{-1}\Big[\frac{1}{\kappa}\Big(p(t) - \frac{\alpha}{2\beta}\Big)\Big] = \kappa\beta(T-t) + \tan^{-1}\Big[\frac{1}{\kappa}\Big(g - \frac{\alpha}{2\beta}\Big)\Big].$$

Then

$$p(t) = \frac{\alpha}{2\beta} + \kappa\tan\Big\{\kappa\beta(T-t) + \tan^{-1}\Big(\frac{2\beta g - \alpha}{2\kappa\beta}\Big)\Big\}.$$

The above is well-defined for $t \in [0, T]$ if and only if

$$\tan^{-1}\frac{2\beta g - \alpha}{2\kappa\beta} + \kappa\beta T < \frac{\pi}{2},$$

which is true for all $T > 0$ if and only if $\beta \leq 0$.

Subcase 3. Suppose

$$\alpha^2 - 4\beta\gamma > 0.$$

Then the Riccati equation becomes

$$\dot{p} + \beta\Big[\Big(p - \frac{\alpha}{2\beta}\Big)^2 - \kappa^2\Big] = 0.$$

If

$$(2\beta g - \alpha - 2\kappa\beta)(2\beta g - \alpha + 2\kappa\beta) \equiv 4\beta\Big(q - \frac{a}{2\beta} - \kappa\Big)\Big(q - \frac{a}{2\beta} + \kappa\Big) = 0, \tag{A1}$$

then one of the following

$$p(t) \equiv \frac{\alpha}{2\beta} \pm \kappa, \qquad t \in [0,T],$$

is the unique global solution to the Riccati equation. We now let

$$(2\beta g - \alpha - 2\kappa\beta)(2\beta g - \alpha + 2\kappa\beta) \equiv 4\beta\Big(g - \frac{a}{2\beta} - \kappa\Big)\Big(g - \frac{a}{2\beta} + \kappa\Big) \neq 0.$$

Then

$$\frac{dp}{(p - \frac{\alpha}{2\beta})^2 - \kappa^2} = -\beta dt.$$

Hence,

$$\frac{1}{2\kappa} \ln\Big|\frac{p(t) - \frac{\alpha}{2\beta} - \kappa}{p(t) - \frac{\alpha}{2\beta} + \kappa}\Big| = -\beta t + \widetilde{C},$$

which implies

$$\frac{p(t) - \frac{\alpha}{2\beta} - \kappa}{p(t) - \frac{\alpha}{2\beta} + \kappa} = Ce^{-2\kappa\beta t},$$

with

$$C = e^{2\kappa\beta T}\frac{g - \frac{\alpha}{2\beta} - \kappa}{g - \frac{\alpha}{2\beta} + \kappa} = e^{2\kappa\beta T}\frac{2\beta g - \alpha - 2\kappa\beta}{2\beta g - \alpha + 2\kappa\beta}.$$

Then

$$\frac{p(t) - \frac{\alpha}{2\beta} - \kappa}{p(t) - \frac{\alpha}{2\beta} + \kappa} = e^{2\kappa\beta(T-t)}\frac{2\beta g - \alpha - 2\kappa\beta}{2\beta g - \alpha + 2\kappa\beta}.$$

Consequently,

$$p(t) - \frac{\alpha}{2\beta} - \kappa = e^{2\kappa\beta(T-t)}\frac{2\beta g - \alpha - 2\kappa\beta}{2\beta g - \alpha + 2\kappa\beta}\Big[p(t) - \frac{\alpha}{2\beta} + \kappa\Big].$$

Thus, $p(\cdot)$ globally exists on $[0,T]$ if and only if

$$e^{2\kappa\beta(T-t)}\frac{2\beta g - \alpha - 2\kappa\beta}{2\beta g - \alpha + 2\kappa\beta} - 1 \neq 0, \qquad \forall t \in [0,T],$$

which is equivalent to

$$\psi(t) \equiv e^{2\kappa\beta(T-t)}(2\beta g - \alpha - 2\kappa\beta) - (2\beta g - \alpha + 2\kappa\beta) \neq 0, \qquad \forall t \in [0,T].$$

Since $\psi'(t)$ does not change sign on $[0,T]$, the above is equivalent to the following:

$$0 < \psi(0)\psi(T) = \Big[e^{2\kappa\beta T}(2\beta g - \alpha - 2\kappa\beta) - (2\beta g - \alpha + 2\kappa\beta)\Big](-4\kappa\beta),$$

which is equivalent to

$$\Big[e^{2\kappa\beta T}(2\beta g - \alpha - 2\kappa\beta) - (2\beta g - \alpha + 2\kappa\beta)\Big]\beta < 0.$$

Note that when (A1) holds, the above is true. In the case $\beta > 0$, the above reads

$$e^{2\kappa\beta T}(2\beta g - \alpha - 2\kappa\beta) < 2\beta g - \alpha + 2\kappa\beta,$$

which is true for all $T > 0$ if and only if

$$2\beta g - \alpha - 2\kappa\beta \leq 0. \tag{A2}$$

Finally, if $\beta < 0$, then

$$\begin{aligned}
0 &< e^{2\kappa\beta T}(2\beta g - \alpha - 2\kappa\beta) - (2\beta g - \alpha + 2\kappa\beta) \\
&= e^{-2\kappa|\beta|T}(-2|\beta|g - \alpha + 2\kappa|\beta|) - (-2|\beta|g - \alpha - 2\kappa|\beta|) \\
&= e^{-2\kappa|\beta|T}\Big[-\Big(2|\beta|g + \alpha - 2\kappa|\beta|\Big) + e^{2\kappa|\beta|}\Big(2|\beta|g + \alpha + 2\kappa|\beta|\Big)\Big],
\end{aligned}$$

which is true for all $T > 0$ if and only if

$$0 \leq 2|\beta|g + \alpha + 2\kappa|\beta| = -(2\beta g - \alpha - 2\kappa\beta),$$

which has the same form as (A2). This completes the proof. □