# RESONANT UNIQUENESS OF RADIAL SEMICLASSICAL SCHRÖDINGER OPERATORS

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Abstract. We prove that radial, monotonic, superexponentially decaying potentials in $C^\infty(\mathbb{R}^n)$, $n \geq 1$ odd, are determined by the resonances of the associated semiclassical Schrödinger operator among all superexponentially decaying potentials in $C^\infty(\mathbb{R}^n)$.

## 1. Introduction

Given $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$, the semiclassical inverse spectral problem asks: what information about $V$ can be recovered from the asymptotics of the spectrum of $-h^2\Delta + V$ as $h \to 0$? In the case when the spectrum is discrete, various positive results have been given since the paper of Sjöstrand [Sjö92] (see also Iantchenko-Sjöstrand-Zworski [IaSjZw02]), including many sufficient conditions under which the potential can be determined, beginning with the work of Guillemin-Uribe [GuUr07] (see also [Hez09, CoGu08p, Col08p, GuWa09p, DaHeVe11]).

In this paper we consider potentials $V \in C^\infty(\mathbb{R}^n;[0,\infty)$ satisfying

$$|V(x)| \leq A\exp(-B|x|^{1+\varepsilon}), \qquad |\partial^\alpha V(x)| \leq C_\alpha. \tag{1.1}$$

For such potentials the spectrum of $-h^2\Delta + V$ is continuous and equals $[0,\infty)$, and hence contains no (further) information about $V$. In this setting *resonances* replace the discrete data of eigenvalues. For $n$ odd they are defined as the poles of the meromorphic continuation of $R_V(\lambda) = (-h^2\Delta + V - \lambda^2)^{-1} \colon L^2_{\text{comp}} \to L^2_{\text{loc}}$ from $\operatorname{Im}\lambda > 0$ to $\mathbb{C}$.

We prove that potentials which are radial and monotonic are determined by their resonances among all potentials satisfying (1.1).

**Theorem.** *Let $n \geq 1$ be odd, and let $V_0, V \in C^\infty(\mathbb{R}^n;[0,\infty))$ satisfy (1.1). Suppose $V_0(x) = R(|x|)$, and $R'(r)$ vanishes only at $r = 0$ and whenever $R(r) = 0$. Suppose that the resonances of $-h^2\Delta + V_0(x)$ agree with the resonances of $-h^2\Delta + V(x)$, up to $o(h^2)$, for $h \in \{h_j\}_{j=1}^\infty$ for some sequence $h_j \to 0$. Then there exists $x_0 \in \mathbb{R}^n$ such that $V(x) = V_0(x - x_0)$.*

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The strong symmetry assumption on $V_0$ is crucial to our argument. To our knowledge, such symmetry assumptions are always needed for uniqueness results for inverse resonance problems (and also for inverse spectral problems, see the introduction of [DaHeVe11] for a discussion). The strongest previous results are those for the nonsemiclassical Schrödinger problem when $n = 1$. In [Zwo01] Zworski proves that a compactly supported even potential $V$ is determined from the resonances of $-\frac{d^2}{dx^2} + V$, and in [Kor05] Korotyaev shows that a potential which is not necessarily even is determined by some additional scattering data. In the present paper we study the semiclassical problem for general odd dimensions, for not necessarily compactly supported potentials, and assume a priori only that $V_0$ and not necessarily $V$ is radial, and we use only resonances to determine $V$.

Analogous results hold in the case of obstacle scattering. Hassell and Zworski [HaZw99] show that a ball is determined by its Dirichlet resonances among all compact obstacles in $\mathbb{R}^3$. Christiansen [Chr08] extends this result to multiple balls, to higher odd dimensions, and to Neumann resonances. As in the present paper, the proofs use two trace invariants and isoperimetric-type inequalities, although the invariants and inequalities are different here. The case of an analytic obstacle with two mutually symmetric connected components is treated by Zelditch [Zel04] by using the singularities of the wave trace generated by the bouncing ball between the two components. There is also a large literature of inverse scattering results where data other than the resonances are used: see for example [Mel95].

Our proof is based on recovering and analyzing first two integral invariants of the Helffer-Robert semiclassical trace formula ([HeRo83, Proposition 5.3], see also [GuSt11p, §10.5]):

$$\begin{aligned}&\operatorname{Tr}(f(-h^2\Delta + V)) - f(-h^2\Delta)) = \qquad (1.2)\\ &\quad \frac{1}{(2\pi h)^n}\left(\int_{\mathbb{R}^{2n}} f(|\xi|^2 + V) - f(|\xi|^2) dx d\xi + \frac{h^2}{12}\int_{\mathbb{R}^{2n}} |\nabla V|^2 f^{(3)}(|\xi|^2 + V) dx d\xi + \mathcal{O}(h^4)\right).\end{aligned}$$

This is analogous to the approach taken by Colin de Verdière [Col08p], by Guillemin-Wang [GuWa09p], and by Ventura and the authors [DaHeVe11] for the problem of recovering the potential from the discrete spectrum.

To express the left hand side of (1.2) in terms of the resonances of $-h^2\Delta + V$, we use Melrose's Poisson formula ([Mel82]), an extension of the formula of Bardos-Guillot-Ralston ([BaGuRa82]), adapted to $V$ satisfying (1.1) by Sá Barreto-Zworski ([SáZw95, SáZw96]):

$$2\operatorname{Tr}\left(\cos(t\sqrt{-h^2\Delta + V}) - \cos(t\sqrt{-h^2\Delta})\right) = \sum_{\lambda \in \mathrm{Res}} e^{-i|t|\lambda}, \qquad t \neq 0, \qquad (1.3)$$

where Res denotes the set of resonances of $-h^2\Delta + V$, included according to multiplicity, and the equality is in the sense of distributions on $\mathbb{R} \setminus 0$. When $n = 1$ a stronger trace formula, valid for all $t \in \mathbb{R}$, is known: see for example [Zwo96, page 3]. When $n$ is even, the meromorphic continuation of the resolvent is not to $\mathbb{C}$ but to the Riemann surface of the logarithm, and as a result Poisson formulæ for resonances are more complicated and contain error terms which we have not been able to treat.

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## 2. Proof of the theorem

From (1.3), it follows that if

$$\hat{g} \in C_0^\infty(\mathbb{R} \setminus 0) \text{ is even,} \tag{2.1}$$

then

$$\operatorname{Tr}(g(\sqrt{-h^2\Delta + V}) - g(\sqrt{-h^2\Delta})) = \frac{1}{4\pi} \sum_{\lambda \in \operatorname{Res}} \int_{\mathbb{R}} e^{-i|t|\lambda} \hat{g}(t) dt. \tag{2.2}$$

Now setting the right hand sides of (2.2) and (1.2) equal and taking $h \to 0$, we find that

$$\int_{\mathbb{R}^{2n}} f(|\xi|^2 + V) - f(|\xi|^2) dx d\xi, \qquad \int_{\mathbb{R}^{2n}} |\nabla V|^2 f^{(3)}(|\xi|^2 + V) dx d\xi \tag{2.3}$$

are resonant invariants (i.e. are determined by knowledge of the resonances up to $o(h^2)$) provided that $f(\tau^2) = g(\tau)$ for all $\tau$ and for some $g$ as in (2.1). Taylor expanding, we write the first invariant as

$$\sum_{k=1}^m \frac{1}{k!} \int_{\mathbb{R}^n} f^{(k)}(|\xi|^2) d\xi \int_{\mathbb{R}^n} V(x)^k dx + \int_{\mathbb{R}^{2n}} \frac{V(x)^{m+1}}{m!} \int_0^1 (1-t)^m f^{(m+1)}(|\xi|^2 + tV(x)) dt dx d\xi.$$

Replacing $f$ by $f_\lambda$, where $f_\lambda(\tau) = f(\tau/\lambda)$ (note that $g_\lambda(\tau) = f_\lambda(\tau^2)$ satisfies (2.1)) gives

$$\sum_{k=1}^m \lambda^{n/2-k} \frac{1}{k!} \int_{\mathbb{R}^n} f^{(k)}(|\xi|^2) d\xi \int_{\mathbb{R}^n} V(x)^k dx + \mathcal{O}(\lambda^{n/2-m-1})$$

Taking $\lambda \to \infty$ and $m \to \infty$ we obtain the invariants

$$\int_{\mathbb{R}^n} f^{(k)}(|\xi|^2) d\xi \int_{\mathbb{R}^n} V(x)^k dx,$$

for every $k \geq 1$.

**Lemma 2.1.** *There exists $g$ satisfying* (2.1) *such that if $f(\tau^2) = g(\tau)$, then*

$$\int_{\mathbb{R}^n} f^{(k)}(|\xi|^2) d\xi \neq 0,$$

*provided $k \geq n$.*

*Proof.* Passing to polar coordinates, and writing $f^{(k)}(\tau^2) = \sum_{j=1}^k c_j g^{(j)}(\tau)\tau^{j-2k}$, we obtain

$$\int_{\mathbb{R}^n} f^{(k)}(|\xi|^2) d\xi = \lim_{\varepsilon \to 0^+} \sum_{j=1}^k c_j \int_\varepsilon^\infty g^{(j)}(\tau) \tau^{j-2k+n-1} d\tau.$$

We next integrate each integral by parts $2k - j - n$ times to obtain

$$\int_{\mathbb{R}^n} f^{(k)}(|\xi|^2) d\xi = A \int_0^\infty g^{(2k-n)}(\tau)\tau^{-1} d\tau + B g^{(2k-n)}(0) = A \int_0^\infty g^{(2k-n)}(\tau)\tau^{-1} d\tau \tag{2.4}$$

for some constants $A, B$. Note that all negative powers of $\varepsilon$ in the boundary terms must cancel when summed in $j$, since the left hand side is a finite integral, and that $g^{(2k-n)}(0) = 0$

since $2k-n$ is odd. To prove that $A \neq 0$, we observe that the identity (2.4) holds for $f(\tau)=e^{-\tau}$, $g(\tau)=e^{-\tau^2}$, and that in that case $\int f^{(k)}(|\xi|^2)d\xi = (-1)^k\pi^{n/2}$. Now,

$$2\int_0^\infty g^{(2k-n)}(\tau)\tau^{-1}d\tau = \int_{-\infty}^\infty g^{(2k-n)}(\tau)\tau^{-1}d\tau = \frac{i^{2k-n+1}}{2}\int_{-\infty}^\infty t^{2k-n}\hat{g}(t)\,\mathrm{sgn}\,t dt,$$

where we used the oddness of $g^{(2k-n)}$ followed by Plancherel's theorem. To make the final expression nonzero it suffices to take $g$ such that $\hat{g}$ is nonnegative and not identically 0. □

This shows that

$$\int_{\mathbb{R}^n} V(x)^k dx = \int_{\mathbb{R}^n} V_0(x)^k dx \tag{2.5}$$

for every $k \geq n$, and a similar analysis of the second invariant of (2.3) proves that

$$\int_{\mathbb{R}^n} V(x)^k|\nabla V(x)|^2 dx = \int_{\mathbb{R}^n} V_0(x)^k|\nabla V_0(x)|^2 dx \tag{2.6}$$

for every $k \geq n$.

We rewrite the invariant (2.5) using $V_*dx$, the pushforward of Lebesgue measure by $V$, as

$$\int_{\mathbb{R}^n} V(x)^k dx = \int_{\mathbb{R}} s^k (V_*dx)_s = i^k \widehat{V_*dx}^{(k)}(0). \tag{2.7}$$

Since $V$ and $V_0$ are both bounded functions, the pushforward measures are compactly supported and hence have entire Fourier transforms, and we conclude that

$$V_*dx = V_{0*}dx + \sum_{k=0}^{n-1} c_k\delta_0^{(k)} = V_{0*}dx + c_0\delta_0.$$

For the first equality we used the invariants (2.7), and for the second the fact that $V_*dx$ is a measure. In other words

$$\mathrm{vol}(\{V>\lambda\}) = \mathrm{vol}(\{V_0>\lambda\}) \tag{2.8}$$

whenever $\lambda > 0$. Moreover, this shows that $V_*dx$ is absolutely continuous on $(0,\infty)$, and so by Sard's lemma the critical set of $V$ is Lebesgue-null on $V^{-1}((0,\infty))$. As a result we may use the coarea formula[1] to write

$$V_*dx = \int_{\{V=s\}} |\nabla V|^{-1} dS ds, \qquad \text{on } (0,\infty)$$

and to conclude that

$$\int_{\{V=s\}} |\nabla V|^{-1} dS = \int_{\{V_0=s\}} |\nabla V_0|^{-1} dS \tag{2.9}$$

for almost every $s>0$. Similarly, rewriting the invariants (2.6) as

$$\int_{\mathbb{R}^n} V(x)^k|\nabla V(x)|^2 dx = \int_{\mathbb{R}} s^k \int_{\{V=s\}} |\nabla V| dS ds,$$

we find that

$$\int_{\{V=s\}} |\nabla V| dS = \int_{\{V_0=s\}} |\nabla V_0| dS, \qquad s>0. \tag{2.10}$$

[1]If $n=1$ we put $\int_{\{V=s\}} |\nabla V|^{-1}dS = \sum_{x\in V^{-1}(s)} |V'(x)|^{-1}$.

From the Cauchy-Schwarz inequality, (2.9) and (2.10) we find that

$$\left(\int_{\{V=s\}} 1 dS\right)^2 \leq \int_{\{V=s\}} |\nabla V|^{-1} dS \int_{\{V=s\}} |\nabla V| dS = \left(\int_{\{V_0=s\}} 1 dS\right)^2, \tag{2.11}$$

for almost every $s > 0$, where for the last equality we used the fact that $\nabla V_0 = R'$ is constant on level sets of $V_0$. By assumption these level sets $\{V_0 = s\}$ are spheres, and by (2.8) the volumes of their interiors equal those of $\{V = s\}$. Hence by the isoperimetric inequality[2] the level sets $\{V = s\}$ for almost every $s$ are spheres also, and furthermore equality is attained in (2.11). Consequently, from the Cauchy-Schwarz equality we conclude that $|\nabla V|$ and $|\nabla V|^{-1}$ are proportional on these level sets $\{V = s\}$, with

$$|\nabla V(x)|^2 = R'(R^{-1}(V(x))). \tag{2.12}$$

Because by assumption the right hand side does not vanish for $x$ such that $V(x) \in (0, \max V_0)$, we may conclude that the same is true of the left hand side and that (2.12) holds for all $x \in V^{-1}((0, \max V_0))$. Solving (2.12) along gradient flowlines as in [DaHeVe11, §3] gives the conclusion.

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