INDUCED METRIC AND MATRIX INEQUALITIES ON UNITARY MATRICES

H. F. CHAU[1*], CHI-KWONG LI[2], YIU-TUNG POON[3], AND NUNG-SING SZE[4]

Abstract. We show that any symmetric norm on $\mathbb{R}^n$ induces a metric on $U(n)$, the group of all $n \times n$ unitary matrices. Using the same technique, we prove an inequality concerning the eigenvalues of a product of two unitary matrices which generalizes a few inequalities obtained earlier by Chau [arXiv:1006.3614v1].

1. Introduction

Evolution of quantum states are described by unitary operators acting on Hilbert spaces. And in quantum information science, it is instructive to measure the cost needed to evolve a quantum system [9] as well as to quantify the difference between two quantum evolutions on a system [4]. Recently, Chau [2, 3] gave partial answers to these questions by introducing two families of real-valued functions in the domain $U(n) \times U(n)$, where $U(n)$ is the group of all $n \times n$ unitary matrices. Actually, for any given $U, V \in U(n)$, the families of functions he considered are certain weighted sums of the absolute value of the argument of the eigenvalues of the matrix UV^{-1}. Using eigenvalue perturbation method, he showed that the two families of functions are in fact families of metric and pseudo-metric on $U(n)$, respectively [2, 3].

From quantum information science point of view, the significance of studying metrics of $U(n)$ that are functions of the eigenvalues of UV^{-1} is that they are indicators of the resources in terms of the product of time and energy needed to convert unitary operator V to U [2, 3].

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*Corresponding author.

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Hence, it is instructive to find this type of metrics of $U(n)$. Here we prove that a symmetric norm of $\mathbb{R}^n$ induces a metric on $U(n)$ of the required type. Moreover, the proof can be adapted to prove an inequality concerning the eigenvalues of a product of two unitary matrices. This inequality is a generalization of several inequalities first proven in Ref. [2] using eigenvalue perturbation technique.

2. Metric Induced By A Symmetric Norm

To show that a symmetry norm on $\mathbb{R}^n$ induces a metric on $U(n)$, we make use of the following result by Thompson [11] as well as Agnihotri and Woodward [1]:

Theorem 1 (Thompson)**.** If A and B are Hermitian matrices, there exist unitary matrices U and V (depending on A and B) such that

$$\exp(iA)\exp(iB) = \exp\left(iUAU^{-1} + iVBV^{-1}\right). \tag{1}$$

Proposition 2. For any given symmetric norm $g : \mathbb{R}^n \to [0, \infty)$, that is, $g(\mathbf{v}) = g(\mathbf{v}P)$ for any $\mathbf{v} \in \mathbb{R}^{1\times n}$, and any permutation matrix or diagonal orthogonal matrix P, we can define a metric on $U(n)$ as follows:

$$d_g(U, V) = \inf_{x\in\mathbb{R}} g(|a_1(x)|, \ldots, |a_n(x)|), \tag{2}$$

where $e^{ix}UV^{-1}$ has eigenvalues $e^{ia_1(x)}, \ldots, e^{ia_n(x)}$ with $\pi \geq a_1(x) \geq \cdots \geq a_n(x) > -\pi$.

Note that the infimum above is actually a minimum as we can search the infimum in any compact interval of the form $[x_0, x_0 + 2\pi]$.

Proof. If $U \neq V$, then $UV^{-1} \neq I$ so that $e^{ix}UV^{-1}$ has eigenvalues $e^{ia_1(x)}, \ldots, e^{ia_n(x)}$ with $\pi \geq a_1(x) \geq \cdots \geq a_n(x) > -\pi$ such that $(a_1(x), \ldots, a_n(x)) \neq \mathbf{0}$. Hence, $d_g(U, V) > 0$.

Clearly, $e^{ix}UV^{-1}$ has eigenvalues $\pi \geq a_1(x) \geq \cdots \geq a_n(x) > -\pi$ if and only if $e^{-ix}VU^{-1}$ has eigenvalues $\pi > -a_n(x) \geq \cdots \geq -a_1(x) \geq -\pi$. Thus, $g(|a_1(x)|, \ldots, |a_n(x)|) = g(|-a_n(x)|, \ldots, |-a_1(x)|)$; and hence $d_g(U, V) = d_g(V, U)$.

Finally, we verify the triangle inequality. Let $X, Y, Z \in U(n)$. Suppose $d_g(X, Y) = g(|a_1|, \ldots, |a_n|)$ and $d_g(Y, Z) = g(|b_1|, \ldots, |b_n|)$ where $e^{ia_1}, \ldots, e^{ia_n}$ are the eigenvalues of $e^{ir}XY^{-1}$, and $e^{ib_1}, \ldots, e^{ib_n}$ are the eigenvalues of $e^{is}YZ^{-1}$. Suppose $e^{i(r+s)}XZ^{-1}$ has eigenvalues $e^{ic_1}, \ldots, e^{ic_n}$ with $\pi \geq c_1 \geq \cdots \geq c_n > -\pi$. Then by Theorem 1, we know that there are Hermitian matrices $A, B, C = A+B$ with eigenvalues $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$ and $\tilde{c}_1 \geq \cdots \geq \tilde{c}_n$ such that if we replace $\tilde{c}_j$ by $\tilde{c}_j - 2\pi$ if

$\tilde{c}_j > \pi$ and replace $\tilde{c}_j$ by $\tilde{c}_j + 2\pi$ if $\tilde{c}_j \leq -\pi$, then the resulting n entries will be the same as $c_1, \ldots, c_n$ if they are arranged in descending order. Consequently, if $\|\mathbf{v}\|_k$ is the sum of the k largest entries of $\mathbf{v} \in \mathbb{R}^{1\times n}$ for $k = 1, \ldots, n$, then

$$\begin{aligned} \|(|c_1|, \ldots, |c_n|)\|_k &\leq \|(|\tilde{c}_1|, \ldots, |\tilde{c}_n|)\|_k \\ &\leq \|(|a_1|, \ldots, |a_n|)\|_k + \|(|b_1|, \ldots, |b_n|)\|_k \\ &= \|(|a_1| + |b_1|, \ldots, |a_n| + |b_n|)\|_k. \end{aligned} \tag{3}$$

Note that to arrive at the second inequality above, we have used the fact that for any $n \times n$ complex-valued matrices A, B, we have

$$\|A + B\|_k \leq \|A\|_k + \|B\|_k, \quad k = 1, \ldots, n. \tag{4}$$

Here $\|X\|_k$ is the Ky Fan k-norm, which is defined as the sum of the k largest singular values of X [5].

Since $g(\mathbf{u}) \leq g(\mathbf{v})$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{1\times n}$ if and only if $\|\mathbf{u}\|_k \leq \|\mathbf{v}\|_k$ for $k = 1, \ldots, n$ [6, 8], it follows that

$$\begin{aligned} d_g(X, Z) &\leq g(|c_1|, \ldots, |c_n|) \\ &\leq g(|a_1| + |b_1|, \ldots, |a_n| + |b_n|) \\ &\leq g(|a_1|, \ldots, |a_n|) + g(|b_1|, \ldots, |b_n|) \\ &= d_g(X, Y) + d_g(Y, Z). \end{aligned} \tag{5}$$

The proof is complete. □

Example 3. For any $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$, define the μ-norm by

$$\|\mathbf{v}\|_\mu = \max\left\{ \sum_{j=1}^{n} |\mu_j v_{i_j}| \colon \{i_1, \ldots, i_n\} = \{1, \ldots, n\} \right\}. \tag{6}$$

Clearly this is a family of symmetric norms; and the induced metrics on $U(n)$ is the family of metrics introduced by Chau in Refs. [2, 3].

Example 4. One may pick g to be the ℓ_p norm defined by $\ell_p(\mathbf{v}) = \left(\sum_{j=1}^{n} |v_j|^p\right)^{1/p}$ for any $p \in [1, \infty]$. The induced metric on $U(n)$ has some interesting quantum information theoretical interpretations [7].

Remark 5. In the perturbation theory context, we consider $\tilde{U} = UE$, where E is very close to the identity. Suppose $U = e^{iA}$, where A has eigenvalues $\pi - \varepsilon > a_1 \geq \cdots \geq a_n > -\pi + \varepsilon$, and $E = e^{iB}$ such that the eigenvalues of B lie in $[-\varepsilon, \varepsilon]$ for an $\varepsilon > 0$. Then we may conclude that $\tilde{U}$ has eigenvalues $\pi > c_1 \geq \cdots \geq c_n > -\pi$ such that $|c_j - a_j| \leq \varepsilon$.

3. Several Inequalities On Products Of Two Unitary Matrices

The proof technique used in Proposition 2 can be used to show an inequality generalizing a few similar ones originally reported by Chau in Ref. [2].

Proposition 6. Let

$$h(\mathrm{s}^{\downarrow}(A+B), \mathrm{s}^{\downarrow}(A), \mathrm{s}^{\downarrow}(B)) \leq 0 \tag{7}$$

be an inequality valid for all n-dimensional Hermitian matrices A and B, where $\mathrm{s}^{\downarrow}(A)$ denotes the sequence of singular values of A arranged in descending order. Suppose further that h is a Schur-convex function of its first argument whenever the second and third arguments are kept fixed. (That is to say, $h(\mathbf{u},\mathbf{v},\mathbf{w}) \leq h(\mathbf{u}',\mathbf{v},\mathbf{w})$ whenever $\mathbf{u}$ is weakly sub-majorized by $\mathbf{u}'$.) Then,

$$h(\mathrm{AAE}^{\downarrow}(UV), \mathrm{AAE}^{\downarrow}(U), \mathrm{AAE}^{\downarrow}(V)) \leq 0 \tag{8}$$

where $\mathrm{AAE}^{\downarrow}(U)$ denotes the sequence of absolute value of the principle value of argument of the eigenvalues of an $n \times n$ unitary matrix U arranged in descending order.

Proof. Let U, V be two $n\times n$ unitary matrices. And write $U = \exp(iA)$, $V = \exp(iB)$ and $UV = \exp(iC)$ where the eigenvalues of the Hermitian matrices A, B, C are all in the range $(-\pi, \pi]$. By Theorem 1, we can find a Hermitian matrix $\tilde{C}$ and $UV = \exp(i\tilde{C})$, where $\tilde{C} = W_1 A W_1^{-1} + W_2 B W_2^{-1}$ for some unitary matrices W_1 and W_2. Hence, $h(\mathrm{s}^{\downarrow}(\tilde{C}), \mathrm{s}^{\downarrow}(A), \mathrm{s}^{\downarrow}(B)) \leq 0$.

Note that the eigenvalues of $\tilde{C}$ need not lie on the interval $(-\pi, \pi]$. Yet, we can transform $\tilde{C}$ to C by replacing those eigenvalues a_j's of $\tilde{C}$ by $a_j + 2\pi$ if $a_j \leq -\pi$ and replacing them by $a_j - 2\pi$ if $a_j > \pi$. Obviously, $\mathrm{s}^{\downarrow}(C)$ is weakly sub-majorized by $\mathrm{s}^{\downarrow}(\tilde{C})$. Therefore,

$$\begin{aligned} & h(\mathrm{AAE}^{\downarrow}(UV), \mathrm{AAE}^{\downarrow}(U), \mathrm{AAE}^{\downarrow}(V)) \\ =& h(\mathrm{s}^{\downarrow}(C), \mathrm{s}^{\downarrow}(A), \mathrm{s}^{\downarrow}(B)) \leq h(\mathrm{s}^{\downarrow}(\tilde{C}), \mathrm{s}^{\downarrow}(A), \mathrm{s}^{\downarrow}(B)) \leq 0. \end{aligned} \tag{9}$$

So, we are done. □

Corollary 7. Let $U, V \in U(n)$ and that U, V and UV have eigenvalues e^{ia_j}'s, e^{ib_j}'s and e^{ic_j}'s, respectively with $\pi \geq |a_1| \geq \cdots \geq |a_n| \geq 0$, $\pi \geq |b_1| \geq \cdots \geq |b_n| \geq 0$ and $\pi \geq |c_1| \geq \cdots \geq |c_n| \geq 0$. Then

$$\sum_{\ell=1}^{p} |c_{j_\ell + k_\ell - \ell}| \leq \sum_{\ell=1}^{p} \left(|a_{j_\ell}| + |b_{k_\ell}|\right), \tag{10}$$

for any $1 \leq j_1 < \cdots < j_p \leq n$ and $1 \leq k_1 < \cdots < k_p \leq n$ with $j_p + k_p - p \leq n$.

Proof. Eq. (10) is the direct consequences of Proposition 6 and the inequality

$$\sum_{\ell=1}^{p} \lambda^{\downarrow}_{j_\ell+k_\ell-\ell}(A+B) \leq \sum_{\ell=1}^{p} \left[\lambda^{\downarrow}_{j_\ell}(A) + \lambda^{\downarrow}_{k_\ell}(B) \right] \tag{11}$$

reported in Ref. [10]. Here $\lambda^{\downarrow}_{j}(A)$ denotes the jth eigenvalue of the Hermitian matrix A arranged in descending order. □

References

1. S. Agnihotri and C. Woodward, *Eigenvalues of products of unitary matrices and quantum Schubert calculus*, Math. Res. Lett. **5** (1998), no. 6, 817–836.
2. H. F. Chau, *Metrics on unitary matrices, bounds on eigenvalues of product of unitary matrices, and measures of non-commutativity between two unitary matrices*, 2010, arXiv:1006.3614v1.
3. ______, *Metrics on unitary matrices and their application to quantifying the degree of non-commutativity between unitary matrices*, Quant. Inform. & Comp. (2011), in press.
4. A. Chefles, A. Kitagawa, M. Takeoka, M. Sasaki, and J. Twamley, *Unambiguous discrimination among oracle operators*, J. Phys. A **40** (2007), no. 33, 10183–10213.
5. K. Fan, *Maximum properties and inequalities for the eigenvalues of completely continuous operators*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), no. 11, 760–766.
6. K. Fan and A. J. Hoffman, *Some metric inequalities in the space of matrices*, Proc. Amer. Math. Soc. **6** (1955), no. 1, 111–116.
7. K. Y. Lee and H. F. Chau, 2011, in preparation.
8. C.-K. Li and N.-K. Tsing, *On the unitarily invariant norms and some related results*, Linear and Multilinear Algebra **20** (1987), no. 2, 107–119.
9. S. Lloyd, *Ultimate physical limits to computation*, Nature **406** (2000), no. 9, 1047–1054.
10. R. C. Thompson, *Singular value inequalities for matrix sums and minors*, Linear Algebra Appl. **11** (1975), no. 3, 251–269.
11. ______, *Proof of a conjectured exponential formula*, Linear and Multilinear Algebra **19** (1986), no. 2, 187–197.

(1) Department of Physics and Center of Theoretical and Computational Physics, University of Hong Kong, Pokfulam Road, Hong Kong
E-mail address: hfchau@hku.hk

(2) Department of Mathematics, College of William & Mary, Williamsburg, VA 23187-8795, USA
Current address: (for year 2011) Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong
E-mail address: ckli@math.wm.edu

(3) Department of Mathematics, Iowa State University, Ames, IA 50051, USA
E-mail address: ytpoon@iastate.edu

(4) Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong
E-mail address: raymond.sze@inet.polyu.edu.hk