Asymptotic description of solutions of the exterior Navier Stokes problem in a half space

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Abstract

We consider the problem of a body moving within an incompressible fluid at constant speed parallel to a wall, in an otherwise unbounded domain. This situation is modeled by the incompressible Navier-Stokes equations in an exterior domain in a half space, with appropriate boundary conditions on the wall, the body, and at infinity. We focus on the case where the size of the body is small. We prove in a very general setup that the solution of this problem is unique and we compute a sharp decay rate of the solution far from the moving body and the wall.

Contents

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1 Introduction

In the present paper we discuss solutions of the Navier-Stokes equations for the stationary flow around a body that moves with constant speed parallel to a wall in an otherwise unbounded space filled with a fluid. The mathematical formulation of the problem is as follows. Let $\Omega_+ = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \mid y > 1\}$, and let $B_t = \{(x, y) \in \mathbb{R}^2 \mid (x, y) + t\mathbf{e}_1 \in B\}$, where $\mathbf{e}_1 = (0, 1)$ and where B is a bounded open connected subset of Ω_+ such that $\overline{B} \subset \Omega_+$. As a function of $t \geq 0$, the set B_t corresponds to a body which is immersed in a fluid and moves at constant speed from right to left parallel to the wall $\partial\Omega_+$. The flow around this body is modeled by the Navier-Stokes equations

$$\begin{cases} \partial_t \mathbf{U} &= -(\mathbf{U} \cdot \nabla)\mathbf{U} + \Delta\mathbf{U} - \nabla P \ , \\ \nabla \cdot \mathbf{U} &= 0 \ , \end{cases} \tag{1}$$

in $\Omega_t = \Omega_+ \setminus \overline{B_t}$ with the boundary conditions (the boundary $\partial\Omega_+$ is at rest and we choose no slip boundary conditions at the surface of the body),

$$\mathbf{U}|_{\partial\Omega_+} = 0 \ , \quad \lim_{\substack{\mathbf{x} \in \Omega_t \\ |\mathbf{x}| \to \infty}} \mathbf{U}(\mathbf{x}, t) = 0 \ , \quad \mathbf{U}|_{\partial B_t} = -\mathbf{e}_1 \ . \tag{2}$$

We are interested in the construction of solutions of equations (1)-(2) that are stationary when viewed in a reference frame attached to the moving body. We therefore set

$$\mathbf{U}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x} + t\mathbf{e}_1) \ , \quad P(\mathbf{x}, t) = p(\mathbf{x} + t\mathbf{e}_1) \ ,$$

and get the following stationary problem:

$$\begin{cases} -(\mathbf{u} \cdot \nabla)\mathbf{u} - \partial_x\mathbf{u} + \Delta\mathbf{u} - \nabla p &= 0 \ , \\ \nabla \cdot \mathbf{u} &= 0 \ , \end{cases} \tag{3}$$

in $\Omega_+ \setminus \overline{B}$, with the boundary conditions

$$\mathbf{u}|_{\partial B} = -\mathbf{e}_1 \ , \quad \mathbf{u}|_{\partial\Omega_+} = 0 \ , \quad \lim_{\substack{\mathbf{x} \in \Omega_+ \\ |\mathbf{x}| \to \infty}} \mathbf{u}(\mathbf{x}) = 0 \ . \tag{4}$$

Note that we have set without restriction of generality all the physical constants and the speed of the moving body equal to one. This can always be achieved by an appropriate scaling. With this choice of normalization the Reynolds number of the moving body corresponds to the diameter ε of B. The problem contains a second length-scale, which is the distance h of (the center of) B from the wall $\partial\Omega_+$. In this paper, we are interested in the regime where ε is small, and in particular small with respect to h.

The system (3) with boundary conditions (4) is related to the so-called exterior Navier-Stokes problem:

$$\begin{cases} -\lambda\left((\mathbf{u} - \mathbf{u}_\infty) \cdot \nabla\right)\mathbf{u} + \Delta\mathbf{u} - \nabla p &= 0 \ , \\ \nabla \cdot \mathbf{u} &= 0 \ , \end{cases} \quad \text{in } \mathbb{R}^n \setminus \overline{B} \tag{5}$$

$$\mathbf{u}|_{\partial B} = \mathbf{u}^* \ , \quad \lim_{|\mathbf{x}| \to \infty} \mathbf{u}(\mathbf{x}) = 0 \ . \tag{6}$$

where B is a bounded open connected subset of $\mathbb{R}^n$ with smooth boundary, $\lambda \in \mathbb{R}$ is the Reynolds number, $\mathbf{u}_\infty \in \mathbb{R}^n$ is a prescribed asymptotic velocity and $\mathbf{u}^* \in H^{1/2}(\partial B)$ is a given boundary condition. Most of the methods for solving this problem are extensively described in the fundamental book of G.P. Galdi [9]. We give a brief outline of some results in the following lines.

The first methods to solve such exterior problems go back to the pioneering work of J. Leray [15]. In this reference, the author introduces an *invading domain* method yielding existence of at least one *weak solution* to (5)-(6) whose velocity-field $\mathbf{u}$ satisfies $\|\nabla\mathbf{u}\,;\, L^2(\mathbb{R}^n \setminus \overline{B})\| < \infty$. A comparable result is obtained by H. Fujita [7]. Similar weak solutions are constructed also for exterior Navier Stokes system (5) with other types of boundary conditions on ∂B (see [18] and [19]). The only shortcoming of these weak solutions is that insufficient information is obtained on the behavior at infinity. In the case $n = 2$ with $\mathbf{u}_\infty = 0$, it is still not known whether the vanishing condition at infinity is satisfied by weak solutions or not (see [1, 10] and [16] for recent developments in this question). This difficulty is linked to the famous Stokes Paradox which holds in two space-dimensions. For the geometry of the present paper, existence of weak solutions for (3) decaying at infinity, combined with other boundary conditions, is studied in [12].

In the case $\mathbf{u}_\infty \neq 0$, a more refined description of the asymptotic behavior of solutions to (5)-(6) is given in a second series of papers. These results rely on the idea that the dominating system at infinity is the Oseen system :

$$\left\{ \begin{array}{rcl} \lambda \mathbf{u}_\infty \cdot \nabla \mathbf{u} + \Delta \mathbf{u} - \nabla p & = & 0 \ , \\ \nabla \cdot \mathbf{u} & = & 0 \ . \end{array} \right. \tag{7}$$

A detailed comprehension of the asymptotics of solutions to this linear system enables to construct solution to (5)-(6) via a standard perturbation technique and then to compute the asymptotics of the constructed solutions. Such an analysis is performed by K.I. Babenko in the 3D-setting [3], and by R. Finn and D.R. Smith [5] and L.I. Sazonov [17] in the 2D-setting. This method is transposed to the geometry studied in the present paper by T. Fischer, G.C. Hsiao and W.L. Wendland in [6]. In this last case the difficulty linked to the Stokes paradox is less limitative. In particular, the Stokes problem :

$$\left\{ \begin{array}{rcl} \Delta \mathbf{u} - \nabla p & = & 0 \ , \\ \nabla \cdot \mathbf{u} & = & 0 \ , \end{array} \right. \qquad \text{in } \Omega_+ \setminus \overline{B}. \tag{8}$$

$$\mathbf{u}|_{\partial B} = \mathbf{u}^* \ , \quad \mathbf{u}|_{\partial \Omega_+} = 0 \ , \quad \lim_{|\mathbf{x}| \to \infty} \mathbf{u}(\mathbf{x}) = 0 \ , \tag{9}$$

is well-posed. In [6] existence of solutions to (3)-(4) is obtained via a perturbation method based on this linear Stokes problem. Nevertheless, the dominating system at infinity in our case is still the Oseen system with $\mathbf{u}_\infty = \mathbf{e}_1$ so that no precise asymptotics of the constructed solutions is given in [6]. This computation requires a very careful analysis of the Oseen linear system in the half space, the analysis of which is not yet available with these former methods. For completness, we mention here that the properties of the Stokes system in the geometry of the present paper is studied in the more general framework of weighted Sobolev spaces in [2]. No equivalent study for the Oseen system is provided to our knowledge.

The present paper uses a dynamical-system approach for studying the asymptotics of solutions to an exterior Navier Stokes problem. In this method, the first idea is to interpret one coordinate as a time. Then, one rewrites (3) as a system of nonlinear evolution equations. Solutions are constructed via a perturbation method in function spaces enabling to compute the exact long-time behavior. In return, one obtains solutions to (3)-(4) with detailed asymptotics. This program is applied successfully to the case of the 3D exterior Navier-Stokes system in [20] and of the 2D half-space problem, with the solid B replaced by a smooth source term with compact support, in a previous publication of the authors [13]. In this last reference, the solutions to the system of nonlinear evolution equations are computed performing a Fourier transform in the transversal direction (*i.e.*, with respect to x in our case). This is the reason why we replace the obstacle by a source term with compact support in [13].

In the present paper, we prove existence of solutions to (3)-(4) with a detailed asymptotics by combining the invading method of Leray and the dynamical-system approach. Since we apply in part perturbation methods, our results hold only for small Reynolds numbers. The role of Reynolds number is played by the diameter of the solid B in our setting. More precisely, let S be a bounded open subset of $\mathbb{R}^2$ containing the origin, with a smooth boundary, and let h be a positive parameter which fixes the center of the body with respect to the boundary. Then, we set $S_\varepsilon := (0, 1+h) + \varepsilon S$ and rewrite our system as :

$$\left\{ \begin{array}{rcl} -(\mathbf{u} \cdot \nabla)\, \mathbf{u} - \partial_x \mathbf{u} + \Delta \mathbf{u} - \nabla p & = & 0 \ , \\ \nabla \cdot \mathbf{u} & = & 0 \ , \end{array} \right. \tag{10}$$

in $\Omega \setminus \overline{S}_\varepsilon$, with the boundary conditions

$$\mathbf{u}|_{\partial S_\varepsilon} = -\mathbf{e}_1 \ , \quad \mathbf{u}|_{\partial \Omega_+} = 0 \ , \quad \lim_{\substack{\mathbf{x} \in \Omega_+ \\ |\mathbf{x}| \to \infty}} \mathbf{u}(\mathbf{x}) = 0 \ . \tag{11}$$

In what follows (10) together with boundary conditions (11) is referred to as Problem 1. The following theorem is our main result.

Theorem 1 *For ε sufficiently small, there exists a unique weak solution* $\mathbf{u}$ *of Problem 1. Furthermore, there exists a constant $C_\varepsilon < \infty$ such that, for all $(x,y) \in \Omega_+ \setminus \overline{S_\varepsilon}$,*

$$|\mathbf{u}(x,y)| \leq \frac{C_\varepsilon}{y^{\frac{3}{2}}} \ . \tag{12}$$

A precise definition of weak solutions for Problem 1 is given in Section 1. For the sake of simplicity we only give a bound for the decay of the weak solution in (12). Nevertheless, a precise first order for the asymptotics is available with our techniques. Such computations are performed in an independent paper (see [4]). This bound is the critical ingredient for proving uniqueness in the frame of weak solutions to Problem 1.

Our strategy to obtain detailed information on weak solutions of Problem 1 at infinity is divided in five steps. First, we show the existence of weak solutions for Problem 1 by the invading method of Leray. Second, we use a cut-off function to obtain, from a weak solution $(\mathbf{u},p)$ of Problem 1, a weak solution $(\tilde{\mathbf{u}},\tilde{p})$ to

$$\begin{gathered} -(\tilde{\mathbf{u}}\cdot\nabla)\,\tilde{\mathbf{u}} - \partial_x \tilde{\mathbf{u}} + \Delta\tilde{\mathbf{u}} - \nabla\tilde{p} = \mathbf{f}\ , \\ \nabla\cdot\tilde{\mathbf{u}} = 0\ , \end{gathered} \tag{13}$$

with the boundary conditions

$$\tilde{\mathbf{u}}|_{\partial\Omega_+} = 0\ , \quad \lim_{\substack{\mathbf{x}\in\Omega_+ \\ \mathbf{x}\to\infty}} \tilde{\mathbf{u}}(x,y) = 0\ , \tag{14}$$

The system (13) together with boundary conditions (14) is referred to as Problem 2 in what follows. Note that, in this new system we keep the divergence-free condition: the cut-off function is applied to the stream function of $\mathbf{u}$. This enables to compute explicitly the source term $\mathbf{f}$. So, in the third step, we show that, for ε small enough, the function $\mathbf{f}$ satisfies the smallness condition formulated in our previous paper [13], so that there exists at least one α-solution $(\mathbf{u}_\alpha, p_\alpha)$ for Problem 2 (see Section 3 for the definition of α-solutions). In the fourth step, we prove a weak-strong uniqueness result for Problem 2. Once again, this weak-strong argument applies to weak-solutions and α-solutions constructed for ε small enough. The uniqueness of solutions for Problem 2 does not directly imply the uniqueness of solutions for Problem 1, because different solutions of Problem 1 may lead to different functions $\mathbf{f}$. So in a last step, we prove uniqueness of weak solutions for Problem 1 for ε small enough.

1.1 Sets and function spaces

In the whole paper, we use the standard notations for function spaces such as $L^p(\mathcal{O})$ for Lebesgue spaces and $W^{m,p}(\mathcal{O})$ or $H^m(\mathcal{O})$ for Sobolev spaces. We denote by $\mathcal{C}^m(\mathcal{O})$ the spaces of continuous functions having m continuous derivative (m might be infinite). We use the subscript c to specify that function have compact support in the set $\mathcal{O}$. Given a Banach space X and $p \in X$, the norm of p in X is denoted by $\|p;X\|$ and, if X is a function space containing the constants:

$$\|p;X/\mathbb{R}\| := \inf\{\|p+c;X\|,\ c\in\mathbb{R}\}\ . \tag{15}$$

This latter notation is very useful for pressures which are defined up to an additive constant in systems such as (3).

In some proofs, we shall need a smooth covering of Ω_+ or of $\Omega_+ \setminus \{(0,1+h)\}$. For this purpose, we introduce here some particular subsets of Ω_+. First, we denote $B(\lambda)$ the open balls with center $(0,1+h)$ *i.e.* given $\lambda > 0$, we denote by

$$B(\lambda) = \Big\{(x,y)\in\Omega_+ \text{ such that } |(x,y)-(0,1+h)| < \lambda\Big\}\ .$$

We note in particular, that, since S is bounded, there exists $\varepsilon_0 > 0$, such that $S_\varepsilon \subset B(h/3)$ for $\varepsilon < \varepsilon_0$. We keep the classical convention $B((x,y),r)$ for balls with center $(x,y)\in\mathbb{R}^2$ and radius $r>0$. We introduce $(\Delta_n)_{n\in\mathbb{N}}$ an increasing covering of Ω_+ such that, for all $n\in\mathbb{N}$:

- Δ_n has a smooth boundary

- $B(\,(1+n)h\,) \subset \Delta_n \subset B(\,(2+n)h\,)$.

Furthermore we define, for $n \in \mathbb{N}$, the sets $\mathcal{A}_n$ by $\mathcal{A}_n = \Delta_n \setminus \overline{B(2^{-n}h)}$. Therefore, for all $n \in \mathbb{N}$, $\mathcal{A}_n$ has a smooth boundary.

2 Weak solutions for Problem 1

In this section, we consider the theory of weak solutions for Problem 1. The main result of this section is the following theorem.

Theorem 2 *There exists a family $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ which is defined for ε sufficiently small and such that:*

(i) for all $\varepsilon > 0$, $\mathbf{u}_\varepsilon$ is a weak solution of Problem 1 for S_ε,

(ii) given $\eta > 0$ there exists $0 < \varepsilon_\eta$ such that, for all $\varepsilon < \varepsilon_\eta$ there holds $\|\mathbf{u}_\varepsilon; D\| \leq \eta$,

(iii) there exists a pressure p_ε such that $(\mathbf{u}_\varepsilon, p_\varepsilon)$ satisfies (13) in $\Omega_+ \setminus \overline{S}_\varepsilon$ and, given $m \in \mathbb{N}$, there holds:

$$\|\mathbf{u}_\varepsilon; \mathcal{C}^{m+1}(\overline{\mathcal{A}_2})\| + \|p_\varepsilon; \mathcal{C}^m(\overline{\mathcal{A}_2})/\mathbb{R}\| \leq C_m \|\mathbf{u}_\varepsilon; D\| \ . \tag{16}$$

for some universal constant C_m depending only on m.

We refer the reader to the introduction for the definition of $\mathcal{A}_2$. We introduce function spaces and the definition of weak solutions for Problem 1 just below.

The proof of this result is divided in three steps. First, we recall the method of Leray for the construction of weak solutions. We obtain in this way a family of weak solutions which satisfy a particular uniform bound with respect to the (small) size of the obstacle. Eventually, we prove that this family of solutions tends to 0 in the sense of **Theorem 2**.

2.1 Definition of weak solutions

To begin with, the size ε of the obstacle is fixed such that $S_\varepsilon \subset B(h/3)$. Let $(\mathbf{u}, p)$ be a smooth solution of Problem 1 for S_ε. We extend $\mathbf{u}$ from $\Omega_+ \setminus \overline{S_\varepsilon}$ to the whole of Ω_+ by setting $\mathbf{u} = -\mathbf{e}_1$ on $\overline{S_\varepsilon}$. Let $\mathbf{w}$ be a smooth divergence-free vector-field with compact support in Ω_+ which is equal to a given constant vector field $\mathbf{W}$ on S_ε. Then, if we multiply equation (10) by $\mathbf{w}$ and integrate over $\Omega_+ \setminus \overline{S_\varepsilon}$ we get

$$\int_{\Omega_+ \setminus \overline{S_\varepsilon}} (\Delta \mathbf{u} - \nabla p) \cdot \mathbf{w} \; d\mathbf{x} = \int_{\Omega_+ \setminus \overline{S_\varepsilon}} [(\mathbf{u} + \mathbf{e}_1) \cdot \nabla \mathbf{u}] \cdot \mathbf{w} \; d\mathbf{x} \ . \tag{17}$$

In order to unburden notations we have suppressed in (17), and in what follows, the arguments of functions when no confusion is possible. Applying Green's identity to the left-hand side of (17) leads to the equality

$$\int_{\Omega_+ \setminus \overline{S_\varepsilon}} (\Delta \mathbf{u} - \nabla p) \cdot \mathbf{w} \; d\mathbf{x} = \int_{\partial(\Omega_+ \setminus \overline{S_\varepsilon})} T(\mathbf{u}, p)\mathbf{n} \cdot \mathbf{w} \; d\sigma - \frac{1}{2} \int_{\Omega_+ \setminus \overline{S_\varepsilon}} \left(\nabla \mathbf{u} + [\nabla \mathbf{u}]^\top \right) : \left(\nabla \mathbf{w} + [\nabla \mathbf{w}]^\top \right) \; d\mathbf{x} \ ,$$

where $T(\mathbf{u}, p) = (\nabla \mathbf{u} + [\nabla \mathbf{u}]^\top) - pI$ and where $\mathbf{n}$ is the outward normal on $\partial(\Omega_+ \setminus \overline{S_\varepsilon})$. Using the boundary conditions for $\mathbf{u}$, which imply in particular that $\nabla \mathbf{u}$ vanishes on S_ε, and using that $\mathbf{w} = \mathbf{W}$ on S_ε, we obtain that $\mathbf{u}$ satisfies

$$\int_{\Omega_+} \nabla \mathbf{u} : \nabla \mathbf{w} \; d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u} + \mathbf{e}_1) \cdot \nabla \mathbf{u}] \cdot \mathbf{w} \; d\mathbf{x} = -\mathbf{\Sigma} \cdot \mathbf{W} \ , \tag{18}$$

with the vector

$$\mathbf{\Sigma} = -\int_{\partial S_\varepsilon} T(\mathbf{u}, p)\mathbf{n} \; d\sigma \ . \tag{19}$$

The vector $\mathbf{\Sigma}$ is the force which the fluid exerts on S_ε. If we replace, on a formal level, $\mathbf{w}$ by $\mathbf{u}$ in (18), we obtain, as $\mathbf{W} = -\mathbf{e}_1$ in this case :

$$\int_{\Omega_+} |\nabla \mathbf{u}|^2 \, d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u} + \mathbf{e}_1) \cdot \nabla \mathbf{u}] \cdot \mathbf{u} \, d\mathbf{x} = \mathbf{\Sigma} \cdot \mathbf{e}_1 \ . \tag{20}$$

Integrating by parts yields, as $\mathbf{u}$ is divergence-free:

$$\int_{\Omega_+} [(\mathbf{u} + \mathbf{e}_1) \cdot \nabla \mathbf{u}] \cdot \mathbf{u} \, d\mathbf{x} = \frac{1}{2} \int_{\Omega_+} \left[\mathbf{u} \cdot \nabla |\mathbf{u}|^2 + \partial_x |\mathbf{u}|^2 \right] \, d\mathbf{x} = 0 \ , \tag{21}$$

and therefore (19) reduces to

$$\int_{\Omega_+} |\nabla \mathbf{u}|^2 \, d\mathbf{x} = \mathbf{\Sigma} \cdot \mathbf{e}_1 \ . \tag{22}$$

We conclude that if $(\mathbf{u}, p)$ is a solution of Problem 1 which decays sufficiently rapidly at infinity, then $\mathbf{u}$ satisfies the integral equation (18) and we have the identity (22), which means that $\nabla \mathbf{u} \in L^2(\Omega_+)$.

The above discussion motivates the following functional setting for weak solutions of Problem 1. Let $\mathcal{D}$ be the vector space of smooth divergence-free vector-fields with compact support in Ω_+. We equip $\mathcal{D}$ with the scalar product

$$((\mathbf{w}_1, \mathbf{w}_2)) = \int_{\Omega_+} \nabla \mathbf{w}_1 : \nabla \mathbf{w}_2 \, d\mathbf{x} \ . \tag{23}$$

For functions in $\mathcal{D}$ we have

$$\int_{\Omega_+} \nabla \mathbf{w}_1 : \nabla \mathbf{w}_2 \, d\mathbf{x} = \frac{1}{2} \int_{\Omega_+} (\nabla \mathbf{w}_1 + [\nabla \mathbf{w}_1]^\top) : (\nabla \mathbf{w}_2 + [\nabla \mathbf{w}_2]^\top) \, d\mathbf{x} \ . \tag{24}$$

Let D be the Hilbert space with respect to the scalar product (23) obtained by completion of $\mathcal{D}$. Let $\mathcal{D}^\varepsilon \subset \mathcal{D}$ be the vector-fields $\mathbf{w} \in \mathcal{D}$ which are constant on S_ε and let D^ε be the closure of $\mathcal{D}^\varepsilon$ in D. On D^ε we define the function Γ by

$$\begin{aligned} \Gamma : \quad D^\varepsilon &\longrightarrow \mathbb{R}^2 \\ \mathbf{w} &\longmapsto \mathbf{W} = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \mathbf{w}(\mathbf{x}) \, d\mathbf{x} \ . \end{aligned} \tag{25}$$

It follows from Hardy's inequality (see Proposition 18 below) that Γ is bounded. For convenience later on we define, for all $\mathbf{W} \in \mathbb{R}^2$,

$$\mathcal{D}^\varepsilon_{\mathbf{W}} = \{ \mathbf{w} \in \mathcal{D}^\varepsilon \mid \mathbf{w}_{|S_\varepsilon} = \mathbf{W} \} \ , \quad D^\varepsilon_{\mathbf{W}} = \{ \mathbf{w} \in D^\varepsilon \mid \mathbf{w}_{|S_\varepsilon} = \mathbf{W} \} \ . \tag{26}$$

Such spaces have been studied extensively in [9, Chapter III.5]. In particular, we emphasize that with our smoothness assumptions on ∂S_ε we have that $\overline{\mathcal{D}^\varepsilon_{\mathbf{W}}} = D^\varepsilon_{\mathbf{W}}$.

Following the work of Leray, we now define weak solutions for Problem 1:

Definition 3 *A vector-field* $\mathbf{u}$ *is called a weak solution of Problem 1, if*

(i) $\mathbf{u} \in D^\varepsilon_{-\mathbf{e}_1}$,

(ii) there exists a vector $\mathbf{\Sigma} \in \mathbb{R}^2$*, such that for all* $\mathbf{w} \in \mathcal{D}^\varepsilon$

$$\int_{\Omega_+} \nabla \mathbf{u} : \nabla \mathbf{w} \, d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u} + \mathbf{e}_1) \cdot \nabla \mathbf{u}] \cdot \mathbf{w} \, d\mathbf{x} = -\mathbf{\Sigma} \cdot \Gamma(\mathbf{w}) \ , \tag{27}$$

and

$$\int_{\Omega_+} |\nabla \mathbf{u}|^2 \, d\mathbf{x} \leq \mathbf{\Sigma} \cdot \mathbf{e}_1 \ . \tag{28}$$

The following standard lemma shows that weak solutions are well defined.

Lemma 4 *Let* $(\mathbf{u},\mathbf{v}) \in D^2$ *and let* $\mathbf{w} \in D$ *with* $\mathrm{Supp}(\mathbf{w}) \subset \mathcal{O} \subset\subset \Omega_+$[1]*. Then,*

$$\int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{v}]\cdot\mathbf{w}\ d\mathbf{x} = -\int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{w}]\cdot\mathbf{v}\ d\mathbf{x}\ , \tag{29}$$

and

$$\left|\int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{v}]\cdot\mathbf{w}\ d\mathbf{x}\right| \le C(\mathcal{O})\left(\|\mathbf{u};L^4(\mathcal{O})\|\ \|\mathbf{v};D\|\ \|\mathbf{w};L^4(\mathcal{O})\| + \|\mathbf{v};D\|\ \|\mathbf{w};L^2(\mathcal{O})\|\right)\ . \tag{30}$$

Below we show that, given a weak solution $\mathbf{u}$ of Problem 1, one can construct a function p such that the couple $(\mathbf{u},p)$ satisfies the equation (10) in the classical sense. We will call p the pressure associated with the weak solution $\mathbf{u}$. Using the ellipticity of the Stokes operator together with the smoothness of the boundary of the fluid domain, it is possible to prove that $(\mathbf{u},p)\in\mathcal{C}^\infty(\overline{\Omega_+}\setminus S_\varepsilon)$, and that the boundary conditions (11) on S_ε and on $\partial\Omega_+$ are satisfied in the classical sense. Therefore, weak solutions have all the requested properties of classical solutions, and the only difficulty with weak solutions is that their rate of decay at infinity remains unknown. A bound on the decay rate, like (12), is crucial in order to prove uniqueness of solutions.

2.2 Existence of weak solutions

In this section, we prove:

Theorem 5 *There exist constants* $K<\infty$ *and* $\varepsilon_1>0$ *such that if* $\varepsilon<\varepsilon_1$*, there exists at least one weak solution* $\mathbf{u}$ *for Problem 1 for* S_ε*, satisfying the further bound* $\|\mathbf{u};D\|\le K$*.*

The proof is based on the exhaustion method of Leray. Namely, we consider a nested sequence of finite domains that converge to Ω_+ and, for any domain of this sequence, we prove existence of one approximate weak solution having support in this domain and satisfying a suitable estimate. Our result then follows by a compactness argument. Many aspects of the proof are standard, but the uniform bound is new to our knowledge.

2.2.1 Sketch of proof for Theorem 5

In this proof the size ε of the obstacle is again fixed such that $S_\varepsilon\subset B(h/3)$. We mention further assumptions on ε when needed. We consider the sequence $(\Delta_n)_{n\ge1}$ given in the introduction. This sequence satisfies, for all $n\in\mathbb{N}$:

- Δ_n is a bounded open set having a smooth boundary
- $S_\varepsilon \subset\subset \Delta_n \subset \Delta_{n+1}$
- $\bigcup_{n\in\mathbb{N}}\Delta_n = \Omega_+$.

Given Δ_n, we define $D^{\varepsilon,n}$ and $D^{\varepsilon,n}_{\mathbf{W}}$ by

$$D^{\varepsilon,n} = \{\mathbf{w}\in D^\varepsilon \mid \mathbf{w}|_{\Omega_+\setminus\overline{\Delta_n}} = 0\}\ ,\quad D^{\varepsilon,n}_{\mathbf{W}} = \{\mathbf{w}\in D^\varepsilon_{\mathbf{W}} \mid \mathbf{w}|_{\Omega_+\setminus\overline{\Delta_n}} = 0\}\ . \tag{31}$$

With these conventions, the definition of approximate weak solutions for Problem 1 is:

Definition 6 *Let* $n\in\mathbb{N}$*. A vector-field* $\mathbf{u}$ *is called an approximate weak solution on* Δ_n *if:*

(i) $\mathbf{u}\in D^{\varepsilon,n}_{-\mathbf{e}_1}$,

[1]We use the standard notation $A\subset\subset B$ to mean that the closure $\bar{A}$ is a compact subset of B.

(*ii*) *for all* $\mathbf{w} \in D_0^{\varepsilon,n}$,

$$\int_{\Omega_+} \nabla \mathbf{u} : \nabla \mathbf{w} \; d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{u}]\cdot\mathbf{w} \; d\mathbf{x} = 0 \; . \tag{32}$$

Before giving a sketch of the proof of **Theorem 5**, we mention that, since $D_0^{\varepsilon,n}$ is a closed subspace of $D^{\varepsilon,n}$ of codimension two, the Lagrange multiplier theorem implies the existence of a vector $\mathbf{\Sigma} \in \mathbb{R}^2$, such that for all $\mathbf{w} \in D^{\varepsilon,n}$

$$\int_{\Omega_+} \nabla \mathbf{u} : \nabla \mathbf{w} \; d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{u}]\cdot\mathbf{w} \; d\mathbf{x} = -\mathbf{\Sigma}\cdot\Gamma(\mathbf{w}) \; . \tag{33}$$

The vector $\mathbf{\Sigma}$ is the force associated with the approximate weak solution $\mathbf{u}$. Since $\mathbf{u} \in D^{\varepsilon,n}$, we can replace $\mathbf{w}$ by $\mathbf{u}$ in (33), and an integration by parts yields

$$\int_{\Omega_+} |\nabla \mathbf{u}|^2 \; d\mathbf{x} = \mathbf{\Sigma}\cdot\mathbf{e}_1 \; . \tag{34}$$

The energy (in)equality is therefore a consequence of Definition 6, and for this reason we do not need to impose it in the definition of approximate weak solutions in contrast with the definition of weak solutions.

The proof of **Theorem 5** is based on the following two lemmas:

Lemma 7 *There exists a constant* $\varepsilon_1 > 0$ *such that if* $\varepsilon < \varepsilon_1$ *there exists at least one approximate weak solution on* Δ_n, *for all* $n \in \mathbb{N}$.

Lemma 8 *Let* ε *be as in Lemma 7 and* $n \in \mathbb{N}$. *There exists a constant* $K < \infty$, *such that* $\|\mathbf{u}; D\| + |\mathbf{\Sigma}| \leq K$ *for any approximate weak solution* $\mathbf{u}$ *on* Δ_n *with associated force* $\mathbf{\Sigma}$.

Proofs for these lemmas are given in Section 2.2.2. We sketch now the remaining steps of the proof of **Theorem 5** assuming that $\varepsilon < \varepsilon_1$.

(*i*) By Lemma 7, there exists a sequence $(\mathbf{u}_n, \mathbf{\Sigma}_n)_{n\geq 1}$ such that $\mathbf{u}_n$ is an approximate weak solution on Δ_n with associated force $\mathbf{\Sigma}_n$. By Lemma 8 this sequence is bounded in $D \times \mathbb{R}^2$. One can therefore extract a subsequence $(\mathbf{u}_{n_i}, \mathbf{\Sigma}_{n_i})_{i\geq 1}$, such that $(\mathbf{u}_{n_i})_{i\geq 1}$ converges in D weakly to $\mathbf{u}$ and such that $(\mathbf{\Sigma}_{n_i})_{i\geq 1}$ converges in $\mathbb{R}^2$ strongly to $\mathbf{\Sigma}$. By Hardy's inequality the sequence $(\mathbf{u}_{n_i})_{i\geq 1}$ is bounded in $H^1(S_\varepsilon)$. We can therefore extract a subsequence which converges in $L^2(S_\varepsilon)$ strongly to $\mathbf{u}$. Since $\mathbf{u}_n = -\mathbf{e}_1$, for all $n \in \mathbb{N}$, we find that $\mathbf{u} \in D^\varepsilon_{-\mathbf{e}_1}$.

(*ii*) Given $\mathbf{w} \in \mathcal{D}^\varepsilon$ there exists $n_\mathbf{w} > 0$, such that $\mathbf{w} \in D^{\varepsilon,n}$ for all $n \geq n_\mathbf{w}$. Therefore, we have for i sufficiently large

$$\int_{\Omega_+} \nabla \mathbf{u}_{n_i} : \nabla \mathbf{w} \; d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}_{n_i}+\mathbf{e}_1)\cdot\nabla\mathbf{u}_{n_i}]\cdot\mathbf{w} \; d\mathbf{x} = -\mathbf{\Sigma}_{n_i}\cdot\Gamma(\mathbf{w}) \; . \tag{35}$$

Since $H^1(\Delta_{n_\mathbf{w}})$ is compactly imbedded in $L^4(\Delta_{n_\mathbf{w}})$, we find using Lemma 4, that (35) remains valid in the limit. This shows that

$$\int_{\Omega_+} \nabla \mathbf{u} : \nabla \mathbf{w} \; d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{u}]\cdot\mathbf{w} \; d\mathbf{x} = -\mathbf{\Sigma}\cdot\Gamma(\mathbf{w}) \; . \tag{36}$$

In the weak limit we have moreover that

$$\int_{\Omega_+} |\nabla \mathbf{u}|^2 \; d\mathbf{x} \leq \liminf_{i\to\infty} \int_{\Omega_+} |\nabla \mathbf{u}_{n_i}|^2 \; d\mathbf{x} = \liminf_{i\to\infty} \mathbf{\Sigma}_{n_i}\cdot\mathbf{e}_1 = \mathbf{\Sigma}\cdot\mathbf{e}_1 \; . \tag{37}$$

Combining (36) and (37) we conclude that there exists $\mathbf{\Sigma} \in \mathbb{R}^2$ such that, for all $\mathbf{w} \in \mathcal{D}^\varepsilon$,

$$\int_{\Omega_+} \nabla \mathbf{u} : \nabla \mathbf{w} \; d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{u}]\cdot\mathbf{w} \; d\mathbf{x} = -\mathbf{\Sigma}\cdot\Gamma(\mathbf{w}) \; , \tag{38}$$

and that

$$\int_{\Omega_+} |\nabla \mathbf{u}|^2 \; d\mathbf{x} \leq \mathbf{\Sigma}\cdot\mathbf{e}_1 \; . \tag{39}$$

This completes the proof of **Theorem 5**.

2.2.2 Proofs of Lemma 7 and Lemma 8

In these proofs $n \in \mathbb{N}$ is fixed. Since $D^{\varepsilon,n}$ and $D_0^{\varepsilon,n}$ are closed subspaces of D^ε, they are Hilbert spaces with respect to the scalar product (23). The space $D_{-\mathbf{e}_1}^{\varepsilon,n}$ is not empty. Indeed, let χ be a smooth cut-off function that is equal to one outside the disk $B(2h/3)$ and equal to zero inside the disk $B(h/3)$. As $\overline{S_\varepsilon} \subset B(h/3)$ the function $\tilde{\mathbf{U}}_{-\mathbf{e}_1}(x,y) = -\nabla^\perp((1-\chi)\, y)$ satisfies $\tilde{\mathbf{U}}_{-\mathbf{e}_1} \in D_{-\mathbf{e}_1}^{\varepsilon,n}$. We note that $D_{-\mathbf{e}_1}^{\varepsilon,n}$ is an affine subspace of $D^{\varepsilon,n}$ with direction $D_0^{\varepsilon,n}$. For technical reason (see (41)), we also introduce $\mathbf{U}_{-\mathbf{e}_1}$ the unique minimizer of the D-norm amongst the velocity-fields in $D_{-\mathbf{e}_1}^{\varepsilon,n}$. This velocity field satisfies :

1. $D_{-\mathbf{e}_1}^{\varepsilon,n} = \mathbf{U}_{-\mathbf{e}_1} + D_0^{\varepsilon,n}$
2. $((\mathbf{U}_{-\mathbf{e}_1}, \mathbf{w})) = 0$ for all velocity fields $\mathbf{w} \in D_0^{\varepsilon,n}$.

We now reformulate the existence of an approximate weak solution on Δ_n as a fixed point problem for a functional equation. First, we note that **Lemma 4** implies that for all $\mathbf{u} \in D_{-\mathbf{e}_1}^{\varepsilon,n}$, the map

$$\begin{array}{rcl} D_0^{\varepsilon,n} & \to & \mathbb{R} \\ \mathbf{w} & \mapsto & \displaystyle\int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{u}]\cdot\mathbf{w}\; d\mathbf{x}\ , \end{array}$$

is a continuous linear form. By the Riesz-Fréchet theorem we can therefore define a continuous map b_n^* from $D_{-\mathbf{e}_1}^{\varepsilon,n}$ to $D_0^{\varepsilon,n}$ by the formula

$$((b_n^*(\mathbf{u}), \mathbf{w})) = \int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{u}]\cdot\mathbf{w}\; d\mathbf{x}\ , \quad \forall\, \mathbf{w} \in D_0^{\varepsilon,n}\ . \tag{40}$$

With these definitions we find, on the one hand, that $\mathbf{u}$ is an approximate weak solution on Δ_n if and only if $\mathbf{u} = \mathbf{U}_{-\mathbf{e}_1} + \mathbf{v}$, with $\mathbf{v}$ a solution of the functional equation

$$\mathbf{v} = b_n^*(\mathbf{U}_{-\mathbf{e}_1} + \mathbf{v})\ , \qquad \mathbf{v} \in D_0^{\varepsilon,n}\ . \tag{41}$$

On the other hand, (30) together with (29) imply that b_n^* is continuous on $D_{-\mathbf{e}_1}^{\varepsilon,n}$ equipped with the $L^4(\Delta_n)$-norm. Using that $H_0^1(\Delta_n)$ is compactly imbedded in $L^4(\Delta_n)$ yields that b_n^* is completely continuous, *i.e.*, for any given bounded sequence $(\mathbf{v}_i)_{i\geq 1}$ in $D_0^{\varepsilon,n}$, there exists a subsequence $(\mathbf{v}_{i_j})_{j\geq 1}$ such that the sequence $(b_n^*(\mathbf{U}_{-\mathbf{e}_1} + \mathbf{v}_{i_j}))_{j\geq 1}$ converges strongly in $D_0^{\varepsilon,n}$. Hence, the Leray-Schauder fixed point theorem (see [14] or [11, Theorem 11.6, p. 286] for more details) guarantees the existence of a solution of (41) by proving a suitable estimate on *a priori* solutions to an auxiliary problem. This estimate is the content of the following proposition.

Proposition 9 *There exist constants* $\varepsilon_1 > 0$ *and* $C < \infty$*, such that for all* $\varepsilon < \varepsilon_1$*,* $\lambda \in [0,1]$ *and all* $(\mathbf{u}, \mathbf{\Sigma}) \in D_{-\mathbf{e}_1}^{\varepsilon,n} \times \mathbb{R}^2$ *which satisfy*

$$\int_{\Omega_+} \nabla\mathbf{u} : \nabla\mathbf{w} + \lambda \int_{\Omega_+} [(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{u}]\cdot\mathbf{w} = -\mathbf{\Sigma}\cdot\Gamma(\mathbf{w})\ , \quad \forall\ \mathbf{w} \in D^{\varepsilon,n}\ , \tag{42}$$

we have the bound $\|\mathbf{u}; D\| + |\mathbf{\Sigma}| \leq C$.

Because of the Leray-Schauder theory, this lemma implies **Lemma 7**. Then, **Lemma 8** is proved assuming $\varepsilon < \varepsilon_1$ and applying this proposition to the constructed approximate solution (in this case $\lambda = 1$).

Proof of Proposition 9. First we note that given $(\mathbf{u}, \mathbf{\Sigma}, n, \lambda)$ as in **Proposition 9** we can set $\mathbf{w} = \mathbf{u}$ in (42), and we obtain (34). Hence, it suffices to find a bound on $\mathbf{\Sigma}$. For this purpose, we introduce an additional family of cut-off functions χ_δ. This family truncates in balls around the point $(0, 1+h)$. Namely, let $\zeta\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\zeta(s) = 1\ , \quad \forall\ s < 0\ , \quad \zeta(s) = 0\ , \quad \forall\ s > 1\ . \tag{43}$$

Then, given $0 < \delta < h/3$, we set for $(x, y) \in \Omega_+$,

$$\chi_\delta(x, y) = \zeta \left(\frac{|(x, y - 1 - h)|}{\delta} - 1 \right) . \tag{44}$$

With this definition, we have $\chi_\delta = 1$ in $B(\delta)$ while $\chi_\delta = 0$ in the exterior of $B(2\delta)$. Now, given $(\mathbf{u}, \mathbf{\Sigma}, n, \lambda)$ and an obstacle S_ε, we set $\delta(\varepsilon) = \lambda_0\ \varepsilon$, with $\lambda_0 = \sup\{|(x, y)|,\ (x, y) \in S\}$, and define, for arbitrary $\mathbf{W} \in \mathbb{R}^2$, the test-function:

$$\mathbf{w}_\varepsilon = -\nabla^\perp \left(\chi_{\delta(\varepsilon)}(x, y)\ \left[\mathbf{W}^\perp \cdot ((x, y) - (0, 1 + h)) \right] \right) . \tag{45}$$

Since S_ε tends homothetically to a point when $\varepsilon \to 0$, we can choose ε_0 (say $\varepsilon_0 = h/(3\lambda_0)$ for instance) such that $\mathbf{w}_\varepsilon$ is equal to $\mathbf{W}$ on S_ε and equal to zero outside $B(2h/3)$ for $\varepsilon < \varepsilon_0$. Thus, we can use $\mathbf{w}_\varepsilon$ as a test-function in (42). By construction of $\mathbf{w}_\varepsilon$, there exists a universal constant C_1 such that

$$\|\mathbf{w}_\varepsilon; D\| + \|\mathbf{w}_\varepsilon; L^\infty(\mathbb{R}^2)\| \leq C_1 |\mathbf{W}| , \tag{46}$$

and we get from (42) the inequality

$$\begin{aligned} |\mathbf{\Sigma} \cdot \mathbf{W}| &\leq \|\mathbf{u}; D\|\ \|\mathbf{w}_\varepsilon; D\| + \lambda \|\mathbf{u} + \mathbf{e}_1; L^2(B(2\delta(\varepsilon)))\|\ \|\mathbf{u}; D\|\ \|\mathbf{w}_\varepsilon; L^\infty(\Omega_+)\| \\ &\leq C_1 |\mathbf{W}| \left(\|\mathbf{u}; D\| + \|\mathbf{u} + \mathbf{e}_1; L^2(B(2\delta(\varepsilon)))\|\ \|\mathbf{u}; D\| \right) . \end{aligned}$$

Since $\mathbf{u}_{|S_\varepsilon} = -\mathbf{e}_1$, Poincaré's inequality implies that there exists a constant $\widetilde{C_2}$ such that

$$\|\mathbf{u} + \mathbf{e}_1; L^2(B(2\delta(\varepsilon)))\| \leq \widetilde{C_2}\ \|\mathbf{u}; D\| .$$

A scaling argument shows $\widetilde{C_2} = \varepsilon C_2$ with a constant C_2 independent of ε and $\mathbf{u}$ (see [8, Exercise 4.10] for a construction of C_2). Therefore,

$$|\mathbf{\Sigma}| \leq C_1 \left(\|\mathbf{u}; D\| + C_2 \varepsilon \|\mathbf{u}; D\|^2 \right) . \tag{47}$$

From (47) and (34) we find that if ε satisfies moreover $\varepsilon < 1/(2C_1C_2)$, we have a bound on $|\mathbf{\Sigma}|$ and $\|\mathbf{u}; D\|$ which is independent of n, λ, and ε, namely

$$\|\mathbf{u}; D\| \leq |\mathbf{\Sigma}|^{1/2} \leq 2C_1 . \tag{48}$$

■

2.3 Weak solutions for obstacles of vanishing size

From now on, we choose once and for all $(\mathbf{u}_\varepsilon)_{\varepsilon<\varepsilon_1}$ a bounded family of D such that $\mathbf{u}_\varepsilon$ is a weak solution of Problem 1 for S_ε for all $\varepsilon < \varepsilon_1$. Such a sequence exists according to **Theorem 5**. In this section we complete the proof of **Theorem 2** by showing that the sequence $(\mathbf{u}_\varepsilon)_{\varepsilon<\varepsilon_1}$ converges to zero when the size of the obstacle tends to zero. As a by-product, we also obtain the pressure p associated with a weak solution.

The convergence is proved in the family of spaces $(\mathcal{C}^m(\mathcal{A}_n))_{(m,n)\in\mathbb{N}^2}$. We refer the reader to Section 1.1 for the definition of sets $(\mathcal{A}_n)_{n\in\mathbb{N}}$. We recall here that they satisfy the following fundamental properties

- for all $n \in \mathbb{N}$, $\mathcal{A}_n \subset \overline{\mathcal{A}_n} \subset \mathcal{A}_{n+1}$,
- for all $n \in \mathbb{N}$, $\mathcal{A}_n$ has a smooth boundary,
- $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n = \Omega_+ \setminus (0, 1 + h)$.

Theorem 2 is a straightforward consequence of the following two lemmas which we prove in the following subsections.

Lemma 10 *Given $\eta > 0$ there exists $0 < \varepsilon_\eta$ such that $\|\mathbf{u}_\varepsilon; D\| \leq \eta$ for all $\varepsilon < \varepsilon_\eta$.*

Lemma 11 *Let* $(n,m) \in \mathbb{N}^2$ *and* $\varepsilon < \varepsilon_1$ *such that* $S_\varepsilon \subset\subset [\ \mathbb{R}^2_+ \setminus \overline{\mathcal{A}_{n+m+1}}\ ]$*. Then, there exists a constant* $C_{m,n}$*, depending only on* m *and* n*, for which any weak solution* $\mathbf{u}$ *of Problem 1 for* S_ε *such that* $\|\mathbf{u}; D\| \leq 1$ *satisfies*

(i) there exists a pressure p *such that* $(\mathbf{u}, p)$ *is solution to* (10)

(ii) the following estimate holds true

$$\|\mathbf{u}; H^{m+1}(\mathcal{A}_n)\| + \|p; H^m(\mathcal{A}_n)/\mathbb{R}\| \leq C_{m,n}\ \|\mathbf{u}; D\|\ . \tag{49}$$

Remark:
One might be tempted to assume that the smallness estimate of **Theorem 2** is straightforward, since the fluid is moving only due to the no-slip boundary condition on ∂S_ε. The smaller the body, the smaller should be the fluid flow which is induced by this boundary condition so that the flow should be zero in the limit of a body of vanishing size. The following scaling argument shows that, because of the Stokes paradox, things are not quite as simple. Let $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ a family of weak solutions for S_ε and $\Omega^\varepsilon_+ = \{(x,y) \in \mathbb{R}^2 \mid (\varepsilon x, \varepsilon y - (h+1)) \in \Omega_+\}$, and let $\mathbf{v}_\varepsilon(x,y) = \mathbf{u}_\varepsilon(\varepsilon x, \varepsilon y - (h+1))$ and $q_\varepsilon(x,y) = p_\varepsilon(\varepsilon x, \varepsilon y - (h+1))$. This scaling does not affect the $D-$norm, so that $\|\nabla \mathbf{v}_\varepsilon; L^2(\Omega^\varepsilon_+)\| = \|\nabla \mathbf{u}_\varepsilon; L^2(\Omega_+)\|$. Therefore, if the family $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is bounded in D, the family $(\|\nabla \mathbf{v}_\varepsilon; L^2(\Omega^\varepsilon_+)\|)_{\varepsilon>0}$ is also bounded, and the functions $\mathbf{v}_\varepsilon$ satisfy in Ω^ε_+ the equation

$$\begin{aligned} -\varepsilon\,(\mathbf{v}_\varepsilon \cdot \nabla)\,\mathbf{v}_\varepsilon - \varepsilon \partial_x \mathbf{v}_\varepsilon + \Delta \mathbf{v}_\varepsilon - \nabla q_\varepsilon = 0\ ,\\ \nabla \cdot \mathbf{v}_\varepsilon = 0\ , \end{aligned} \tag{50}$$

with the boundary condition $\mathbf{v}_\varepsilon = -\mathbf{e}_1$ on ∂S_1. Using the same line of arguments as in the previous section we can therefore extract a subsequence converging in the topology induced by the $D-$norm to some function $\mathbf{v}$ for which $\|\nabla \mathbf{v}; L^2(\mathbb{R}^2)\|$ is finite and which solves the Stokes equations in $\mathbb{R}^2 \setminus \overline{S_1}$ with the boundary condition $\mathbf{v} = -\mathbf{e}_1$ on ∂S_1. By the Stokes paradox this implies that $\mathbf{v} = -\,\mathbf{e}_1$ (see [9, Theorem 2.2 p. 253]), and therefore the sequence $\mathbf{v}_\varepsilon$ does not converge to zero with ε. Note that this remark does not contradict **Theorem 2**. It only means that, when ε goes to zero, $\mathbf{v}_\varepsilon$ does take values close to $-\mathbf{e}_1$ on a part of the domain that increases in size and covers eventually all of $\mathbb{R}^2$. The diameter of this region remains however small compared with $1/\varepsilon$, and its size therefore converges to zero in the un-scaled variables.

2.3.1 Proof of Lemma 10

We prove **Lemma 10** by contradiction. First, we assume that there exists $\eta_0 > 0$, sequences $(\varepsilon_i)_{i\in\mathbb{N}} \in (0,\varepsilon_1)^\mathbb{N}$ and $(\mathbf{u}_i, \mathbf{\Sigma}_i)_{i\in\mathbb{N}} \in (D \times \mathbb{R}^2)^\mathbb{N}$ such that $\lim \varepsilon_i = 0$, that $\mathbf{u}_i = \mathbf{u}_{\varepsilon_i}$ has associated force $\mathbf{\Sigma}_i$ and is such that $\|\mathbf{u}_i; D\| \geq \eta_0$ for all $i \in \mathbb{N}$. Then, the sequence $(\mathbf{u}_i, \mathbf{\Sigma}_i)_{i\in\mathbb{N}}$ is bounded. This implies the existence of a pair $(\mathbf{u}, \mathbf{\Sigma}) \in D \times \mathbb{R}^2$ and of a subsequence $(\mathbf{u}_{i_j}, \mathbf{\Sigma}_{i_j})_{j\in\mathbb{N}}$ such that $\mathbf{u}_{i_j} \rightharpoonup_{j\to\infty} \mathbf{u}$ weakly in D and such that $\mathbf{\Sigma}_{i_j} \to_{j\to\infty} \mathbf{\Sigma}$ strongly in $\mathbb{R}^2$.

We now proceed as in the proof of **Proposition 9**. Let χ_δ be the cut-off function defined in (44) and define, as in (45) for $0 < \delta < h/3$ and arbitrary $\mathbf{W} \in \mathbb{R}^2$ the test-function $\mathbf{w}_\delta$

$$\mathbf{w}_\delta(x,y) = -\nabla^\perp \left(\chi_\delta(x,y)\ \left[\mathbf{W}^\perp \cdot ((x,y) - (0, 1+h))\right]\right)\ ,$$

which has support in $B(2h/3)$. As in (46) there exists a universal constant $C_1 < \infty$ such that

$$\|\mathbf{w}_\delta; D\| + \|\mathbf{w}_\delta; L^\infty(\Omega_+)\| \leq C_1 |\mathbf{W}|\ . \tag{51}$$

Since $\lim \varepsilon_i = 0$, there exists i_δ such that for $i \geq i_\delta$ the function $\mathbf{w}_\delta$ is an admissible test-function, and we have :

$$\int_{\Omega_+} \nabla \mathbf{u}_i : \nabla \mathbf{w}_\delta\ d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u}_i + \mathbf{e}_1) \cdot \nabla \mathbf{u}_i] \cdot \mathbf{w}_\delta\ d\mathbf{x} = -\mathbf{\Sigma}_i \cdot \mathbf{W}\ .$$

In the limit as i goes to infinity we therefore get

$$\int_{\Omega_+} \nabla \mathbf{u} : \nabla \mathbf{w}_\delta \, d\mathbf{x} + \int_{\Omega_+} [(\mathbf{u} + \mathbf{e}_1) \cdot \nabla \mathbf{u}] \cdot \mathbf{w}_\delta \, d\mathbf{x} = -\boldsymbol{\Sigma} \cdot \mathbf{W} \ ,$$

so that,

$$|\boldsymbol{\Sigma}| \leq C_1 \|\nabla \mathbf{u}; L^2(B(2\delta/3))\| \left[1 + \|\mathbf{u} + \mathbf{e}_1; L^2(B(2\delta/3))\|\right] \ .$$

Letting δ go to 0, yields $\boldsymbol{\Sigma} = 0$, *i.e.*, $\boldsymbol{\Sigma}_i \to_{i\to\infty} 0$ which by the energy estimate (28) implies that $\lim \|\mathbf{u}_i; D\| = 0$, in contradiction with our assumption.

2.3.2 Proof of Lemma 11

Let $(n, m) \in \mathbb{N}^2$ and $\varepsilon, \mathbf{u}$ be given as in **Lemma 11**. At first, we recall how to construct the pressure associated with $\mathbf{u}$. We test (27) with smooth divergence free vector-fields having compact support in $\Omega_+ \setminus \overline{S_\varepsilon}$. This shows that $\mathbf{u}$ is a generalized solution in the sense of [8, Definition IV.1.1, p. 185] of the Stokes equation in $\Omega_+ \setminus \overline{S_\varepsilon}$ with source term

$$\mathbf{f} = (\mathbf{u} + \mathbf{e}_1) \cdot \nabla \mathbf{u} \ .$$

Since, for all $\Omega' \subset\subset (\Omega_+ \setminus \overline{S_\varepsilon})$, we have $\mathbf{f} \in H^{-1}(\Omega')$ with the bound

$$\|\mathbf{f}; H^{-1}(\Omega')\| \leq C(\Omega') \left[\|\mathbf{u}; H^1(\Omega')\|^2 + \|\mathbf{u}; D\|\right] \tag{52}$$

we can apply [8, lemma IV.1.1,p. 186] to construct a function $p \in L^2_{loc}(\Omega_+ \setminus \overline{S_\varepsilon})$ such that, in the sense of distributions,

$$\left\{ \begin{array}{rcl} \Delta \mathbf{u} - \nabla p &=& \mathbf{f} \ , \\ \nabla \cdot \mathbf{u} &=& 0 \ , \end{array} \right. \tag{53}$$

in $\Omega_+ \setminus \overline{S_\varepsilon}$. Classically, this pressure p is unique up to a finite number of constants (equal to the number of connected components of $\Omega_+ \setminus \overline{S_\varepsilon}$). We call p the *pressure associated with* $\mathbf{u}$ and we indeed have that $(\mathbf{u}, p)$ satisfies (10).

The remainder of **Lemma 11** is obtained via an induction argument (with respect to $m \in \mathbb{N}$). Namely, we prove that, for all $k \leq m$ the following statement holds true:

There exist constants $C_{m,k}$ depending only on m and k such that :

$$\|\mathbf{u}; H^{k+1}(\mathcal{A}_{n+m-k})\| + \|p; H^k(\mathcal{A}_{n+m-k})/\mathbb{R}\| \leq C_{m,k} \ \|\mathbf{u}; D\| \ . \tag{$\mathcal{P}_k$}$$

Proof, initialization: The restriction of $(\mathbf{u}, p)$ to $\mathcal{A}_{n+m}$ is a solution of the Stokes equations with source term $\mathbf{f} = (\mathbf{u} + \mathbf{e}_1) \cdot \nabla \mathbf{u}$ and boundary data $\mathbf{u} = \mathbf{u}_{|\partial \mathcal{A}_{n+m}}$. Hence, combining [8, theorem 1.1 p. 188] and (52), and using that $\|\mathbf{u}; D\| \leq 1$, we find that

$$\|\mathbf{u}; H^1(\mathcal{A}_{n+m})\| + \|p; L^2(\mathcal{A}_{n+m})/\mathbb{R}\| \leq \widetilde{C}_0 \left[\|\mathbf{u}; D\| + \|\mathbf{u}; D\|^2\right] \leq C_{m,0} \|\mathbf{u}; D\| \ . \tag{54}$$

Our statement holds true for $k = 0$.

Before the inductive step of the proof, we need to compute an L^∞ estimate on $\mathbf{u}$ inside $\mathcal{A}_{n+m}$. To this end, we recall that we have by construction that $\mathcal{A}_{n+m} \subset (\overline{\mathcal{A}_{n+m}} \cap \Omega_+) \subset \mathcal{A}_{n+m+1}$. Hence there exists a smooth truncation function $\chi \in \mathcal{C}^\infty(\overline{\Omega}_+)$, such that $\chi = 1$ on $\mathcal{A}_{n+m}$ and $\chi = 0$ outside $\mathcal{A}_{n+m+1}$. We make the dependance of χ upon n implicit for legibility. Let $\tilde{\mathbf{u}} = \chi \mathbf{u}$ and $\tilde{p} = \chi p$. Then $(\tilde{\mathbf{u}}, \tilde{p})$ is a solution of the Stokes system

$$\left\{ \begin{array}{rl} \Delta \tilde{\mathbf{u}} - \nabla \tilde{p} = \widetilde{\mathbf{f}} & \text{on } \mathcal{A}_{n+m+1} \ , \\ \nabla \cdot \tilde{\mathbf{u}} = \tilde{g} & \text{on } \mathcal{A}_{n+m+1} \ , \end{array} \right. \quad \text{with } \tilde{\mathbf{u}} = 0 \text{ on } \partial \mathcal{A}_{n+m+1} \ , \tag{55}$$

where

$$\begin{cases} \widetilde{\mathbf{f}} &= \chi(\mathbf{u}+\mathbf{e}_1)\cdot\nabla\mathbf{u} + 2\nabla\chi\cdot\nabla\mathbf{u} + (\Delta\chi)\,\mathbf{u} - p\nabla\chi \;, \\ \tilde{g} &= \mathbf{u}\cdot\nabla\chi \;. \end{cases} \tag{56}$$

Using the bound (54) on $(\mathbf{u}, p)$, we find that, for a given $q < 2$ (say $q = 3/2$), there holds $\widetilde{\mathbf{f}} \in L^q(\mathcal{A}_{n+m+1})$ and $\tilde{g} \in W^{1,q}(\mathcal{A}_{n+m+1})$. Furthermore we have,

$$\|\widetilde{\mathbf{f}}; L^q(\mathcal{A}_{n+m+1})\| + \|\tilde{g}; W^{1,q}(\mathcal{A}_{n+m+1})\| \\ \leq C_{m,q}\left[\|\mathbf{u}; H^1(\mathcal{A}_{n+m+1})\| + \|\mathbf{u}; H^1(\mathcal{A}_{n+m+1})\|^2 + \|p; L^2(\mathcal{A}_{n+m+1})\|\right] \;.$$

Applying [8, Exercise IV.6.2, p. 232] we get that $\tilde{\mathbf{u}} \in W^{2,q}(\mathcal{A}_{n+m+1})$ and $\tilde{p} \in W^{1,q}(\mathcal{A}_{n+m+1})$, and that

$$\|\tilde{\mathbf{u}}; W^{2,q}(\mathcal{A}_{n+m+1})\| + \|\tilde{p}; W^{1,q}(\mathcal{A}_{n+m+1})/\mathbb{R}\| \\ \leq C_{m,q}\left[\|\mathbf{u}; H^1(\mathcal{A}_{n+m+1})\| + \|\mathbf{u}; H^1(\mathcal{A}_{n+m+1})\|^2 + \|p; L^2(\mathcal{A}_{n+m+1})\|\right] \;. \tag{57}$$

Note that we can always replace p by $p+c$ before truncation, so that we can replace $\|p; L^2(\mathcal{A}_{n+m+1})\|$ by $\|p; L^2(\mathcal{A}_{n+m+1})/\mathbb{R}\|$ in the right-hand side of the last inequality, as well as in the estimates that follow. Combining (57) with (54), we get

$$\|\tilde{\mathbf{u}}; W^{2,q}(\mathcal{A}_{n+m+1})\| + \|\tilde{p}; W^{1,q}(\mathcal{A}_{n+m+1})/\mathbb{R}\| \leq C_{m,q}\|\mathbf{u}; D\| \;. \tag{58}$$

Therefore, we have in particular that $\mathbf{u} \in W^{2,q}(\mathcal{A}_{n+m}) \subset L^\infty(\mathcal{A}_{n+m})$ with

$$\|\mathbf{u}; L^\infty(\mathcal{A}_{n+m}))\| \leq K_{m,n}\|\mathbf{u}; D\| \;.$$

Proof, inductive step:
Assuming that for $k \leq m$, there exist constants $C_{m,k}$ depending only on m and k such that :

$$\|\mathbf{u}; H^{k+1}(\mathcal{A}_{n+m-k})\| + \|p; H^k(\mathcal{A}_{n+m-k})/\mathbb{R}\| \leq C_{m,k}\;\|\mathbf{u}; D\| \;,$$

we apply again the same truncation technique as described above. Namely, we introduce $\chi \in \mathcal{C}^\infty(\overline{\Omega}_+)$ a smooth truncation function such that $\chi = 1$ on $\mathcal{A}_{n+m-k-1}$ and $\chi = 0$ outside $\mathcal{A}_{n+m-k}$ and we let $\tilde{\mathbf{u}} = \chi\mathbf{u}$ and $\tilde{p} = \chi p$. Then $(\tilde{\mathbf{u}}, \tilde{p})$ is a solution of the Stokes system (55), on $\mathcal{A}_{n+m-k}$ with homogeneous boundary condition. Hence, we get by the ellipticity of the Stokes operator that

$$\|\tilde{\mathbf{u}}; H^{k+2}(\mathcal{A}_{n+m-k})\| + \|\tilde{p}; H^{k+1}(\mathcal{A}_{n+m-k})/\mathbb{R}\| \leq \tilde{C}_{m,k}\left[\|\widetilde{\mathbf{f}}; H^k(\mathcal{A}_{n+m-k}))\| + \|\tilde{g}; H^{k+1}(\mathcal{A}_{n+m-k})\|\right] \;. \tag{59}$$

We also have

$$\|\widetilde{\mathbf{f}}; H^k(\mathcal{A}_{n+m-k}))\| + \|\tilde{g}; H^{k+1}(\mathcal{A}_{n+m-k})\| \\ \leq \tilde{C}_{m,k}\Big[\|\mathbf{u}; L^\infty(\mathcal{A}_{n+m})\|\;\|\mathbf{u}; H^{k+1}(\mathcal{A}_{n+m-k})\| + \|\mathbf{u}; H^{k+1}(\mathcal{A}_{n+m-k})\|^2 \\ + \|\mathbf{u}; H^{k+1}(\mathcal{A}_{n+m-k})\| + \|p; H^k(\mathcal{A}_{n+m-k})/\mathbb{R}\|\Big] \;,$$

and therefore there exists, by the induction assumption, a constant $C_{m,k+1}$, such that

$$\|\mathbf{u}; H^{k+2}(\mathcal{A}_{n+m-k-1})\| + \|p; H^{k+1}(\mathcal{A}_{n+m-k-1})/\mathbb{R}\| \leq C_{m,k+1}\;\|\mathbf{u}; D\| \;.$$

This completes the inductive step and ends the proof.

3 Behavior of weak solutions at large distance from the obstacle.

In this section we show that the weak solutions of Problem 1 constructed above decay at infinity with the expected rate. Namely, we prove:

Theorem 12 *There exists $\varepsilon_e > 0$, such that, for all $\varepsilon < \varepsilon_e$, the weak solution $\mathbf{u}_\varepsilon$ satisfies the decay estimate,*

$$|\mathbf{u}_\varepsilon(x,y)| \leq \frac{C_\varepsilon}{y^{\frac{3}{2}}} \ . \quad \forall\, (x,y) \in \Omega_+ \setminus \overline{S_\varepsilon} \ . \tag{60}$$

for some $C_\varepsilon < \infty$.

This result is proved in three steps by comparing weak solutions with α-solutions. First, we show how to construct solutions for Problem 2 by truncating a weak solution for Problem 1. We prove in particular that, when the solid is sufficiently small, weak solutions to Problem 1 provided by **Theorem 2** yield weak solutions to Problem 2 with a source term which is arbitrary small, so that we are able to construct α-solutions. We conclude by proving that any weak solution coincides with the α-solution when the source-term is sufficiently small.

3.1 Truncation procedure

We start this section by describing how to construct a solution for Problem 2 by truncating a weak solution for Problem 1. Let

$$\begin{array}{rrcl} \Pi: & D \cap \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4)) & \longrightarrow & \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4)) \\ & \mathbf{w} & \longmapsto & \psi(x,y) = -\displaystyle\int_1^y \mathbf{w}(x,z) \cdot \mathbf{e}_1 \, dz \ . \end{array}$$

The divergence-free condition satisfied by $\mathbf{w}$ implies that $\nabla^\perp \Pi[\mathbf{w}] = \mathbf{w}$, and that

$$\Pi[\mathbf{w}](x,y) = \int_\gamma \mathbf{w}^\perp \cdot d\gamma \ ,$$

for any path γ such that $\gamma(0) = (0,1)$ and $\gamma(1) = (x,y)$. Hence, it is sufficient that $\mathbf{w}$ is smooth in $\overline{\Omega_+} \setminus B(h/3)$ in order for the associated stream-function $\Pi[\mathbf{w}]$ to be smooth in $\overline{\Omega_+} \setminus B(h/4)$. More precisely, for all $m \in \mathbb{N}$, there exists a constant C_m, such that,

$$\|\Pi[\mathbf{w}]; \mathcal{C}^m(\overline{B(2h/3)} \setminus B(h/3))\| \leq C_m \|\mathbf{w}; \mathcal{C}^{m-1}(\overline{\mathcal{A}_2})\| \quad \forall\, \mathbf{w} \in D \cap \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4)) \ . \tag{61}$$

We introduce a truncation function $\chi \in \mathcal{C}^\infty(\mathbb{R}^2)$ which satisfies

$$\chi(x,y) = \begin{cases} 0 & \text{if } |(x,y) - (0,1+h)| < h/3 \\ \in [0,1], & \text{if } |(x,y) - (0,1+h)| \in (h/3, 2h/3) \\ 1 & \text{if } |(x,y) - (0,1+h)| > 2h/3 \end{cases}$$

and define truncation operators $\mathbf{T}_v$ and T_π for the velocity and the pressure as follows

$$\begin{array}{rrcl} \mathbf{T}_v: & D \cap \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4)) & \longrightarrow & \mathcal{C}^\infty(\overline{\Omega_+}) \\ & \mathbf{w} & \longmapsto & \nabla^\perp\left[\chi \Pi[\mathbf{w}]\right] \end{array}$$

and

$$\begin{array}{rrcl} T_\pi: & \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4)) & \longrightarrow & \mathcal{C}^\infty(\overline{\Omega_+}) \\ & q & \longmapsto & \chi q \end{array}$$

These operators are well-defined, since the truncation function χ vanishes identically in $B(h/4)$. For any $\mathbf{w} \in D \cap \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4))$ and $q \in \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4))$, we have by a straightforward application of (61) that

(T-i) $\mathbf{T}_v[\mathbf{w}] \in D \cap \mathcal{C}^\infty(\overline{\Omega_+})$, and $T_\pi[q] \in \mathcal{C}^\infty(\overline{\Omega_+})$,

(T-ii) $\mathbf{T}_v[\mathbf{w}] = \mathbf{w}$ and $T_\pi[q] = q$ in $\overline{\Omega_+} \setminus B(2h/3)$,

(T-iii) Given $m \in \mathbb{N}$, there exists a constant C_m such that

$$\begin{aligned} \|\mathbf{T}_v[\mathbf{w}]; \mathcal{C}^{m+1}(\overline{B(2h/3)} \setminus B(h/3))\| &\leq C_m \|\mathbf{w}; \mathcal{C}^{m+1}(\overline{\mathcal{A}_2})\| \ , \\ \|T_\pi[q]; \mathcal{C}^{m}(\overline{B(2h/3)} \setminus B(h/3))\| &\leq C_m \|q; \mathcal{C}^{m}(\overline{\mathcal{A}_2})\| \ . \end{aligned}$$

Next for $(\mathbf{w}, q) \in (D \cap \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4))) \times \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4))$ we define the function $\mathbf{f} \in \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4))$ by

$$\mathbf{f} = (\mathbf{w} + \mathbf{e}_1) \cdot \nabla \mathbf{w} - \Delta \mathbf{w} + \nabla q \ ,$$

and we define $\mathbf{f} = 0$ inside $B(h/4)$. Finally we define the function $TNS[\mathbf{w}, q]$ on Ω_+ by

$$TNS[\mathbf{w}, q] = -\chi \mathbf{f} + [(\tilde{\mathbf{w}} + \mathbf{e}_1) \cdot \nabla \tilde{\mathbf{w}} - \Delta \tilde{\mathbf{w}} + \nabla \tilde{q}] \ ,$$

where $(\tilde{\mathbf{w}}, \tilde{q}) = (\mathbf{T}_v[\mathbf{w}], T_\pi[q])$. Given $(\mathbf{w}, q) \in (D \cap \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4))) \times \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4))$, the above properties of the truncation operators $\mathbf{T}_v$ and T_π imply that the function $TNS[\mathbf{w}, q]$ satisfies:

(S-i) $TNS[\mathbf{w}, q]$ is smooth and has compact support in $\overline{B(2h/3)} \setminus B(h/3)$,

(S-ii) The truncated functions $\tilde{\mathbf{w}} = \mathbf{T}_v[\mathbf{w}]$ and $\tilde{q} = T_\pi[q]$ satisfy:

$$\left\{ \begin{aligned} (\tilde{\mathbf{w}} + \mathbf{e}_1) \cdot \nabla \tilde{\mathbf{w}} - \Delta \tilde{\mathbf{w}} + \nabla \tilde{q} &= \chi \mathbf{f} + TNS[\mathbf{w}, q] \ , & \text{in } \Omega_+ \ , \\ \nabla \cdot \tilde{\mathbf{w}} &= 0 \ , & \text{in } \Omega_+ \ . \end{aligned} \right.$$

with $\mathbf{f} = (\mathbf{w} + \mathbf{e}_1) \cdot \nabla \mathbf{w} - \Delta \mathbf{w} + \nabla q$,

(S-iii) Given $m \in \mathbb{N}$, there exists a constant C_m such that

$$\begin{aligned} &\|TNS[\mathbf{w}, q]; \mathcal{C}^m(\overline{B(2h/3)} \setminus B(h/3))\| \\ &\qquad \leq C_m \left[\left(1 + \|\mathbf{w}; \mathcal{C}^{m+2}(\overline{\mathcal{A}_2})\|\right) \|\mathbf{w}; \mathcal{C}^{m+2}(\overline{\mathcal{A}_2})\| + \|q; \mathcal{C}^{m+1}(\overline{\mathcal{A}_2})/\mathbb{R}\| \right] \ . \end{aligned} \tag{62}$$

Applying this construction to any weak solution of Problem 1 yields a solution of Problem 2 for the source term computed with TNS. To prepare the last weak-strong uniqueness argument of this section, we show that such solutions of Problem 2 obtained by truncation satisfy a further energy property. This is the content of the next proposition.

Proposition 13 *Given ε such that $S_\varepsilon \subset\subset B(h/4)$ and a weak solution $\mathbf{u}$ of Problem 1 for S_ε with associated pressure p, the vector-field $\tilde{\mathbf{u}} = \mathbf{T}_v[\mathbf{u}]$ satisfies*

(i) $\tilde{\mathbf{u}} \in D$,

(ii) for all $\mathbf{w} \in \mathcal{D}$, there holds:

$$\int_{\Omega_+} \nabla \tilde{\mathbf{u}} : \nabla \mathbf{w} \, d\mathbf{x} + \int_{\Omega_+} [(\tilde{\mathbf{u}} + \mathbf{e}_1) \cdot \nabla \tilde{\mathbf{u}}] \cdot \mathbf{w} \, d\mathbf{x} = \int_{\Omega_+} \widetilde{\mathbf{f}} \cdot \mathbf{w} \, d\mathbf{x} \ , \tag{63}$$

and

$$\int_{\Omega_+} |\nabla \tilde{\mathbf{u}}|^2 \, d\mathbf{x} \leq \int_{\Omega_+} \widetilde{\mathbf{f}} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} \ , \tag{64}$$

with $\widetilde{\mathbf{f}} = TNS[\mathbf{u}, p]$.

As for the case of Problem 1, we emphasize that $\widetilde{\mathbf{f}}$ and the test-functions $\mathbf{w}$ have compact support so that the integrals in (63) and (64) are well-defined. A velocity-field $\tilde{\mathbf{u}}$ satisfying (i) and (ii) for a given $\widetilde{\mathbf{f}} \in \mathcal{C}^\infty_c(\Omega_+)$ is called a **weak solution** for Problem 2 with source term $\widetilde{\mathbf{f}}$.

Proof. First we recall that ellipticity estimates for the Stokes system imply that any weak solution $\mathbf{u}$ for S_ε with associated pressure p satisfies

$$(\mathbf{u}, p) \in \left(D \cap \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4))\right) \times \mathcal{C}^\infty(\overline{\Omega_+} \setminus B(h/4)) \ .$$

Hence $\tilde{\mathbf{u}} = \mathbf{T}_v[\mathbf{u}]$, $\tilde{p} = T_\pi[p]$ and $TNS[\mathbf{u}, p]$ are well-defined. Moreover $(\mathbf{u}, p)$ is a classical solution of the Navier Stokes equations outside S_ε and in particular in $\overline{\Omega_+} \setminus B(h/4)$.

In order to show that $\tilde{\mathbf{u}}$ is a weak solution of Problem 2 we first use (T-i) to conclude that $\tilde{\mathbf{u}} \in D$. Then, since $(\mathbf{u}, p)$ is a classical solution to the Navier Stokes equations in $\Omega_+ \setminus \overline{B(h/4)}$, the second point (S-$ii$) implies that we have

$$(\tilde{\mathbf{u}} + \mathbf{e}_1) \cdot \nabla \tilde{\mathbf{u}} - \Delta \tilde{\mathbf{u}} + \nabla \tilde{p} = TNS[\mathbf{u}, p]$$

in Ω_+. If we multiply this equality by $\mathbf{w} \in \mathcal{D}$ and integrate by parts we obtain (63) for $\tilde{\mathbf{u}}$, with $\widetilde{\mathbf{f}} = TNS[\mathbf{u}, p]$.

The main difficulty of the proof is to obtain the energy estimate (64) for $\tilde{\mathbf{u}}$. For this purpose, we multiply the Navier Stokes equations satisfied by $(\tilde{\mathbf{u}}, \tilde{p})$ on $B(5h/6)$ by $\tilde{\mathbf{u}}$. Integrating by parts yields

$$\int_{B(5h/6)} |\nabla \tilde{\mathbf{u}}|^2 \, d\mathbf{x} = \int_{B(5h/6)} \widetilde{\mathbf{f}} \cdot \tilde{\mathbf{u}} \, d\mathbf{x} + \int_{\partial B(5h/6)} \left[T(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n} \cdot \tilde{\mathbf{u}} + \frac{|\tilde{\mathbf{u}}|^2}{2} (\tilde{\mathbf{u}} + \mathbf{e}_1) \cdot \mathbf{n} \right] d\sigma \ . \tag{65}$$

Next, multiplying the Navier Stokes equation satisfied by $(\mathbf{u}, p)$ on $B(5h/6) \setminus \overline{S_\varepsilon}$ by $\mathbf{u}$ and integrating by parts gives

$$\int_{B(5h/6)} |\nabla \mathbf{u}|^2 \, d\mathbf{x} = \mathbf{\Sigma} \cdot \mathbf{e}_1 + \int_{\partial B(5h/6)} \left[T(\mathbf{u}, p)\mathbf{n} \cdot \mathbf{u} + \frac{|\mathbf{u}|^2}{2} (\mathbf{u} + \mathbf{e}_1) \cdot \mathbf{n} \right] d\sigma \ , \tag{66}$$

with $\mathbf{\Sigma}$ the associated force applied on S_ε. By definition, we have

$$\int_{\Omega_+} |\nabla \mathbf{u}|^2 \, d\mathbf{x} \leq \mathbf{\Sigma} \cdot \mathbf{e}_1 \ . \tag{67}$$

Subtracting (66) from (67) yields

$$\int_{\Omega_+ \setminus \overline{B(5h/6)}} |\nabla \mathbf{u}|^2 \, d\mathbf{x} \leq - \int_{\partial B(5h/6)} \left[T(\mathbf{u}, p)\mathbf{n} \cdot \mathbf{u} + \frac{|\mathbf{u}|^2}{2} (\mathbf{u} + \mathbf{e}_1) \cdot \mathbf{n} \right] d\sigma \ . \tag{68}$$

Since outside $B(2h/3)$ we have by construction that $\mathbf{u} = \tilde{\mathbf{u}}$ and $p = \tilde{p}$, we get by combining (65) and (68)

$$\int_{\Omega_+} |\nabla \tilde{\mathbf{u}}|^2 \, d\mathbf{x} \leq \int_{\Omega_+} TNS[\mathbf{u}, p] \cdot \tilde{\mathbf{u}} \, d\mathbf{x} \ .$$

This completes the proof. ■

3.2 Existence of α-solutions

The second step of the proof of **Theorem 12** is to construct an α-solution for Problem 2 with the source term $\widetilde{\mathbf{f}}_\varepsilon$ obtained by truncation of a weak solutions $\mathbf{u}_\varepsilon$. To keep this paper self-contained, we recall the definition and the main properties of α-solutions. See [13], for details.

Definition 14 *We define for fixed α, $r \geq 0$ the function $\mu_{\alpha,r} \colon \mathbb{R} \times [1, \infty) \to (0, \infty)$ by*

$$\mu_{\alpha,r}(k, t) = \frac{1}{1 + (|k| \, t^r)^\alpha} \ . \tag{69}$$

We define, for fixed $\alpha \geq 0$, and p, $q \geq 0$, $\mathcal{B}_{\alpha,p,q}$ to be the Banach space of functions $\hat{f} \in \mathcal{C}(\mathbb{R}_0 \times [1,\infty), \mathbb{C})$, $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, for which the norm

$$\left\| \hat{f};\ \mathcal{B}_{\alpha,p,q} \right\| = \sup_{t\geq 1} \sup_{k\in\mathbb{R}_0} \frac{\left|\hat{f}(k,t)\right|}{\frac{1}{t^p}\mu_{\alpha,1}(k,t) + \frac{1}{t^q}\mu_{\alpha,2}(k,t)}$$

is finite. Furthermore, we set $\mathcal{U}_\alpha = \mathcal{B}_{\alpha,\frac{5}{2},1} \times \mathcal{B}_{\alpha,\frac{1}{2},0} \times \mathcal{B}_{\alpha,\frac{1}{2},1}$.

Formally, it is possible to compute the velocity-field $\mathbf{u} = (u, v)$, of a solution $(\mathbf{u}, p)$ to Problem 2 with source term $\widetilde{\mathbf{f}} := (F_1, F_2)$, as the inverse fourier transform, with respect to x, of a pair $(\hat{u}, \hat{v})$:

$$u(x,y) = \int_{\mathbb{R}} e^{ikx}\hat{u}(k,y)\mathrm{d}k, \qquad v(x,y) = \int_{\mathbb{R}} e^{ikx}\hat{v}(k,y)\mathrm{d}k, \quad \forall\,(x,y) \in \Omega_+,$$

the pair $(\hat{u}, \hat{v})$ satisfying:

$$\hat{u}(k,y) = -\hat{\eta}(k,y) + \hat{\phi}(k,y) \qquad \hat{v}(k,y) = \hat{\omega}(k,y) + \hat{\psi}(k,y), \quad \forall\,(k,y) \in \mathbb{R} \times (1,\infty),$$

with $(\hat{\omega}, \hat{\eta}, \hat{\phi}, \hat{\psi})$ a solution to

$$\partial_y\hat{\omega} = -ik\hat{\eta} + \hat{Q}_1, \tag{70}$$

$$\partial_y\hat{\eta} = (ik+1)\hat{\omega} + \hat{Q}_0, \tag{71}$$

$$\partial_y\hat{\psi} = ik\hat{\phi} - \hat{Q}_1, \tag{72}$$

$$\partial_y\hat{\phi} = -ik\hat{\psi} + \hat{Q}_0. \tag{73}$$

The source terms $(\hat{Q}_0, \hat{Q}_1)$ is computed as follows :

$$\hat{Q}_0 = \frac{1}{2\pi}(\hat{u} * \hat{\omega}) + \hat{F}_2, \tag{74}$$

$$\hat{Q}_1 = \frac{1}{2\pi}(\hat{v} * \hat{\omega}) - \hat{F}_1. \tag{75}$$

Here $\hat{F}_1$ and $\hat{F}_2$ stand for the fourier transform, with respect to x, of F_1 and F_2 respectively. When the solution $(\hat{\omega}, \hat{u}, \hat{v})$ given by the solution of (70)–(75) satisfies $(\hat{\omega}, \hat{u}, \hat{v}) \in \mathcal{U}_\alpha$ with $\alpha > 3$, the velocity-field $\mathbf{u} = (u, v)$ constructed this way is a weak solution to Problem 2 in the sense of **Proposition 13**.

In [13] the following existence theorem is proved:

Theorem 15 *Let $\alpha > 3$, $\mathbf{f} \in \mathcal{C}_c^\infty(\Omega_+)$, and let $\widehat{\mathbf{f}}$ be the Fourier transform with respect to x of $\mathbf{f}$. If $\|\widehat{\mathbf{f}}; \mathcal{W}_\alpha\|$ is sufficiently small, then there exists an α-solution $\bar{\mathbf{u}}$ being the inverse Fourier transform (with respect to x) of $\hat{\mathbf{u}} \in \mathcal{U}_\alpha$, with $\hat{\mathbf{u}}$ satisfying $\|\hat{\mathbf{u}}; \mathcal{U}_\alpha\| \leq C_\alpha \|\widehat{\mathbf{f}}; \mathcal{W}_\alpha\|$, for some constant C_α depending only on the choice of α.*

The α-solution $\bar{\mathbf{u}}$ satisfies:

1. $\bar{\mathbf{u}} \in H_0^1(\Omega_+)$,
2. there exists a constant C such that:

$$\|\bar{\mathbf{u}}; H_0^1(\Omega_+)\| \leq C\|\hat{\mathbf{u}}; \mathcal{U}_\alpha\| \ , \quad \text{and} \quad |\bar{\mathbf{u}}(x,y)| \leq C\frac{\|\hat{\mathbf{u}}; \mathcal{U}_\alpha\|}{y^{3/2}} \ , \quad \forall\,(x,y) \in \Omega_+ \ .$$

We now show that when the obstacle size is small, the function $\widetilde{\mathbf{f}}_\varepsilon = TNS[\mathbf{u}_\varepsilon, p_\varepsilon]$ satisfies the condition of **Theorem 15**. This reads:

Lemma 16 *Given $\alpha > 3$, there exists $\varepsilon_\alpha > 0$ such that, for all $\varepsilon < \varepsilon_\alpha$ the weak solution $\mathbf{u}_\varepsilon$ with associated pressure p_ε is such that Problem 2 with source term $\widetilde{\mathbf{f}}_\varepsilon = TNS\,[\mathbf{u}_\varepsilon, p_\varepsilon]$ admits an α-solution $\bar{\mathbf{u}}_\varepsilon$. Moreover, there exists $C_\alpha < \infty$ depending only on α such that the α-solution satisfies $\|\hat{\mathbf{u}}_\varepsilon; \mathcal{U}_\alpha\| \le C_\alpha \|\mathbf{u}_\varepsilon; D\|$, where $\hat{\mathbf{u}}_\varepsilon$ is the Fourier transform of $\bar{\mathbf{u}}_\varepsilon$ with respect to x.*

Proof. First, let η_0 be a sufficiently small parameter to be fixed later on and denote by m the integer part of $\alpha + 1$. Applying **Theorem 2**, there exists $\varepsilon_{\alpha,\eta}$ such that for all $\varepsilon < \varepsilon_{\alpha,\eta}$ the weak solution $\mathbf{u}_\varepsilon$ with associated pressure p_ε satisfy :

$$\|\mathbf{u}_\varepsilon; \mathcal{C}^{m+2}(\overline{\mathcal{A}_2})\| + \|p_\varepsilon; \mathcal{C}^{m+1}(\overline{\mathcal{A}_2})/\mathbb{R}\| \le C_m \|\mathbf{u}_\varepsilon; D\| \le C_\alpha \eta_0 \ .$$

As a consequence, the source-term $\widetilde{\mathbf{f}}_\varepsilon := TNS[\mathbf{u}_\varepsilon, p_\varepsilon]$ obtained after truncation satisfies (see (S-*i*) and (S-*iii*)):

- $\widetilde{\mathbf{f}}_\varepsilon$ has compact support in $\overline{B(3h/4)} \setminus B(h/3)$
- $\|\widetilde{\mathbf{f}}_\varepsilon; \mathcal{C}^{m+2}(\overline{\mathcal{A}_2})\| \le K_\alpha \|\mathbf{u}_\varepsilon;\ D\| \le K_\alpha \eta_0$

Denoting by f any component of $\widetilde{\mathbf{f}}_\varepsilon$ we apply then the following classical computation. The function $f \in \mathcal{C}^\infty_c(\overline{\Omega_+})$ has support in $B(2h/3)$. Hence, the Fourier transform $\hat{f}$ of f is well-defined and continuous on Ω_+. Moreover we have, for $y \ge 1$ and $k \in \mathbb{R}$,

$$\hat{f}(k, y) = \int_{-2h/3}^{2h/3} e^{ikx} f(x, y)\ dx \ .$$

Integration by parts implies the existence of a constant C such that, for $y \ge 1$ and $k \in \mathbb{R}_0$,

$$\left|\hat{f}(k, y)\right| \le C\|f; \mathcal{C}^0(\Omega_+)\| \ , \quad \text{and} \quad \left|\hat{f}(k, y)\right| \le C\frac{\|f; \mathcal{C}^m(\Omega_+)\|}{|k|^m} \ .$$

Using that $\hat{f}$ has compact support in y, we obtain that

$$\left|\hat{f}(k, y)\right| \le C\left[\frac{\|f; \mathcal{C}^m(\Omega_+)\|}{y^p\,(1 + (|k|\,y)^m)} + \frac{\|f; \mathcal{C}^m(\Omega_+)\|}{y^q\,(1 + (|k|\,y^2)^m)}\right] \ ,$$

for arbitrary $m \in \mathbb{N}$. In particular, there holds:

$$\|\hat{f}; \mathcal{B}_{\alpha,p,q}\| \le K^\alpha_{p,q}\|f; \mathcal{C}^\alpha(\Omega_+)\| \ . \tag{76}$$

Keeping the previous notations for the Fourier transform, we have $\|\widehat{\mathbf{f}}_\varepsilon; \mathcal{W}_\alpha\| \le K_\alpha \|\mathbf{u}_\varepsilon;\ D\| \le K_\alpha \eta_0$. Finally, for η_0 sufficiently small we apply **Theorem 15**. This yields an α-solution $\bar{\mathbf{u}}_\varepsilon$ for Problem 2 with source term $\mathbf{f}_\varepsilon$. Furthermore, this solution satisfies:

$$\|\hat{\mathbf{u}}_\varepsilon; \mathcal{U}_\alpha\| \le \tilde{C}_\alpha \|\widehat{\mathbf{f}}_\varepsilon; \mathcal{W}_\alpha\| \le \tilde{C}_\alpha K_\alpha \|\mathbf{u}_\varepsilon; D\| \ .$$

This completes the proof. ∎

3.3 Weak-strong uniqueness of solution for Problem 2

So far, we have shown that a weak solution $\mathbf{u}$ of Problem 1 for S_ε with associated pressure p provides a weak solution $\tilde{\mathbf{u}}$ of Problem 2 for source term $\widetilde{\mathbf{f}} = TNS\,[\mathbf{u}, p]$ by truncation. We have also shown that, for small obstacles, we can construct an α-solution $\bar{\mathbf{u}}_\varepsilon$ for source terms $\widetilde{\mathbf{f}}_\varepsilon$ obtained after truncation of $\mathbf{u}_\varepsilon$. In this section, we prove:

Theorem 17 *Given $\alpha > 3$, there exists $\eta_\alpha > 0$ such that, given an α-solution $\bar{\mathbf{u}}$ for source-term $\widetilde{\mathbf{f}} \in \mathcal{C}^\infty_c(\Omega_+)$ such that $\|\hat{\mathbf{u}}; \mathcal{U}_\alpha\| < \eta_\alpha$, any weak solution $\tilde{\mathbf{u}}$ of Problem 2 with source term $\widetilde{\mathbf{f}}$ coincides with $\bar{\mathbf{u}}$.*

Consequently, choosing $\alpha = 4$, for instance, and a sufficiently small obstacle, we have, by **Lemma 16** that $\|\hat{\mathbf{u}}_\varepsilon;\mathcal{U}_\alpha\| \le \eta_\alpha$. Hence, we can apply this theorem to $\tilde{\mathbf{u}}_\varepsilon := \mathbf{T}_v[\mathbf{u}_\varepsilon]$. This yields that $\tilde{\mathbf{u}}_\varepsilon$ coincides with $\bar{\mathbf{u}}_\varepsilon$. Since by construction the weak solution $\tilde{\mathbf{u}}_\varepsilon$ coincides with $\mathbf{u}_\varepsilon$ outside a compact set, $\mathbf{u}_\varepsilon$ also coincides with the α-solution $\bar{\mathbf{u}}_\varepsilon$ outside a compact set and inherits its asymptotic properties. Thus, this weak-strong uniqueness result ends the proof of **Theorem 12**.

Theorem 17 is a generalization of [13, Theorem 8], where it was shown that for small $\widetilde{\mathbf{f}}$ any weak solution $\tilde{\mathbf{u}} \in H^1_0(\Omega_+)$ of Problem 1, is an α-solution. This theorem was not general enough for the present purposes because weak solutions that are obtained by truncation merely satisfy $\tilde{\mathbf{u}} \in D$. The remainder of this section is devoted to this new uniqueness proof.

3.3.1 Sketch of proof for Theorem 17

We set $\alpha > 3$ and fix $\bar{\mathbf{u}}$ an α-solution with a source-term $\widetilde{\mathbf{f}}$. It has been shown in [13, Section 3], that such an α-solution is also a weak solution of Problem 2 for $\widetilde{\mathbf{f}}$. Hence, we have (63) and (64) for $\tilde{\mathbf{u}}$ and $\bar{\mathbf{u}}$. To estimate $\tilde{\mathbf{u}} - \bar{\mathbf{u}}$, we use the $D-$norm

$$\|\tilde{\mathbf{u}} - \bar{\mathbf{u}}; D\|^2 = \|\tilde{\mathbf{u}}; D\|^2 + \|\bar{\mathbf{u}}; D\|^2 - 2\int_{\Omega_+} \nabla\tilde{\mathbf{u}} : \nabla\bar{\mathbf{u}}\; d\mathbf{x} \;,$$

where, applying (64) :

$$\|\tilde{\mathbf{u}}; D\|^2 \le \int_{\Omega_+} \widetilde{\mathbf{f}} \cdot \tilde{\mathbf{u}}\; d\mathbf{x} \;, \qquad \text{and} \qquad \|\bar{\mathbf{u}}; D\|^2 \le \int_{\Omega_+} \widetilde{\mathbf{f}} \cdot \bar{\mathbf{u}}\; d\mathbf{x} \;.$$

We now assume that

$$\int_{\Omega_+} \nabla\tilde{\mathbf{u}} : \nabla\bar{\mathbf{u}}\; d\mathbf{x} + \int_{\Omega_+} (\tilde{\mathbf{u}} + \mathbf{e}_1) \cdot \nabla\tilde{\mathbf{u}} \cdot \bar{\mathbf{u}}\; d\mathbf{x} = \int_{\Omega_+} \widetilde{\mathbf{f}} \cdot \bar{\mathbf{u}}\; d\mathbf{x} \;, \tag{H1}$$

and that

$$\int_{\Omega_+} \nabla\bar{\mathbf{u}} : \nabla\tilde{\mathbf{u}}\; d\mathbf{x} + \int_{\Omega_+} (\bar{\mathbf{u}} + \mathbf{e}_1) \cdot \nabla\bar{\mathbf{u}} \cdot \tilde{\mathbf{u}}\; d\mathbf{x} = \int_{\Omega_+} \widetilde{\mathbf{f}} \cdot \tilde{\mathbf{u}}\; d\mathbf{x} \;. \tag{H2}$$

These assumptions are proved below. Combining (H1) and (H2) yields

$$\|\tilde{\mathbf{u}} - \bar{\mathbf{u}}; D\|^2 \le \int_{\Omega_+} (\mathbf{u} + \mathbf{e}_1) \cdot \nabla\tilde{\mathbf{u}} \cdot \bar{\mathbf{u}} + \int_{\Omega_+} (\bar{\mathbf{u}} + \mathbf{e}_1) \cdot \nabla\bar{\mathbf{u}} \cdot \tilde{\mathbf{u}} \;.$$

Next we assume that for any α-solution $\bar{\mathbf{u}}$ we have:

$$\int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla\mathbf{w} \cdot \bar{\mathbf{u}}\; d\mathbf{x} = -\int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla\bar{\mathbf{u}} \cdot \mathbf{w}\; d\mathbf{x}, \qquad \forall\, (\mathbf{v}, \mathbf{w}) \in D^2 \;. \tag{H3}$$

This assumption is also proved below. Together with the previous inequality we get

$$\begin{aligned} \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}; D\|^2 &\le -\int_{\Omega_+} (\tilde{\mathbf{u}} + \mathbf{e}_1) \cdot \nabla\bar{\mathbf{u}} \cdot \tilde{\mathbf{u}}\; d\mathbf{x} + \int_{\Omega_+} (\bar{\mathbf{u}} + \mathbf{e}_1) \cdot \nabla\bar{\mathbf{u}} \cdot \tilde{\mathbf{u}}\; d\mathbf{x} \\ &\le \int_{\Omega_+} (\bar{\mathbf{u}} - \tilde{\mathbf{u}}) \cdot \nabla\bar{\mathbf{u}} \cdot \tilde{\mathbf{u}}\; d\mathbf{x} \\ &\le \int_{\Omega_+} (\bar{\mathbf{u}} - \tilde{\mathbf{u}}) \cdot \nabla\bar{\mathbf{u}} \cdot (\tilde{\mathbf{u}} - \bar{\mathbf{u}})\; d\mathbf{x} \;, \end{aligned}$$

as it yields from (H3) :

$$\int_{\Omega_+} (\bar{\mathbf{u}} - \tilde{\mathbf{u}}) \cdot \nabla\bar{\mathbf{u}} \cdot \bar{\mathbf{u}}\; d\mathbf{x} = 0.$$

Finally, we assume that for any α-solution $\bar{\mathbf{u}}$ we have:

$$\left| \int_{\Omega_+} \mathbf{v} \cdot \nabla\bar{\mathbf{u}} \cdot \mathbf{w}\; d\mathbf{x} \right| \le K \|\hat{\mathbf{u}};\mathcal{U}_\alpha\|\; \|\mathbf{v}; D\|\; \|\mathbf{w}; D\| \;, \quad \forall (\mathbf{v}, \mathbf{w}) \in D^2 \;, \tag{H4}$$

and we get

$$\|\tilde{\mathbf{u}} - \bar{\mathbf{u}}; D\|^2 \quad \leq \quad C\|\hat{\mathbf{u}}; \mathcal{U}_\alpha\| \; \|\tilde{\mathbf{u}} - \bar{\mathbf{u}}; D\|^2 \; .$$

For η_α sufficiently small we have $C\|\hat{\mathbf{u}} \; ; \; \mathcal{U}_\alpha\| < 1/2$, so that $\|\tilde{\mathbf{u}} - \bar{\mathbf{u}}; D\| = 0$. This completes the proof up to the technical points (H1)–(H4) which are proved in the following sections.

3.3.2 Proof of (H2) and (H4)

We first establish some additional conditions for the trilinear form

$$\int_{\Omega_+} (\mathbf{u} + \mathbf{e}_1) \cdot \nabla \mathbf{v} \cdot \mathbf{w} \; d\mathbf{x} \tag{77}$$

to be well defined. The main tool is the Hardy inequality for functions in D:

Proposition 18 *For all* $\mathbf{w} \in D$*,*

$$\int_{\Omega_+} \frac{|\mathbf{w}(x,y)|^2}{(y-1)^2} \; dx \; dy \leq 4\|\mathbf{w}; D\|^2 \; .$$

Therefore, we have the following continuity result for the trilinear form:

Proposition 19 *There exists a constant* C *such that for all* $(\mathbf{v}, \mathbf{w}) \in D$ *and all*

$$\bar{\mathbf{u}} \in H^1_{loc}(\Omega_+, d\mathbf{x}) \cap L^2(\Omega_+, d\mathbf{x}) \;\; \textit{such that} \;\; \|y\mathbf{u} \; ; \; L^\infty(\Omega_+)\| < \infty \, ,$$

and

$$\nabla\bar{\mathbf{u}} \in H^1_{loc}(\Omega_+, d\mathbf{x}) \cap L^2(\Omega_+, y^2 d\mathbf{x}) \;\; \textit{such that} \;\; \|y^2\nabla\bar{\mathbf{u}} \; ; \; L^\infty(\Omega_+\| < \infty \, ,$$

respectively, we have

$$\left| \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla \mathbf{w} \cdot \bar{\mathbf{u}} \; d\mathbf{x} \right| \leq C \; (1 + \|\mathbf{v}; D\|) \; \|\mathbf{w}; D\| \; \left(\|\bar{\mathbf{u}}; L^2(\Omega_+)\| + \|y\bar{\mathbf{u}}; L^\infty(\Omega_+)\| \right) \; , \tag{78}$$

and

$$\left| \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla \bar{\mathbf{u}} \cdot \mathbf{w} \; d\mathbf{x} \right| \leq C \; (1 + \|\mathbf{v}; D\|) \; \|\mathbf{w}/y; L^2(\Omega_+)\| \; \left(\|y\nabla\bar{\mathbf{u}}; L^2(\Omega_+)\| + \|y^2\nabla\bar{\mathbf{u}}; L^\infty(\Omega_+)\| \right) \; . \tag{79}$$

Proof. We denote by I_1 and I_2 the integrals in (78) and (79), respectively. Let $(\bar{\mathbf{u}}, \mathbf{v}, \mathbf{w}) \in H^1_{loc}(\Omega_+) \times D^2$. If $\|\bar{\mathbf{u}}; L^2(\Omega_+)\| + \|y\bar{\mathbf{u}}; L^\infty(\Omega_+)\| < \infty$, we can split I_1 into two integrals

$$I_1 = \int_{\Omega_+} \mathbf{v} \cdot \nabla \mathbf{w} \cdot \bar{\mathbf{u}} \; d\mathbf{x} - \int_{\Omega_+} \partial_x \mathbf{w} \cdot \bar{\mathbf{u}} \; d\mathbf{x} \; . \tag{80}$$

The second integral on the right hand side of (80) can be bounded by the Cauchy Schwarz inequality. For the first integral we have

$$\begin{aligned} \left| \int_{\Omega_+} \mathbf{v} \cdot \nabla \mathbf{w} \cdot \bar{\mathbf{u}} \; d\mathbf{x} \right| \quad &= \quad \left| \int_{\Omega_+} \frac{\mathbf{v}}{y} \cdot \nabla \mathbf{w} \cdot y\bar{\mathbf{u}} \; d\mathbf{x} \right| \qquad (81) \\ &\leq \quad \|\mathbf{v}/y; L^2(\Omega_+)\| \; \|\nabla \mathbf{w}; L^2(\Omega_+)\| \; \|y\bar{\mathbf{u}}; L^\infty(\Omega_+)\| \; , \end{aligned}$$

and we obtain the bound on I_1 by applying the Hardy inequality (note that $y \geq y - 1$). The integral I_2 is bounded similarly. ■

This proposition is suitable for α-solutions. Indeed, if $\bar{\mathbf{u}} = (u, v)$, is an α-solution, for $\alpha > 3$, denoting by $\hat{\mathbf{u}} = (\hat{u}, \hat{v})$ (respectively $\hat{\mathbf{u}}_x = (\hat{u}_x, \hat{v}_x)$ and $\hat{\mathbf{u}}_y = (\hat{u}_y, \hat{v}_y)$) the Fourier transform with respect to x of $\bar{\mathbf{u}}$ (respectively $\partial_x\bar{\mathbf{u}}$ and $\partial_y\bar{\mathbf{u}}$) there holds :

- $(\hat{u}, \hat{v}) \in \mathcal{B}_{\alpha,1/2,0} \times \mathcal{B}_{\alpha,1/2,1}$,
- $(\hat{u}_x, \hat{v}_x) \in \mathcal{B}_{\alpha-1,3/2,2} \times \mathcal{B}_{\alpha-1,3/2,3}$
- $(\hat{u}_y, \hat{v}_y) \in \mathcal{B}_{\alpha-1,3/2,1} \times \mathcal{B}_{\alpha-1,3/2,2}$

with :

$$\|\hat{u}_x; \mathcal{B}_{\alpha-1,3/2,2} \times \mathcal{B}_{\alpha-1,3/2,3}\| + \|\hat{u}_y; \mathcal{B}_{\alpha-1,3/2,1} \times \mathcal{B}_{\alpha-1,3/2,2}\| \leq C_\alpha \|\hat{\mathbf{u}}; \mathcal{U}_\alpha\| \ .$$

(see [13, pp. 685-686] for more details). Moreover, we have:

Proposition 20 *Let p, $q > 0$, $\alpha > 1$, $s \in [2, \infty]$ and let f be the inverse Fourier transform of $\hat{f} \in \mathcal{B}_{\alpha,p,q}$. Then there exists a constant C depending only on α and s such that, for any $y > 1$, $f(\cdot, y) \in L^s(\mathbb{R})$, and*

$$\|f(\cdot, y); L^s(\mathbb{R})\| \leq \frac{C}{y^e} \|\hat{f}; \mathcal{B}_{\alpha,p,q}\| \ .$$

where $e = \min(1 - 1/s + p, 2(1 - 1/s) + q)$.

Proof. Given $\hat{f} \in \mathcal{B}_{\alpha,p,q}$, and $y > 1$, the function $f(x, y)$ is the inverse Fourier transform with respect to k of the function $\hat{f}(k, y)$, which is continuous on $\mathbb{R}_0$ and satisfies for $k \in \mathbb{R}_0$

$$|\hat{f}(k, y)| \leq \left(\frac{1}{y^p(1 + (|k|y)^\alpha)} + \frac{1}{y^q(1 + (|k|y^2)^\alpha)} \right) \|\hat{f}; \mathcal{B}_{\alpha,p,q}\| \ .$$

Therefore $\hat{f}(\cdot, y) \in L^r(\mathbb{R})$ for all $r \in [1, 2]$ so that $f \in L^s(\mathbb{R})$ for all $s \in [2, \infty]$. Moreover, given $s \geq 2$ and $r \leq 2$ the conjugate exponent, *i.e.* $\frac{1}{r} + \frac{1}{s} = 1$, there exists a constant C_s, such that

$$\|f(\cdot, y); L^s(\mathbb{R})\| \leq C_s \|\hat{f}(\cdot, y); L^r(\mathbb{R})\| \ . \tag{82}$$

By a scaling argument, we have

$$\int_{-\infty}^{\infty} \frac{1}{(1 + |k|y)^{r\alpha}} dk \leq \frac{C^1_{r,\alpha}}{y} \ , \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{1}{(1 + |k|y^2)^{r\alpha}} dk \leq \frac{C^2_{r,\alpha}}{y^2} \ ,$$

which, together with (82) gives

$$\|f(\cdot, y); L^s(\mathbb{R})\| \leq C_{\alpha,r} \left(\frac{1}{y^{p+\frac{1}{r}}} + \frac{1}{y^{q+\frac{2}{r}}} \right) \|\hat{f}; \mathcal{B}_{\alpha,p,q}\| \ ,$$

as required. ■

Proposition 20 implies that, if $\hat{f} \in \mathcal{B}_{\alpha,p,q}$ for $p > 0$, $q > 0$ and $\alpha > 1$, then $f \in L^2(\Omega_+)$ and we have, for all $(x, y) \in \Omega_+$,

$$|f(x, y)| \leq \frac{C}{y^{\min(p+1,q+2)}} \|\hat{f}; \mathcal{B}_{\alpha,p,q}\| \ .$$

In particular, for an α-solution $\bar{\mathbf{u}} = (u, v)$, we have $\bar{\mathbf{u}} \in L^2(\Omega_+)$, $y\nabla\bar{\mathbf{u}} \in L^2(\Omega_+)$, and according to the remark before the proposition, there holds :

$$|u(x, y)| + |v(x, y)| \leq \frac{C}{y^{3/2}} \|\hat{\mathbf{u}}; \mathcal{U}_\alpha\| \ , \tag{83}$$

$$|\nabla u(x, y)| + |\nabla v(x, y)| \leq \frac{C}{y^{5/2}} \|\hat{\mathbf{u}}; \mathcal{U}_\alpha\| \ , \tag{84}$$

for all $(x, y) \in \Omega_+$. Consequently, we can use the bound (79) for $\mathbf{w} \in \mathcal{D}$, and we find that

$$\left| \int_{\Omega_+} (\bar{\mathbf{u}} + \mathbf{e}_1) \cdot \nabla\bar{\mathbf{u}} \cdot \mathbf{w} \ d\mathbf{x} \right| \leq C \ (\|\hat{\mathbf{u}}; \mathcal{U}_\alpha\| + 1) \ \|\hat{\mathbf{u}}; \mathcal{U}_\alpha\| \ \|\mathbf{w}; D\| \ .$$

This implies that for the α-solution $\bar{\mathbf{u}}$ the linear form $L_{\bar{\mathbf{u}}}$,

$$L_{\bar{\mathbf{u}}}[\mathbf{w}] = \int_{\Omega_+} (\bar{\mathbf{u}} + \mathbf{e}_1) \cdot \nabla \bar{\mathbf{u}} \cdot \mathbf{w} \; d\mathbf{x}$$

is continuous on D, so that the weak formulation (63) for $\bar{\mathbf{u}}$ can be extended to D. This completes the proof of assumption (H2).

With similar arguments we obtain, that for arbitrary $(\mathbf{v}, \mathbf{w}) \in D^2$

$$\begin{aligned} \left| \int_{\Omega_+} \mathbf{v} \cdot \nabla \bar{\mathbf{u}} \cdot \mathbf{w} \; d\mathbf{x} \right| &\leq K \; \|y^2 \nabla \bar{\mathbf{u}}; L^\infty(\Omega_+)\| \; \|\mathbf{v}; D\| \; \|\mathbf{w}; D\| \\ &\leq K \|\hat{\mathbf{u}}; \mathcal{U}_\alpha\| \; \|\mathbf{v}; D\| \; \|\mathbf{w}; D\| \; . \end{aligned}$$

This completes the proof of assumption (H4).

3.3.3 Proof of (H1)

The following proposition shows that α-solutions are approximated by velocity-fields of compact support:

Proposition 21 *Let $\alpha > 3$ and let $\bar{\mathbf{u}} := (u, v)$ be an α-solution. Then there exists a sequence $(\bar{\mathbf{u}}_n)_{n \in \mathbb{N}} \in D^{\mathbb{N}}$, such that for any $n \in \mathbb{N}$*

i) $\bar{\mathbf{u}}_n \in \mathcal{C}^\infty(\Omega_+)$

ii) $\bar{\mathbf{u}}_n = \bar{\mathbf{u}}$ *in* $B((0,0), n) \cap \Omega_+$ *and* $\bar{\mathbf{u}}_n = 0$ *outside* $B((0,0), 2n) \cap \Omega_+$.

iii) There exists a constant $C(\bar{\mathbf{u}})$ such that, for all $n \in \mathbb{N}$,

$$\|\bar{\mathbf{u}} - \bar{\mathbf{u}}_n; L^\infty(\Omega_+)\| + \|y \nabla (\bar{\mathbf{u}} - \bar{\mathbf{u}}_n); L^2(\Omega_+)\| + \|y (\bar{\mathbf{u}} - \bar{\mathbf{u}}_n); L^\infty(\Omega_+)\| + \|y^2 \nabla (\bar{\mathbf{u}} - \bar{\mathbf{u}}_n); L^\infty(\Omega_+)\| \leq C(\bar{\mathbf{u}}) \; .$$

Proof. Let $\alpha > 3$ and $\bar{\mathbf{u}} = (u, v)$ be an α-solution and let $\psi = \Pi[\bar{\mathbf{u}}]$ be the corresponding stream-function. **Proposition 20** implies that $\bar{\mathbf{u}} \in H^1_0(\Omega_+)$ and we have the bounds (83), (84). Therefore we have not only that $\psi \in \mathcal{C}^1(\Omega_+)$ and that $\nabla^\perp \psi(x, y) = \bar{\mathbf{u}}(x, y)$, but also that $\psi \in L^\infty(\Omega_+)$, and that, for all $(x, y) \in \Omega_+$,

$$|\psi(x, y)| \leq C \|\hat{u}; \mathcal{B}_{\alpha, 1/2, 0}\| \; , \tag{85}$$

with the previous notations, and, since $\bar{\mathbf{u}}$ is smooth in Ω_+, the stream-function ψ is also smooth in Ω_+. Let

$$\zeta_n(x, y) = \zeta\left(\frac{|(x, y)|}{n} - 1 \right) \quad , \quad \text{and} \quad \bar{\mathbf{u}}_n = \nabla^\perp \left[\zeta_n \bar{\mathbf{u}} \right] \; .$$

Then, $\bar{\mathbf{u}}_n \in D$ and it satisfies *i)* and *ii)* for any $n \in \mathbb{N}$, and

$$\bar{\mathbf{u}}_n - \bar{\mathbf{u}} = (\zeta_n - 1) \bar{\mathbf{u}} + \psi \nabla^\perp \zeta_n \; .$$

Using a scaling argument, one shows that $\|\nabla \zeta_n; L^2(\Omega_+)\|$ is uniformly bounded for $n \in \mathbb{N}$, and therefore we have the uniform bound,

$$\|\bar{\mathbf{u}}_n - \bar{\mathbf{u}}; L^2(\Omega_+)\| \leq \|\bar{\mathbf{u}}; L^2(\Omega_+)\| + C_\zeta \|\psi; L^\infty(\Omega_+)\| \; .$$

Similarly, $\|y \nabla \zeta_n; L^\infty(\Omega_+)\|$ is uniformly bounded for $n \in \mathbb{N}$, and as a consequence we have for all $(x, y) \in \Omega_+$

$$|y (\bar{\mathbf{u}}_n - \bar{\mathbf{u}})(x, y)| \leq |y \bar{\mathbf{u}}(x, y)| + C_\zeta |\psi(x, y)| \; . \tag{86}$$

Using (83) and (85), we find that the right-hand side in (86) is uniformly bounded for $(x, y, n) \in \Omega_+ \times \mathbb{N}$.

For the derivatives of $\bar{\mathbf{u}}_n$ and $\bar{\mathbf{u}}$, we have

$$|\nabla \bar{\mathbf{u}}_n(x, y) - \nabla \bar{\mathbf{u}}(x, y)| \leq |\nabla \bar{\mathbf{u}}(x, y)| + C \; |\nabla \zeta_n(x, y)| \; |\bar{\mathbf{u}}(x, y)| + |\nabla^2 \zeta_n(x, y)| \; |\psi(x, y)| \; . \tag{87}$$

Applying scaling techniques as above, one obtains for $(x, y, n) \in \Omega_+ \times \mathbb{N}$,

$$\|y\nabla^2\zeta_n; L^2(\Omega_+)\| \le C_\zeta \ , \qquad |y^2\nabla^2\zeta_n(x,y)| \le C_\zeta \ . \tag{88}$$

From (88) and (87) and (83) we get

$$\begin{aligned}&\|y\nabla(\bar{\mathbf{u}}_n - \bar{\mathbf{u}}); L^2(\Omega_+)\| \\ &\qquad \le \|y\nabla\bar{\mathbf{u}}; L^2(\Omega_+)\| + C\ \|\nabla\zeta_n; L^2(\Omega_+)\|\ \|y\bar{\mathbf{u}}; L^\infty(\Omega_+)\| + \|y\nabla^2\zeta_n; L^2(\Omega_+)\|\ \|\psi; L^\infty(\Omega_+)\| \ ,\end{aligned}$$

which yields a uniform bound with respect to n. Finally, we have,

$$y^2|\nabla\bar{\mathbf{u}}_n(x,y) - \nabla\bar{\mathbf{u}}(x,y)| \le |y^2\nabla\bar{\mathbf{u}}(x,y)| + C\ |y\nabla\zeta_n(x,y)|\ |y\bar{\mathbf{u}}(x,y)| + |y^2\nabla^2\zeta_n(x,y)|\ |\psi(x,y)| \ .$$

Therefore, the previous pointwise bound (84) on $\nabla\bar{\mathbf{u}}$ implies that $\|y^2(\nabla\bar{\mathbf{u}} - \nabla\bar{\mathbf{u}}_n); L^\infty(\Omega_+)\|$ is finite and remains uniformly bounded for $n \in \mathbb{N}$. This completes the proof of **Proposition 21**. ■

Combining **Proposition 19** and **Proposition 21** we are now able to prove (H1). Indeed, let $\tilde{\mathbf{u}} \in D$ be a weak solution for $\widetilde{\mathbf{f}}$ and let $\bar{\mathbf{u}}$ be the corresponding α-solution. From **Proposition 21** we get that there exists a sequence $\bar{\mathbf{u}}_n \in D^{\mathbb{N}}$ which approximates $\bar{\mathbf{u}}$, and, since $\bar{\mathbf{u}}_n$ has bounded support, equation (63) is satisfied by $\bar{\mathbf{u}}_n$. Using the bounds satisfied by $\tilde{\mathbf{u}}$ and $\bar{\mathbf{u}}$ we get

$$\left|\int_{\Omega_+} \nabla\tilde{\mathbf{u}} : (\nabla\bar{\mathbf{u}} - \nabla\bar{\mathbf{u}}_n)\right|\ d\mathbf{x} \le C(\bar{\mathbf{u}}) \left(\int_{\Omega_+\setminus B((0,0),n)} |\nabla\tilde{\mathbf{u}}|^2\ d\mathbf{x}\right)^{\frac{1}{2}} \ ,$$

and therefore

$$\lim_{n\to\infty} \int_{\Omega_+} \nabla\tilde{\mathbf{u}} : \nabla\bar{\mathbf{u}}_n\ d\mathbf{x} = \int_{\Omega_+} \nabla\tilde{\mathbf{u}} : \nabla\bar{\mathbf{u}}\ d\mathbf{x} \ .$$

To bound the trilinear form, we now apply (78) and get

$$\begin{aligned}&\left|\int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1)\cdot\nabla\mathbf{w}\cdot(\bar{\mathbf{u}} - \bar{\mathbf{u}}_n)\ d\mathbf{x}\right| \\ \le\ & C\ \ (1 + \|\mathbf{v}; D\|)\ \ \|\nabla\mathbf{w}; L^2(\Omega_+ \setminus B((0,0),n))\|\ \ \big(\|\bar{\mathbf{u}} - \bar{\mathbf{u}}_n; L^2(\Omega_+)\| + \|y\,(\bar{\mathbf{u}} - \bar{\mathbf{u}}_n)\,; L^\infty(\Omega_+)\|\big) \\ \le\ & C(\bar{\mathbf{u}})\ \ (1 + \|\mathbf{v}; D\|)\ \ \|\nabla\mathbf{w}; L^2(\Omega_+ \setminus B((0,0),n))\| \ .\end{aligned}$$

Passing to the limit in n, we get

$$\lim_{n\to\infty} \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1)\cdot\nabla\mathbf{w}\cdot\bar{\mathbf{u}}_n\ d\mathbf{x} = \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1)\cdot\nabla\mathbf{w}\cdot\bar{\mathbf{u}}\ d\mathbf{x} \ .$$

Finally, using that $\widetilde{\mathbf{f}}$ has compact support, we get, for n sufficiently large,

$$\int_{\Omega_+} \widetilde{\mathbf{f}}\cdot\bar{\mathbf{u}}_n\ d\mathbf{x} = \int_{\Omega_+} \widetilde{\mathbf{f}}\cdot\bar{\mathbf{u}}\ d\mathbf{x} \ .$$

Passing to the limit in (63) with $\bar{\mathbf{u}}_n$ we obtain (63) with $\bar{\mathbf{u}}$. This completes the proof of (H1).

3.3.4 Proof of (H3)

With arguments similar to the ones in the previous subsection we show that we have for any $(\mathbf{v}, \mathbf{w}) \in D^2$ and $\bar{\mathbf{u}}$ an α-solution,

$$\lim_{n\to\infty} \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1)\cdot\nabla\mathbf{w}\cdot\bar{\mathbf{u}}_n\ d\mathbf{x} = \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1)\cdot\nabla\mathbf{w}\cdot\bar{\mathbf{u}}\ d\mathbf{x} \ .$$

From (79) we get

$$\left| \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla(\bar{\mathbf{u}} - \bar{\mathbf{u}}_n) \cdot \mathbf{w} \; d\mathbf{x} \right|$$
$$\leq C \, (1 + \|\mathbf{v}; D\|) \, \|\tfrac{\mathbf{w}}{y}; L^2(\Omega_+ \setminus B((0,0),n))\| \Big[\|y^2 \nabla(\bar{\mathbf{u}}_n - \bar{\mathbf{u}}); L^\infty(\Omega_+)\| + \|y \nabla(\bar{\mathbf{u}}_n - \bar{\mathbf{u}}); L^2(\Omega_+)\| \Big]$$
$$\leq C(\bar{\mathbf{u}}) \;\; (1 + \|\mathbf{v}; D\|) \;\; \|\mathbf{w}/y; L^2(\Omega_+ \setminus B((0,0),n))\| \; .$$

The Hardy inequality implies that $\mathbf{w}/y \in L^2(\Omega_+)$. Consequently, we have the following limit,

$$\lim_{n\to\infty} \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla \bar{\mathbf{u}}_n \cdot \mathbf{w} \; d\mathbf{x} = \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla \bar{\mathbf{u}} \cdot \mathbf{w} \; d\mathbf{x} \; .$$

Since the approximation $\bar{\mathbf{u}}_n$ has compact support, for any fixed $n \in \mathbb{N}$, we have

$$\int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla \mathbf{w} \cdot \bar{\mathbf{u}}_n \; d\mathbf{x} = - \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla \bar{\mathbf{u}}_n \cdot \mathbf{w} \; d\mathbf{x} \; .$$

Therefore, the same identity is true for $\mathbf{w}$. This proves (H3).

4 Uniqueness of solutions for Problem 1

To conclude the paper, we sketch the proof that weak solutions for small obstacles are also unique. We note that this result is not included **Theorem 17**. First, this previous theorem applies to $\tilde{\mathbf{u}}_\varepsilon$. Hence, it gives information on $\mathbf{u}_\varepsilon$ only far from S_ε where $\tilde{\mathbf{u}}_\varepsilon$ coincides with $\mathbf{u}_\varepsilon$. Second, the "unique" α-solution $\bar{\mathbf{u}}_\varepsilon$, to which $\tilde{\mathbf{u}}_\varepsilon$ is compared, depends itself on the source term obtained from $\mathbf{u}_\varepsilon$ in the truncation procedure. However another weak solution to Problem 1 could create another source term. Our final result is

Theorem 22 *There exists $\varepsilon^u > 0$, such that, for all $\varepsilon < \varepsilon^u$, if $\mathbf{u}$ is a weak solution to Problem 1 for S_ε then $\mathbf{u} = \mathbf{u}_\varepsilon$.*

Proof. The following proof is very close to the proof of **Theorem 17**. Hence, we only sketch the main ideas. First, we fix $\alpha > 3$ and choose ε_0^u such that, for all $\varepsilon < \varepsilon_0^u$, any weak solution $\mathbf{u}_\varepsilon$ is equal to the α-solution $\bar{\mathbf{u}}_\varepsilon$ outside $B(2h/3)$. Furthermore, there holds (see **Lemma 16**):

$$\|\hat{\mathbf{u}}_\varepsilon; \mathcal{U}_\alpha\| \leq C_\alpha \|\mathbf{u}_\varepsilon; D\| \; .$$

for some constant C_α depending only on α. Here, $\hat{\mathbf{u}}_\varepsilon$ stands once again for the Fourier transform of $\bar{\mathbf{u}}_\varepsilon$ with respect to x. Now, let $\varepsilon < \varepsilon_0^u$ and let $\mathbf{u}$ be a weak solution of Problem 1 for S_ε. Following the sketch of proof of w**Theorem 17**, we obtain that

$$\|\mathbf{u} - \mathbf{u}_\varepsilon; D\|^2 \leq \int_{\Omega_+} (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \nabla \mathbf{u}_\varepsilon \cdot (\mathbf{u} - \mathbf{u}_\varepsilon) \; d\mathbf{x} \; .$$

The technicalities which arise here are analogous to (H1)–(H4), and are justified by splitting integrals as follows:

$$I(\mathbf{v}, \mathbf{w}, \mathbf{z}) := \int_{\Omega_+} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla \mathbf{w} \cdot \mathbf{z} \; d\mathbf{x} = I_{int}(\mathbf{v}, \mathbf{w}, \mathbf{z}) + I_{ext}(\mathbf{v}, \mathbf{w}, \mathbf{z}) \; ,$$

where

$$I_{ext}(\mathbf{v}, \mathbf{w}, \mathbf{z}) := \int_{\Omega_+ \setminus B(2h/3)} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla \mathbf{w} \cdot \mathbf{z} \; d\mathbf{x} \; , \quad I_{int}(\mathbf{v}, \mathbf{w}, \mathbf{z}) := \int_{B(2h/3)} (\mathbf{v} + \mathbf{e}_1) \cdot \nabla \mathbf{w} \cdot \mathbf{z} \; d\mathbf{x} \; .$$

Therefore, one proves, as in sections 3.3.2 to 3.3.4, suitable continuity and antisymmetric properties of the trilinear form I, when applied to $\mathbf{u}_\varepsilon$, and using the fact that in I_{ext} the weak solution $\mathbf{u}_\varepsilon$ coincides with the α-solution $\bar{\mathbf{u}}_\varepsilon$. This yields

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_\varepsilon; D\|^2 &\leq C \Big[\|\hat{\mathbf{u}}_\varepsilon; \mathcal{U}_\alpha\| + \|\mathbf{u}_\varepsilon; D\| \Big] \|\mathbf{u} - \mathbf{u}_\varepsilon; D\|^2 \\ &\leq C_\alpha \|\mathbf{u}_\varepsilon; D\| \; \|\mathbf{u} - \mathbf{u}_\varepsilon; D\|^2 \; . \end{aligned}$$

According to **Theorem 5**, there exists ε_1^u such that, if $\varepsilon < \varepsilon_1^u$, we have $\|\mathbf{u}_\varepsilon; D\| < 1/(2C_\alpha)$. This completes the proof. ■

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