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The improved split-step backward Euler method for stochastic differential delay equations*

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Abstract

A new, improved split-step backward Euler (SSBE) method is introduced and analyzed for stochastic differential delay equations(SDDEs) with generic variable delay. The method is proved to be convergent in mean-square sense under conditions (Assumption 3.1) that the diffusion coefficient $g(x, y)$ is globally Lipschitz in both x and y, but the drift coefficient $f(x, y)$ satisfies one-sided Lipschitz condition in x and globally Lipschitz in y. Further, exponential mean-square stability of the proposed method is investigated for SDDEs that have a negative one-sided Lipschitz constant. Our results show that the method has the unconditional stability property in the sense that it can well reproduce stability of underlying system, without any restrictions on stepsize h. Numerical experiments and comparisons with existing methods for SDDEs illustrate the computational efficiency of our method.

AMS subject classification: 60H35,65C20,65L20.

Key Words: split-step backward Euler method, strong convergence, one-sided Lipschitz condition, exponential mean-square stability, mean-square linear stability

1 Introduction

In this paper we consider the numerical integration of autonomous stochastic differential delay equations (SDDEs) in the Itô's sense

$$dx(t) = f(x(t), x(t - \tau(t)))dt + g(x(t), x(t - \tau(t)))dw(t) \tag{1.1}$$

with initial data $x(t) = \psi(t), t \in [-\tau, 0]$. Here $\tau(t)$ is a delay term satisfying $\tau(t) \geq 0$ and $-\tau := \inf\{t - \tau(t) : t \geq 0\}$, $f : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, g : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times m}$. We assume that

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the initial data is independent of the Wiener measure driving the equations and $w(t)$ is an m-dimensional Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

For a given constant stepsize $h > 0$, we propose a split-step backward Euler (SSBE) method for SDDEs (1.1) as follows

$$y_n^* = y_n + hf(y_n^*, \tilde{y}_n^*), \tag{1.2a}$$

$$y_{n+1} = y_n^* + g(y_n^*, \tilde{y}_n^*)\Delta w_n, \tag{1.2b}$$

where $\Delta w_n = w(t_{n+1}) - w(t_n)$ and for $0 \leq \mu < 1, 1 \leq q_n \in \mathbb{Z}^+$

$$\tilde{y}_n^* = \begin{cases} \psi(t_n - \tau(t_n)), & t_n - \tau(t_n) < 0, \\ \mu y_{n-q_n+1}^* + (1-\mu) y_{n-q_n}^*, & 0 \leq t_n - \tau(t_n) \in [t_{n-q_n}, t_{n-q_n+1}). \end{cases} \tag{1.3}$$

For arbitrary stepsize $h > 0$, y_n denotes the approximation of $x(t)$ at time $t_n = nh, n = 0, 1 \cdots$. We remark that μ in (1.3) depends on how memory values are handled on non-grid points. Generally there are two ways, the first is to use piecewise constant interpolation, corresponding to $\mu \equiv 0$, and the second to use piecewise linear interpolation. In later development, we prefer to assume $0 \leq \mu < 1$ to cover both cases. Also, we mention that the scheme (1.2a)-(1.2b) here is quite different from the SSBE method in [23], which will be explained at the end of this section.

In (1.2a)-(1.2b), y_n^* serves as an intermediate stage value, and in order to continue the process, we have to solve the implicit equation (1.2a) at every step to acquire y_n^*. Existence and uniqueness of solutions to the implicit equations (1.2a) will be discussed in section 4. Here, we always assume that numerical solution of (1.2a) exists uniquely. And one can easily check that y_n^*, y_n is $\mathcal{F}_{t_n}$-measurable.

The key aim in this work is to propose a new SSBE method for SDDEs with variable delay and its convergence and stability in mean-square sense are investigated under a non-globally Lipschitz condition. This situation has been investigated in [7, 8, 9, 10, 11, 13, 12, 24] for stochastic differential equations (SDEs) without delay. For SDEs with delay, most of previous work has been based on the more restrictive assumption that the coefficients f, g satisfies global Lipschitz and linear growth conditions, see, for example, [1, 5, 15, 19, 23]. In [18], the authors showed that the numerical solution produced by Euler-Maruyama (EM) method will converge to the true solution of the SDDEs under the local Lipschitz condition. Note that the proof of the convergence result in this paper is based on techniques used in [7, 18]. In [7], by interpreting the implicit method SSBE as the EM applied to a modified SDE the authors were able to get a strong convergence result. This paper, however, provides an alternative way to get the convergence result for SSBE. That is, by giving a direct continuous-time extension we accomplished the convergence proof for SSBE without considering the modified SDDEs. Also, in deriving moment bounds of numerical solution, due to the delay term of our SSBE, i.e., $\tilde{y}_n^*$ in (1.2a), y_n^* cannot be explicitly dominated by y_n as (3.25) in [7]. Starting with a recurrence of y_n^* given by substituting (1.2b) into (1.2a), we overcome this difficulty and obtained the desired moment bounds. Note that a similar approach is adopted in the stability analysis.

Of course, the most important contribution of this work is to propose an improved SSBE method for SDDEs and to verify its excellent stability property. In [23], the authors proposed a SSBE method for a linear scalar SDDE with constant lag and its convergence and stability are studied there. It is worth emphasizing that our proposed method is a modified version of SSBE in [23]. The changes are in two aspects: firstly, we drop the stepsize restriction $h = \frac{\tau}{\kappa}, \kappa \in \mathbb{Z}^+$ and allow for arbitrary stepsize $h > 0$; secondly and most importantly, the scheme has been modified to a new one. To see this, the two methods are applied to a linear scalar SDDE in section 5. One can observe that the second terms of f, g in the scheme in [23] is the numerical solution $y_{n-\kappa+1}$ (see (5.4) below). While the corresponding terms in our scheme is the intermediate stage value $y^*_{n-\kappa}$ (see (5.3) below). Note that the modifications of the method do not raise the strong order of the numerical solution, but they indeed improve the stability of the method greatly. In fact, it is shown below that our method can well replicate exponential mean-square stability of nonlinear test problem, including the linear test equation as a special case, without any restrictions on stepsize h. The convergence and stability results of SSBE can be regarded as an extension of those in [7, 8] for SDEs without delay to variable delay case. This unconditional stability property of (1.2a)-(1.2b) demonstrates that the proposed method is promising and will definitely be effective in solving systems with stiffness in the drift term, where stability investigations are particularly important.

This article is organized as follows. In next section, a general convergence result (Theorem 2.4) is established. In section 3, a convergence result is derived under a one-sided Lipschitz condition (Assumption 3.1). Section 4 and 5 are devoted to exponential mean-square stability property of the method. Numerical experiments are included in section 6.

2 The general convergence results

Throughout the paper, let $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^d$ and the trace norm(F-norm) in $\mathbb{R}^{d\times m}$. As the standing hypotheses, we make the following assumption.

Assumption 2.1 *The system (1.1) has a unique solution $x(t)$ on $[-\tau, T]$. And the functions $f(x, y)$ and $g(x, y)$ are both locally Lipschitz continuous in x and y, i.e., there exists a constant L_R such that*

$$|f(x_2, y_2) - f(x_1, y_1)|^2 \vee |g(x_2, y_2) - g(x_1, y_1)|^2 \leq L_R(|x_2 - x_1|^2 + |y_2 - y_1|^2), \tag{2.1}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq R$.

Moreover, we assume that [18]

Assumption 2.2 *$\psi(t)$ is Hölder continuous in mean-square with exponent 1/2, that is*

$$E|\psi(t) - \psi(s)|^2 \leq \eta_1 |t - s|, \tag{2.2}$$

and $\tau(t)$ is a continuous function satisfying

$$|\tau(t) - \tau(s)| \leq \eta_2 |t - s|. \tag{2.3}$$

In the following convergence analysis, we find it convenient to use continuous-time approximation solution. Hence we define continuous version $\bar{y}(t)$ as follows

$$\bar{y}(t) := \begin{cases} \psi(t), \quad t \leq 0, \\ y_n + (t - t_n) f(y_n^*, \tilde{y}_n^*) + g(y_n^*, \tilde{y}_n^*)\Delta w_n(t), \quad t \in [t_n, t_{n+1}), n \geq 0, \end{cases} \tag{2.4}$$

where $\Delta w_n(t) = w(t) - w(t_n)$. For $t \in [t_n, t_{n+1})$ we can write it in integral form as follows

$$\bar{y}(t) := y_0 + \int_0^t f(y^*(s), \tilde{y}^*(s))\mathrm{d}s + \int_0^t g(y^*(s), \tilde{y}^*(s))\mathrm{d}w_s, \tag{2.5}$$

where

$$y^*(s) := \sum_{n=0}^{\infty} 1_{\{t_n \leq s < t_{n+1}\}} y_n^*, \quad \tilde{y}^*(s) := \sum_{n=0}^{\infty} 1_{\{t_n \leq s < t_{n+1}\}} \tilde{y}_n^*. \tag{2.6}$$

It is not hard to verify that $\bar{y}(t_n) = y_n$, that is, $\bar{y}(t)$ coincides with the discrete solutions at the grid-points.

In additional to the above two assumptions, we will need another one.

Assumption 2.3 *The exact solution $x(t)$ and its continuous-time approximation solution $\bar{y}(t)$ have p-th moment bounds, that is, there exist constants $p > 2, A > 0$ such that*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |x(t)|^p\right] \vee \mathbb{E}\left[\sup_{0 \leq t \leq T} |\bar{y}(t)|^p\right] \leq A. \tag{2.7}$$

Now we state our convergence theorem here and give a sequence of lemmas that lead to a proof.

Theorem 2.4 *Under Assumptions 2.1,2.2,2.3, if the implicit equation (1.2a) admits a unique solution, then the continuous-time approximate solution $\bar{y}(t)$ (2.4) will converge to the true solution of (1.1) in the mean-square sense, i.e.,*

$$\mathbb{E} \sup_{0 \leq t \leq T} |\bar{y}(t) - x(t)|^2 \to 0, \quad as \quad h \to 0.$$

We need several lemmas to complete the proof of Theorem 2.4.

First, we will define three stopping times

$$\rho_R = \inf\{t \geq 0 : |x(t)| \geq R\}, \quad \tau_R = \inf\{t \geq 0 : |\bar{y}(t)| \geq R \text{ , or } |y^*(t)| \geq R\}, \quad \sigma_R = \rho_R \wedge \tau_R,$$

where as usual $\inf \emptyset$ is set as ∞ ($\emptyset$ denotes the empty set).

Lemma 2.5 *Under Assumption 2.1, 2.2, there exist constants $C_1(R)$, $C_2(R)$ such that for $s \in [t_n, t_{n+1})$ and $h < 1$*

$$\mathbb{E}1_{\{s \leq \sigma_R\}} |\bar{y}(s) - y^*(s)|^2 \leq C_1(R)h, \tag{2.8}$$

$$\mathbb{E}1_{\{s \leq \sigma_R\}} |\bar{y}(s - \tau(s)) - \tilde{y}^*(s)|^2 \leq C_2(R)h. \tag{2.9}$$

Proof. For $s \in [t_n, t_{n+1})$, by definition of $\bar{y}(s)$ and $y^*(s)$,

$$\begin{aligned} \bar{y}(s) - y^*(s) &= y_n + (s - t_n) f(y_n^*, \tilde{y}_n^*) + g(y_n^*, \tilde{y}_n^*)\Delta w_n(s) - y_n^* \\ &= (s - t_{n+1}) f(y_n^*, \tilde{y}_n^*) + g(y_n^*, \tilde{y}_n^*)\Delta w_n(s). \end{aligned} \tag{2.10}$$

Noticing that for $|x| \vee |y| \le R$

$$\begin{aligned} |f(x,y)|^2 &\le 2|f(x,y) - f(0,0)|^2 + 2|f(0,0)|^2 \\ &\le K_R(1 + |x|^2 + |y|^2), \end{aligned} \tag{2.11}$$

with $K_R = 2\max\{L_R, |f(0,0)|\}$. Using linear growth condition of g and moment bounds in (3.7), we have appropriate constant $C_1(R)$ so that

$$\begin{aligned} \mathbb{E}1_{\{s \le \sigma_R\}}|\bar{y}(s) - y^*(s)|^2 &\le 2K_R h^2(1 + \mathbb{E}|y_n^*|^2 + \mathbb{E}|\tilde{y}_n^*|^2) + 2Kh(1 + \mathbb{E}|y_n^*|^2 + \mathbb{E}|\tilde{y}_n^*|^2) \\ &\le C_1(R)h. \end{aligned}$$

As for estimate (2.9), there are four cases as to the location of $t_n - \tau(t_n)$ and $s - \tau(s)$:

- 1) $t_n - \tau(t_n) < 0, s - \tau(s) < 0$,
- 2) $t_n - \tau(t_n) \ge 0, s - \tau(s) \ge 0$,
- 3) $t_n - \tau(t_n) < 0, s - \tau(s) \ge 0$,
- 4) $t_n - \tau(t_n) \ge 0, s - \tau(s) < 0$.

Noticing that the delay $\tau(s)$ satisfies Lipschitz condition (2.3), one sees that

$$|s - \tau(s) - t_n + \tau(t_n)| \le (\eta_2 + 1)h. \tag{2.12}$$

In the case 1), combining Hölder continuity of initial data (2.2) and (2.12) gives the desired assertion. In the case 2), without loss of generality, we assume $s - \tau(s) \in [t_i, t_{i+1})$, $t_n - \tau(t_n) = (1-\mu)t_j + \mu t_{j+1} \in [t_j, t_{j+1})$, $i > j \ge 0$. Thus we have from (1.2a) and (3.8) that

$$\begin{aligned} \bar{y}(s - \tau(s)) - \tilde{y}^*(s) &= y_i + (s - \tau(s) - t_i) f(y_i^*, \tilde{y}_i^*) + g(y_i^*, \tilde{y}_i^*)\Delta w_i(s - \tau(s)) \\ &\quad -(1-\mu)y_j^* - \mu y_{j+1}^* \\ &= (s - \tau(s) - t_{i+1}) f(y_i^*, \tilde{y}_i^*) + g(y_i^*, \tilde{y}_i^*)\Delta w_i(s - \tau(s)) \\ &\quad +(1-\mu)(y_i^* - y_j^*) + \mu(y_i^* - y_{j+1}^*) \\ &= (s - \tau(s) - t_{i+1}) f(y_i^*, \tilde{y}_i^*) + g(y_i^*, \tilde{y}_i^*)\Delta w_i(s - \tau(s)) \\ &\quad +(1-\mu)\sum_{k=j}^{i-1}\left[h f(y_{k+1}^*, \tilde{y}_{k+1}^*) + g(y_k^*, \tilde{y}_k^*)\Delta w_k\right] \\ &\quad +\mu \sum_{k=j+1}^{i-1}\left[h f(y_{k+1}^*, \tilde{y}_{k+1}^*) + g(y_k^*, \tilde{y}_k^*)\Delta w_k\right], \end{aligned} \tag{2.13}$$

where as usual we define the second summation equals zero when $i = j + 1$. Noticing from (2.12) that $i - j \le \eta_2 + 1$, and combining local linear growth bound (2.11) for f, global linear growth condition for g and moment bounds (3.7), we can derive from (2.13) that

$$\mathbb{E}1_{\{s \le \sigma_R\}}|\tilde{y}^*(s) - \bar{y}(s - \tau(s))|^2 \le C_2(R)h.$$

In the case 3) and 4), using an elementary inequality gives

$$\begin{aligned}&\mathbb{E}1_{\{s\le\sigma_R\}}|\tilde{y}^*(s)-\bar{y}(s-\tau(s))|^2\\ \le&2\mathbb{E}1_{\{s\le\sigma_R\}}|\tilde{y}^*(s)-\bar{y}(0)|^2+2\mathbb{E}1_{\{s\le\sigma_R\}}|\bar{y}(0)-\bar{y}(s-\tau(s))|^2.\end{aligned}$$

Then combining this with results obtained in case 1) and 2) gives the required result, with $C_2(R)$ a universal constant independent of h.

Lemma 2.6 *Under Assumption 2.1, 2.2, for stepsize $h<1$, there exists a constant C_R such that*

$$\mathbb{E}[\sup_{0\le t\le T}|\bar{y}(t\wedge\sigma_R)-x(t\wedge\sigma_R)|^2]\le C_R h,$$

with C_R dependent on R, but independent of h.

Proof. For simplicity, denote

$$e(t):=\bar{y}(t)-x(t).$$

From (1.1) and (2.5), we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le s\le t}|e(s\wedge\sigma_R)|^2\right]=\mathbb{E}\left[\sup_{0\le s\le t}|\bar{y}(s\wedge\sigma_R)-x(s\wedge\sigma_R)|^2\right]\\ =&\ \mathbb{E}\left[\sup_{0\le s\le t}\left|\int_0^{s\wedge\sigma_R}f(y^*(r),\tilde{y}^*(r))-f(x(r),x(r-\tau(r)))\mathrm{d}r\right.\right.\\ &\left.\left.+\int_0^{s\wedge\sigma_R}g(y^*(r),\tilde{y}^*(r))-g(x(r),x(r-\tau(r)))\mathrm{d}w(r)\right|^2\right]\\ \le&\ 2T\mathbb{E}\int_0^{t\wedge\sigma_R}|f(y^*(s),\tilde{y}^*(s))-f(x(s),x(s-\tau(s)))|^2\,\mathrm{d}s\\ &+2\mathbb{E}\left[\sup_{0\le s\le t}\left|\int_0^{s\wedge\sigma_R}g(y^*(r),\tilde{y}^*(r))-g(x(r),x(r-\tau(r)))\mathrm{d}w(r)\right|^2\right]\\ \le&\ 2(T+4)L_R\mathbb{E}\int_0^{t\wedge\sigma_R}|y^*(s)-x(s)|^2+|\tilde{y}^*(s)-x(s-\tau(s))|^2\mathrm{d}s,\end{aligned}\tag{2.14}$$

where Hölder's inequality and the Burkholder-Davis-Gundy inequality were used again. Using the elementary inequality $|a+b|^2\le 2|a|^2+2|b|^2$, one computes from (2.14) that

$$\begin{aligned}&\mathbb{E}[\sup_{0\le s\le t}|e(s\wedge\sigma_R)|^2]\\ \le&\ 4(T+4)L_R\mathbb{E}\int_0^{t\wedge\sigma_R}|y^*(s)-\bar{y}(s)|^2+|\bar{y}(s)-x(s)|^2\mathrm{d}s\\ &+4(T+4)L_R\mathbb{E}\int_0^{t\wedge\sigma_R}|\tilde{y}^*(s)-\bar{y}(s-\tau(s))|^2+|\bar{y}(s-\tau(s))-x(s-\tau(s))|^2\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\leq 8(T+4)L_R\int_0^t \mathbb{E}[\sup_{0\leq r\leq s}|\bar{y}(r\wedge\sigma_R)-x(r\wedge\sigma_R)|^2]\mathrm{d}s\\
&\quad +4(T+4)L_R\mathbb{E}\int_0^{t\wedge\sigma_R}|y^*(s)-\bar{y}(s)|^2\mathrm{d}s\\
&\quad +4(T+4)L_R\mathbb{E}\int_0^{t\wedge\sigma_R}|\tilde{y}^*(s)-\bar{y}(s-\tau(s))|^2\mathrm{d}s, \qquad (2.15)
\end{aligned}
$$

where the fact was used that $|\bar{y}(s-\tau(s))-x(s-\tau(s))|^2 \leq \sup_{0\leq r\leq s}|\bar{y}(r))-x(r)|^2$. By taking Lemma 2.5 into account, we derive from (2.15) that, with suitable constants $\tilde{C}_R, \bar{C}_R$

$$
\begin{aligned}
\mathbb{E}[\sup_{0\leq s\leq t}|e(s\wedge\sigma_R)|^2] &\leq 8(T+4)L_R\int_0^t \mathbb{E}[\sup_{0\leq r\leq s}|\bar{y}(r\wedge\sigma_R)-x(r\wedge\sigma_R)|^2]\mathrm{d}s\\
&\quad +4(T+4)TL_RC_1(R)h+4(T+4)TL_RC_2(R)h\\
&= \tilde{C}_R\int_0^t \mathbb{E}[\sup_{0\leq r\leq s}|e(r\wedge\sigma_R)|^2]\mathrm{d}s+\bar{C}_Rh. \qquad (2.16)
\end{aligned}
$$

Hence continuous Gronwall inequality gives the assertion.

Proof of Theorem 2.4. Armed with Lemma 2.6 and Assumption 2.3, the result may be proved using a similar approach to that in [7, Theorem 2.2] and [18, Theorem 2.1], where under the local Lipschitz condition they showed the strong convergence of the EM method for the SODEs and SDDEs, respectively.

Remark 2.7 *Under the global Lipschitz condition and linear growth condition (cf [17]), we can choose uniform constants $C_1(R)$, $C_2(R)$, C_R in previous Lemma 2.5,2.6 to be independent of R. Accordingly we can recover the strong order of 1/2 by deriving*

$$
\mathbb{E}[\sup_{0\leq t\leq T}|\bar{y}(t)-x(t)|^2]\leq Ch,
$$

where C is independent of R and h.

3 Convergence with a one-sided Lipschitz condition

In this section, we will give some sufficient conditions on equations (1.1) to promise a unique global solution of SDDEs and a well-defined solution of the SSBE method. We make the following assumptions on the SDDEs.

Assumption 3.1 *The functions $f(x,y)$ are continuously differentiable in both x and y, and there exist constants $\gamma_1,\gamma_2,\gamma_3,\gamma_4$, such that $\forall x,y,x_1,x_2,\ y_1,y_2\in\mathbb{R}^d$*

$$
\begin{aligned}
\langle x_2-x_1, f(x_2,y)-f(x_1,y)\rangle &\leq \gamma_1|x_2-x_1|^2, \qquad (3.1)\\
|f(x,y_2)-f(x,y_1)| &\leq \gamma_2|y_2-y_1|, \qquad (3.2)\\
|g(x_2,y_2)-g(x_1,y_1)|^2 &\leq \gamma_3|x_2-x_1|^2+\gamma_4|y_2-y_1|^2. \qquad (3.3)
\end{aligned}
$$

The inequalities (3.1),(3.2) indicate that the first argument x of f satisfies one-sided Lipschitz condition and the second satisfies global Lipschitz condition. It is worth noticing that conditions of the same type as (3.1) and (3.2) have been exploited successfully in the analysis of numerical methods for deterministic delay differential equations (DDEs)(see [3] and references therein). As for SDEs without delay, the conditions (3.1) and (3.3) has been used in [7, 8, 9, 12, 24].

We compute from (3.1)-(3.3) that

$$\begin{aligned}\langle x, f(x,y)\rangle &= \langle x, f(x,y)-f(0,y)\rangle + \langle x, f(0,y)-f(0,0)\rangle + \langle x, f(0,0)\rangle \\ &\leq (\gamma_1+1)|x|^2 + \frac{1}{2}\gamma_2|y|^2 + \frac{1}{2}|f(0,0)|^2, \end{aligned} \tag{3.4}$$

$$|g(x,y)|^2 \leq 2|g(x,y)-g(0,0)|^2 + 2|g(0,0)|^2 \leq 2\gamma_3|x|^2 + 2\gamma_4|y|^2 + 2|g(0,0)|^2. \tag{3.5}$$

On choosing the constant K as

$$K = \max\left\{\gamma_1+1, 2\gamma_3, \frac{1}{2}\gamma_2, 2\gamma_4, \frac{1}{2}|f(0,0)|^2, 2|g(0,0)|^2\right\},$$

the following condition holds

$$x^T f(x,y) \vee |g(x,y)|^2 \leq K(1+|x|^2+|y|^2), \quad \forall x, y \in \mathbb{R}^d. \tag{3.6}$$

In what follows we always assume that for $\forall p > 0$ the initial data satisfies

$$\mathbb{E}\|\psi\|^p := \mathbb{E} \sup_{-\tau\leq s\leq 0} |\psi(s)|^p < \infty.$$

Theorem 3.2 *Assume that Assumption 3.1 is fulfilled. Then there exists a unique global solution $x(t)$ to system (1.1). Morever, for any $p \geq 2$, there exists constant $C = C(p,T)$*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |x(t)|^p\right] \leq C(1+\mathbb{E}\|\psi\|^p).$$

Proof. See the Appendix.

Lemma 3.3 *Assume that f, g satisfy the condition (3.6) and $h < 1$ is sufficiently small , then for $p \geq 2$ the following moment bounds hold*

$$\mathbb{E}\left[\sup_{0\leq nh\leq T} |y_n^*|^{2p}\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |\tilde{y}^*(t)|^{2p}\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |\bar{y}(t)|^{2p}\right] \leq A. \tag{3.7}$$

Proof. Inserting (1.2b) into (1.2a) gives

$$y_n^* = y_{n-1}^* + hf(y_n^*, \tilde{y}_n^*) + g(y_{n-1}^*, \tilde{y}_{n-1}^*)\Delta w_{n-1}, \quad n \geq 1. \tag{3.8}$$

Hence

$$|y_n^* - hf(y_n^*, \tilde{y}_n^*)|^2 = |y_{n-1}^* + g(y_{n-1}^*, \tilde{y}_{n-1}^*)\Delta w_{n-1}|^2.$$

Expanding it and employing (3.6) yields

$$|y_n^*|^2 - 2Kh(1+|y_n^*|^2+|\tilde{y}_n^*|^2) \;\; \leq \;\; |y_{n-1}^*|^2 + 2\left\langle y_{n-1}^*, g(y_{n-1}^*, \tilde{y}_{n-1}^*)\Delta w_{n-1}\right\rangle + |g(y_{n-1}^*, \tilde{y}_{n-1}^*)\Delta w_{n-1}|^2. \tag{3.9}$$

By definition of $\tilde{y}_n^*$, one obtains $|\tilde{y}_n^*|^2 \leq |y_n^*|^2 + \max_{0\leq i\leq n-1}|y_i^*|^2 + \|\psi\|^2$. Taking this inequality into consideration and letting $h < h_0 < 1/(4K)$, we have from (3.9) that

$$\begin{aligned}(1-4Kh)|y_n^*|^2 &\leq |y_{n-1}^*|^2 + 2Kh(1+\max_{0\leq i\leq n-1}|y_i^*|^2+\|\psi\|^2)\\ &+2\left\langle y_{n-1}^*, g(y_{n-1}^*, \tilde{y}_{n-1}^*)\Delta w_{n-1}\right\rangle + |g(y_{n-1}^*, \tilde{y}_{n-1}^*)\Delta w_{n-1}|^2.\end{aligned} \tag{3.10}$$

Denoting $\alpha = 1/(1-4Kh_0)$, one computes that

$$\begin{aligned}|y_n^*|^2 \;\; \leq \;\; & |y_{n-1}^*|^2 + 6K\alpha h \max_{0\leq i\leq n-1}|y_i^*|^2 + 2K\alpha h + 2K\alpha h\|\psi\|^2\\ &+2\alpha\left\langle y_{n-1}^*, g(y_{n-1}^*, \tilde{y}_{n-1}^*)\Delta w_{n-1}\right\rangle + \alpha|g(y_{n-1}^*, \tilde{y}_{n-1}^*)\Delta w_{n-1}|^2.\end{aligned} \tag{3.11}$$

By recursive calculation, we obtain

$$\begin{aligned}|y_n^*|^2 \;\; \leq \;\; & |y_0^*|^2 + 6K\alpha h\sum_{j=0}^{n-1}\max_{0\leq i\leq j}|y_i^*|^2 + 2K\alpha T + 2K\alpha T\|\psi\|^2\\ &+2\alpha\sum_{j=0}^{n-1}\left\langle y_j^*, g(y_j^*, \tilde{y}_j^*)\Delta w_j\right\rangle + \alpha\sum_{j=0}^{n-1}|g(y_j^*, \tilde{y}_j^*)\Delta w_j|^2.\end{aligned}$$

Raising both sides to the power p gives

$$\begin{aligned}|y_n^*|^{2p} \;\; \leq \;\; & 5^{p-1}\Bigg\{|y_0^*|^{2p} + (6K\alpha h)^p n^{p-1}\sum_{j=0}^{n-1}\max_{0\leq i\leq j}|y_i^*|^{2p} + \left[2K\alpha T + 2K\alpha T\|\psi\|^2\right]^p\\ &+(2\alpha)^p\left[\sum_{j=0}^{n-1}\left\langle y_j^*, g(y_j^*, \tilde{y}_j^*)\Delta w_j\right\rangle\right]^p + \alpha^p n^{p-1}\sum_{j=0}^{n-1}|g(y_j^*, \tilde{y}_j^*)\Delta w_j|^{2p}\Bigg\}.\end{aligned}$$

Thus

$$\begin{aligned}&\mathbb{E}\max_{1\leq n\leq M}|y_n^*|^{2p}\\ &\leq 5^{p-1}\Bigg\{\mathbb{E}|y_0^*|^{2p} + (6K\alpha)^p T^{p-1} h\mathbb{E}\sum_{j=0}^{M-1}\max_{0\leq i\leq j}|y_i^*|^{2p} + \mathbb{E}\left(2K\alpha T + 2K\alpha T\|\psi\|^2\right)^p\\ &+(2\alpha)^p\mathbb{E}\max_{1\leq n\leq M}\left[\sum_{j=0}^{n-1}\langle y_j^*, g(y_j^*, \tilde{y}_j^*)\Delta w_j\rangle\right]^p + \alpha^p M^{p-1}\mathbb{E}\sum_{j=0}^{M-1}|g(y_j^*, \tilde{y}_j^*)\Delta w_j|^{2p}\Bigg\}.\end{aligned} \tag{3.12}$$

Here $1 \leq M \leq N$, where N is the largest integer number such that $Nh \leq T$. Now, using the Burkholder-Davis-Gundy inequality (Theorem 1.7.3 in [17]) gives

$$\mathbb{E}\max_{1\leq n\leq M}\left[\sum_{j=0}^{n-1}\left\langle y_j^*, g(y_j^*, \tilde{y}_j^*)\Delta w_j\right\rangle\right]^p \leq C_p\mathbb{E}\left[\sum_{j=0}^{M-1}|y_j^*|^2|g(y_j^*, \tilde{y}_j^*)|^2 h\right]^{p/2}$$

$$\leq C_p(Kh)^{p/2}M^{p/2-1}\mathbb{E}\left[\sum_{j=0}^{M-1}|y_j^*|^p\left(1+|y_j^*|^2+|\tilde{y}_j^*|^2\right)^{p/2}\right]$$
$$\leq \frac{1}{2}C_pK^{p/2}T^{p/2-1}h\mathbb{E}\left[\sum_{j=0}^{M-1}\left(|y_j^*|^{2p}+3^{p-1}(1+|y_j^*|^{2p}+|\tilde{y}_j^*|^{2p})\right)\right]. \tag{3.13}$$

Noticing that

$$\mathbb{E}|\tilde{y}_j^*|^{2p}\leq \mathbb{E}\max_{0\leq i\leq j}|y_i^*|^{2p}+\mathbb{E}\|\psi\|^{2p}, \tag{3.14}$$

inserting it into (3.13), we can find out appropriate constants $\bar{C}=\bar{C}(p,K,T)$ such that

$$\begin{aligned}&\mathbb{E}\max_{0\leq n\leq M}\left[\sum_{j=0}^{n-1}\langle y_j^*,g(y_j^*,\tilde{y}_j^*)\Delta w_j\rangle\right]^p\\ \leq\ &\bar{C}h\sum_{j=0}^{M-1}\mathbb{E}\max_{0\leq i\leq j}|y_i^*|^{2p}+\bar{C}(\mathbb{E}\|\psi\|^{2p}+1).\end{aligned} \tag{3.15}$$

At the same time, noting the fact $y_n^*,\tilde{y}_n^*\in\mathcal{F}_{t_n}$ and Δw_n is independent of $\mathcal{F}_{t_n}$, one can compute that, with $\hat{C}=\hat{C}(p,T)$ a constant that may change line by line

$$\begin{aligned}\mathbb{E}\sum_{j=0}^{M-1}|g(y_j^*,\tilde{y}_j^*)\Delta w_j|^{2p} &\leq \sum_{j=0}^{M-1}\mathbb{E}|g(y_j^*,\tilde{y}_j^*)|^{2p}\mathbb{E}|\Delta w_j|^{2p}\\ &\leq \hat{C}h^p\sum_{j=0}^{M-1}\left[1+\mathbb{E}|y_j^*|^{2p}+\mathbb{E}|\tilde{y}_j^*|^{2p}\right]\\ &\leq \hat{C}h^{p-1}(\mathbb{E}\|\psi\|^{2p}+1)+\hat{C}h^p\sum_{j=0}^{M-1}\mathbb{E}\max_{0\leq i\leq j}|y_i^*|^{2p}.\end{aligned} \tag{3.16}$$

By definition (1.2a), one sees that

$$|y_0^*-hf(y_0^*,\tilde{y}_0^*)|^2=|y_0|^2.$$

Then using a similar approach used before, we can find out a constant $c_0=c_0(p,K)$ to ensure that

$$\mathbb{E}|y_0^*|^{2p}<c_0(\mathbb{E}\|\psi\|^{2p}+1)<\infty. \tag{3.17}$$

Inserting (3.15),(3.16) into (3.12) and considering (3.17) and $h<1$, we have, with suitable constants $C'=C'(p,K,T),C''=C''(p,K,T)$

$$\begin{aligned}\mathbb{E}\max_{0\leq n\leq M}|y_n^*|^{2p} &\leq \mathbb{E}|y_0^*|^{2p}+\mathbb{E}\max_{1\leq n\leq M}|y_n^*|^{2p}\\ &\leq C'(\mathbb{E}\|\psi\|^{2p}+1)+C''h\sum_{j=0}^{M-1}\mathbb{E}\max_{0\leq i\leq j}|y_i^*|^{2p}.\end{aligned} \tag{3.18}$$

Thus using the discrete-type Gronwall inequality, we derive from (3.18) that $\mathbb{E}\left[\sup_{0\le nh\le T}|y_n^*|^{2p}\right]$ is bounded by a constant independent of N. Then by considering the elementary inequality $|\mu x+(1-\mu)y|^{2p}\le\mu|x|^{2p}+(1-\mu)|y|^{2p}$, boundedness of $\mathbb{E}\left[\sup_{0\le t\le T}|\tilde{y}^*(t)|^{2p}\right]$ is immediate.

To bound $\mathbb{E}\left[\sup_{0\le t\le T}|\bar{y}(t)|^{2p}\right]$, we shall first bound $\mathbb{E}\left[\sup_{0\le nh\le T}|y_n|^{2p}\right]$. From (1.2b), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le nh\le T}|y_n|^{2p}\right] &\le 2^{2p-1}\left\{\mathbb{E}\left[\sup_{0\le nh\le T}|y_n^*|^{2p}\right]+\mathbb{E}\left[\sup_{0\le nh\le T}|g(y_n^*,\tilde{y}_n^*)\Delta w_n|^{2p}\right]\right\}\\ &\le 2^{2p-1}\left\{\mathbb{E}\left[\sup_{0\le nh\le T}|y_n^*|^{2p}\right]+\mathbb{E}\sum_{j=0}^{N}|g(y_j^*,\tilde{y}_j^*)\Delta w_j|^{2p}\right\}.\end{aligned}$$

Now (3.16) and bound of $\mathbb{E}\left[\sup_{0\le nh\le T}|y_n^*|^{2p}\right]$ gives the bound of $\mathbb{E}\left[\sup_{0\le nh\le T}|y_n|^{2p}\right]$.

To bound $\mathbb{E}\left[\sup_{0\le t\le T}|\bar{y}(t)|^{2p}\right]$, we denote by n_t the integer for which $t\in[t_{n_t},t_{n_t+1})$. By definitions of (1.2a) and (2.4), for $t\ge 0$,

$$\begin{aligned}\bar{y}(t) &= y_{n_t}+(t-t_{n_t})f(y_{n_t}^*,\tilde{y}_{n_t}^*)+g(y_{n_t}^*,\tilde{y}_{n_t}^*)\Delta w_{n_t}(t)\\ &= y_{n_t}+\gamma(y_{n_t}^*-y_{n_t})+g(y_{n_t}^*,\tilde{y}_{n_t}^*)\Delta w_{n_t}(t)\\ &= (1-\gamma)y_{n_t}+\gamma y_{n_t}^*+g(y_{n_t}^*,\tilde{y}_{n_t}^*)\Delta w_{n_t}(t),\end{aligned}\tag{3.19}$$

where $\gamma=(t-t_{n_t})/h<1$. Thus

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|\bar{y}(t)|^{2p}\right] \le\ & 2^{2p-1}\left\{\gamma\mathbb{E}\left[\sup_{0\le nh\le T}|y_n^*|^{2p}\right]+(1-\gamma)\mathbb{E}\left[\sup_{0\le nh\le T}|y_n|^{2p}\right]\right.\\ &\left.+\mathbb{E}\left[\sup_{0\le t\le T}|g(y_{n_t}^*,\tilde{y}_{n_t}^*)\Delta w_{n_t}(t)|^{2p}\right]\right\}.\end{aligned}\tag{3.20}$$

Using Doob's martingale inequality [17, Theorem 1.3.8], we derive that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|g(y_{n_t}^*,\tilde{y}_{n_t}^*)\Delta w_{n_t}(t)|^{2p}\right] &\le\sum_{n=0}^{N}\mathbb{E}\left[\sup_{0\le s\le h}|g(y_n^*,\tilde{y}_n^*)\Delta w_n(s)|^{2p}\right]\\ &\le\left(\frac{2p}{2p-1}\right)^{2p}\sum_{n=0}^{N}\mathbb{E}\left[|g(y_n^*,\tilde{y}_n^*)\Delta w_n(h)|^{2p}\right].\end{aligned}\tag{3.21}$$

Thus the last term in (3.20) is bounded by considering (3.16) and bounds of $\mathbb{E}[\sup_{0\le nh\le T}|y_n^*|^{2p}]$, $\mathbb{E}[\sup_{0\le nh\le T}|y_n|^{2p}]$. Now boundedness of $\mathbb{E}[\sup_{0\le t\le T}|\bar{y}(t)|^{2p}]$ follows immediately.

Lemma 3.4 *Under Assumption 3.1, if $(\gamma_1+\gamma_2)h<1$, the implicit equation in (1.2a) admits a unique solution.*

Proof. Let $\tilde{f}(c):=f(c,\mu c+(1-\mu)b)$, then the implicit equation (1.2a) takes the form as

$$c=h\tilde{f}(c)+d=hf(c,\mu c+(1-\mu)b)+d,$$

where at each step, $0 \le \mu < 1, b, d$ are known. Observing that

$$\begin{aligned}\langle c_1 - c_2, \tilde{f}(c_1) - \tilde{f}(c_2)\rangle &= \langle c_1 - c_2, f(c_1, \mu c_1 + (1-\mu)b) - f(c_2, \mu c_1 + (1-\mu)b)\rangle \\ &\quad + \langle c_1 - c_2, f(c_2, \mu c_1 + (1-\mu)b) - f(c_2, \mu c_2 + (1-\mu)b)\rangle \\ &\le \gamma_1 |c_1 - c_2|^2 + \mu\gamma_2 |c_1 - c_2|^2 \\ &\le (\gamma_1 + \gamma_2)|c_1 - c_2|^2,\end{aligned}$$

the assertion follows immediately from Theorem 14.2 of [6].

Corollary 3.5 *Under Assumption 2.2,3.1, if $(\gamma_1 + \gamma_2)h < 1$, then the numerical solution produced by (1.2a)-(1.2b) is well-defined and will converge to the true solution in the mean-square sense, i.e.,*

$$\mathbb{E} \sup_{0 \le t \le T} |\bar{y}(t) - x(t)|^2 \to 0, \quad as \quad h \to 0.$$

Proof. Noticing that Assumption 3.1 implies Assumptions 2.1,2.3 by Theorem 3.2 and Lemma 3.3, and taking Lemma 3.4 into consideration, the result follows directly from Theorem 2.4.

Remark 3.6 *We remark that the problem class satisfying condition (2.3) includes plenty of important models. In particular, stochastic pantograph differential equations (see, e.g., [5]) with $\tau(t) = (1-q)t, 0 < q < 1$ and SDDEs with constant lag fall into this class and therefore corresponding convergence results follow immediately.*

4 Mean-square stability with bounded delay

In this section, we will investigate how SSBE shares exponential mean-square stability of general nonlinear systems. In deterministic case, nonlinear stability analysis of numerical methods are carried on under a one-sided Lipschitz condition. This phenomenon has been well studied in the deterministic case ([3, 6] and references therein) and stochastic case without delay [7, 8, 9, 12, 24]. In what follows, we choose the test problem satisfying conditions (3.1)-(3.3). Moreover, we assume that variable delay is bounded , that is, there exists $\tau > 0$, for $1 \le \kappa \in \mathbb{Z}^+$, $0 \le \delta < 1$

$$0 \le \tau(t) \le \tau, \quad \tau = (\kappa - \delta)h. \tag{4.1}$$

We remark that this assumption does not impose additional restrictions on the stepsize h and admits arbitrary large h on choosing $\kappa = 1$ and $0 \le \delta < 1$ close to 1. To begin with, we shall first give a sufficient condition for exponential mean-square stability of analytical solution to underlying problem.

Theorem 4.1 *Under the conditions (3.1),(3.2),(3.3) and (4.1), and with $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ obeying*

$$\beta := 2\gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4 < 0, \tag{4.2}$$

any two solutions $x(t;\psi)$ and $y(t;\phi)$ with $\mathbb{E}\|\psi\|^2 < \infty$ and $\mathbb{E}\|\phi\|^2 < \infty$ satisfy

$$\mathbb{E}|x(t) - y(t)|^2 \le \mathbb{E}\|\phi - \psi\|^2 \exp\{-\nu^+ t\},$$

where $\nu^+ \in (0, -\beta]$ *is the zero of* $\mathcal{L}(\nu) = \nu + \beta_1 + \beta_2 \exp\{\nu\tau\}$*, with* $\beta_1 = 2\gamma_1 + \gamma_2 + \gamma_3$*,* $\beta_2 = \gamma_2 + \gamma_4$*.*

Proof. By Itô formula, we have

$$\begin{aligned}
& \mathbb{E}|x(t+\delta) - y(t+\delta)|^2 - \mathbb{E}|x(t) - y(t)|^2 \\
= & \int_t^{t+\delta} 2\mathbb{E}\langle x(s) - y(s), f(x(s), x(s-\tau(s))) - f(y(s), y(s-\tau(s)))\rangle \mathrm{d}s \\
& + \int_t^{t+\delta} \mathbb{E}|g(x(s), x(s-\tau(s))) - g(y(s), y(s-\tau(s)))|^2 \mathrm{d}s \\
\leq & (2\gamma_1 + \gamma_3) \int_t^{t+\delta} \mathbb{E}|x(s) - y(s)|^2 \mathrm{d}s + \gamma_4 \int_t^{t+\delta} \mathbb{E}|x(s-\tau(s)) - y(s-\tau(s))|^2 \mathrm{d}s \\
& + 2\int_t^{t+\delta} \mathbb{E}\langle x(s) - y(s), f(y(s), x(s-\tau(s))) - f(y(s), y(s-\tau(s)))\rangle \mathrm{d}s \\
\leq & \beta_1 \int_t^{t+\delta} \mathbb{E}|x(s) - y(s)|^2 \mathrm{d}s + \beta_2 \int_t^{t+\delta} \sup_{r \in [s-\tau, s]} \mathbb{E}|x(r) - y(r)|^2 \mathrm{d}s. \qquad (4.3)
\end{aligned}$$

Letting $u(t) = \mathbb{E}|x(t) - y(t)|^2$ and noticing that $u(t)$ exists for $t \geq -\tau$ and is continuous, we derive from (4.3) that

$$\mathrm{D}^+ u(t) \leq \beta_1 u(t) + \beta_2 \sup_{s \in [t-\tau, t]} u(s),$$

where the upper Dini derivative $\mathrm{D}^+ u(t)$ is defined as

$$\mathrm{D}^+ u(t) := \limsup_{\delta \to 0+} \frac{u(t+\delta) - u(t)}{\delta}.$$

Using Theorem 7 in [2] leads to the desired result.

Based on this stability result, we are going to investigate stability of the numerical method.

Theorem 4.2 *Under the conditions (3.1),(3.2),(3.3) and (4.1), if* $\beta < 0$*, then for all* $h > 0$*, any two solutions* X_n, Y_n *produced by SSBE (1.2a)-(1.2b) with* $\mathbb{E}\|\psi\|^2 < \infty$ *and* $\mathbb{E}\|\phi\|^2 < \infty$ *satisfy*

$$\mathbb{E}|X_n - Y_n|^2 \leq \mathbb{E}\|\phi - \psi\|^2 \exp\{-\nu_h^+ nh\}, \quad \textit{as} \quad n \to \infty,$$

where $\nu_h^+ > 0$ *is defined as*

$$\nu_h^+ = \frac{1}{2(\kappa+1)h} \ln\left(\frac{1 - 2h\gamma_1 - h\gamma_2}{1 + h\gamma_2 + h\gamma_3 + h\gamma_4}\right) > 0. \qquad (4.4)$$

Proof. Under $\beta < 0$, the first part is an immediate result from Lemma 3.4. For the second part, in order to state conveniently, we introduce some notations

$$W_n^* = X_n^* - Y_n^*,\ \Delta f_n^* = f(X_n^*, \tilde{X}_n^*) - f(Y_n^*, \tilde{Y}_n^*),\ \Delta g_n^* = g(X_n^*, \tilde{X}_n^*) - g(Y_n^*, \tilde{Y}_n^*). \qquad (4.5)$$

From (3.8), we have

$$W_n^* = W_{n-1}^* + h\Delta f_n^* + \Delta g_{n-1}^* \Delta w_{n-1}. \qquad (4.6)$$

Thus

$$|W_n^* - h\Delta f_n^*|^2 = |W_{n-1}^*|^2 + 2\langle W_{n-1}^*, \Delta g_{n-1}^* \Delta w_{n-1}\rangle + |\Delta g_{n-1}^* \Delta w_{n-1}|^2.$$

Taking expectation and using (3.3) yields

$$\mathbb{E}|W_n^*|^2 - 2h\mathbb{E}\langle W_n^*, \Delta f_n^*\rangle \leq (1 + h\gamma_3)\mathbb{E}|W_{n-1}^*|^2 + h\gamma_4 \mathbb{E}|\tilde{X}_{n-1}^* - \tilde{Y}_{n-1}^*|^2. \tag{4.7}$$

Now using the Cauchy-Schwarz inequality and conditions (3.1)-(3.2), we have

$$\begin{aligned} 2\mathbb{E}\langle W_n^*, \Delta f_n^*\rangle &= 2\mathbb{E}\langle W_n^*, f(X_n^*, \tilde{X}_n^*) - f(Y_n^*, \tilde{X}_n^*)\rangle \\ &\quad + 2\mathbb{E}\langle W_n^*, f(Y_n^*, \tilde{X}_n^*) - f(Y_n^*, \tilde{Y}_n^*)\rangle \\ &\leq 2\gamma_1 \mathbb{E}|W_n^*|^2 + 2\gamma_2 \mathbb{E}|W_n^*||\tilde{X}_n^* - \tilde{Y}_n^*| \\ &\leq (2\gamma_1 + \gamma_2)\mathbb{E}|W_n^*|^2 + \gamma_2 \mathbb{E}|\tilde{X}_n^* - \tilde{Y}_n^*|^2. \end{aligned}$$

Inserting it into (4.7) gives

$$\begin{aligned} (1 - 2h\gamma_1 - h\gamma_2)\mathbb{E}|X_n^* - Y_n^*|^2 &\leq (1 + h\gamma_3)\mathbb{E}|X_{n-1}^* - Y_{n-1}^*|^2 \\ &\quad + h\gamma_4 \mathbb{E}|\tilde{X}_{n-1}^* - \tilde{Y}_{n-1}^*|^2 + h\gamma_2 \mathbb{E}|\tilde{X}_n^* - \tilde{Y}_n^*|^2. \end{aligned} \tag{4.8}$$

Here we have to consider which approach is chosen to treat memory values on non-grid points, piecewise constant interpolation ($\mu \equiv 0$) or piecewise linear interpolation. In the latter case, let us consider two possible cases:

- If $\tau(t_n) = \tilde{\mu}h$, $0 \leq \tilde{\mu} < 1$, then

$$\begin{aligned} \mathbb{E}|\tilde{X}_n^* - \tilde{Y}_n^*|^2 &= \mathbb{E}|\tilde{\mu}\tilde{X}_{n-1}^* + (1-\tilde{\mu})\tilde{X}_n^* - \tilde{\mu}\tilde{Y}_{n-1}^* - (1-\tilde{\mu})\tilde{Y}_n^*|^2 \\ &\leq \tilde{\mu}\mathbb{E}|\tilde{X}_{n-1}^* - \tilde{Y}_{n-1}^*|^2 + (1-\tilde{\mu})\mathbb{E}|\tilde{X}_n^* - \tilde{Y}_n^*|^2. \end{aligned} \tag{4.9}$$

Inserting (4.9), we derive from (4.8) that

$$\begin{aligned} &[1 - 2h\gamma_1 - (2-\tilde{\mu})h\gamma_2]\mathbb{E}|X_n^* - Y_n^*|^2 \\ &\leq (1 + h\gamma_3 + \tilde{\mu}h\gamma_2)\mathbb{E}|X_{n-1}^* - Y_{n-1}^*|^2 + h\gamma_4 \mathbb{E}|\tilde{X}_{n-1}^* - \tilde{Y}_{n-1}^*|^2. \end{aligned}$$

Hence using the fact $\beta < 0$ in (4.2) gives

$$\begin{aligned} \mathbb{E}|X_n^* - Y_n^*|^2 &\leq \frac{1 + h\gamma_3 + \tilde{\mu}h\gamma_2 + h\gamma_4}{1 - 2h\gamma_1 - (2-\tilde{\mu})h\gamma_2} \max_{n-\kappa-1 \leq i \leq n-1} \mathbb{E}|X_i^* - Y_i^*|^2 \\ &\leq \frac{1 + h\gamma_2 + h\gamma_3 + h\gamma_4}{1 - 2h\gamma_1 - h\gamma_2} \max_{n-\kappa-1 \leq i \leq n-1} \mathbb{E}|X_i^* - Y_i^*|^2. \end{aligned} \tag{4.10}$$

- If $\tau(t_n) \geq h$, it follows from (4.8) and $\beta < 0$ that

$$\mathbb{E}|X_n^* - Y_n^*|^2 \leq \frac{1 + h\gamma_2 + h\gamma_3 + h\gamma_4}{1 - 2h\gamma_1 - h\gamma_2} \max_{n-\kappa-1 \leq i \leq n-1} \mathbb{E}|X_i^* - Y_i^*|^2. \tag{4.11}$$

Therefore, it is always true that inequality (4.11) holds for piecewise linear interpolation case. Obviously (4.11) also stands in piecewise constant interpolation case.

Further, from (1.2a) one sees

$$|X_0^* - Y_0^* - h(f(X_0^*, \tilde{X}_0^*) - f(Y_0^*, \tilde{Y}_0^*))|^2 = |X_0 - Y_0|^2.$$

Using a similar approach as before, one can derive

$$\mathbb{E}|X_0^* - Y_0^*|^2 \leq \frac{1 + h\gamma_2}{1 - 2h\gamma_1 - h\gamma_2}\mathbb{E}\|\psi - \phi\|^2 \leq \mathbb{E}\|\psi - \phi\|^2. \tag{4.12}$$

Denote

$$\beta_h := \frac{1 + h\gamma_2 + h\gamma_3 + h\gamma_4}{1 - 2h\gamma_1 - h\gamma_2}. \tag{4.13}$$

Noticing that $\beta < 0$, one can readily derive $0 < \beta_h < 1$, we can deduce from (4.11) and (4.12) that

$$\mathbb{E}|X_{n-1}^* - Y_{n-1}^*|^2 \leq \beta_h^{\lfloor \frac{n-2}{\kappa+1} \rfloor + 1} E\|\psi - \phi\|^2 \leq \beta_h^{\frac{n-2}{\kappa+1}} E\|\psi - \phi\|^2.$$

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.
Finally from (1.2b), we have for large n such that $\frac{\gamma_3+\gamma_4}{n} + \frac{n-\kappa-2}{nh(\kappa+1)} \ln \beta_h < \frac{\ln \beta_h}{2(\kappa+1)h}$

$$\begin{aligned}
\mathbb{E}|X_n - Y_n|^2 &\leq (1 + h\gamma_3)\mathbb{E}|X_{n-1}^* - Y_{n-1}^*|^2 + h\gamma_4\mathbb{E}|\tilde{X}_{n-1}^* - \tilde{Y}_{n-1}^*|^2 \\
&\leq (1 + h\gamma_3)\beta_h^{\frac{n-2}{\kappa+1}} E\|\psi - \phi\|^2 + h\gamma_4\beta_h^{\frac{n-\kappa-2}{\kappa+1}} E\|\psi - \phi\|^2 \\
&\leq \mathrm{e}^{(\gamma_3+\gamma_4)h}\beta_h^{\frac{n-\kappa-2}{\kappa+1}} E\|\psi - \phi\|^2 \\
&\leq \mathbb{E}\|\phi - \psi\|^2 \exp\{-\nu_h^+ nh\},
\end{aligned} \tag{4.14}$$

where ν_h^+ is defined as in (4.4).

The stability result indicates that the method (1.2a)-(1.2b) can well reproduce long-time stability of the continuous system satisfying conditions stated in Theorem 4.1. Note that the exponential mean-square stability under non-global Lipschitz conditions has been studied in [8] in the case of nonlinear SDEs without delay. The preceding results can be regarded as an extension of those in [8] to delay case.

5 Mean-square linear stability

Although the main focus of this work is on nonlinear SDDEs, in this section we show that the SSBE (1.2a)-(1.2b) has a very desirable linear stability property. Hence, we consider the scalar, linear test equation [15, 23] given by

$$\mathrm{d}x(t) = (ax(t) + bx(t - \tau))\mathrm{d}t + (cx(t) + dx(t - \tau))\mathrm{d}w(t). \tag{5.1}$$

Note that (5.1) is a special case of (1.1) with $\tau(t) = \tau$, and satisfies conditions (3.1)-(3.3) with

$$\gamma_1 = a, \quad \gamma_2 = |b|, \quad \gamma_3 = c^2 + |cd|, \quad \gamma_4 = d^2 + |cd|.$$

By Theorem 4.1, (5.1) is mean-square stable if

$$a < -|b| - \frac{1}{2}(|c| + |d|)^2. \tag{5.2}$$

For constraint stepsize $h = \tau/\kappa, 1 \le \kappa \in \mathbb{Z}^+$, i.e., $\delta = 0$ in (4.1), the SSBE proposed in our work applied to (5.1) produces

$$\begin{cases} y_n^* &= y_n + h[ay_n^* + by_{n-\kappa}^*], \\ y_{n+1} &= y_n^* + [cy_n^* + dy_{n-\kappa}^*]\Delta w_n. \end{cases} \tag{5.3}$$

In [23], the authors constructed a different SSBE for the linear test equation (5.1) and their method applied to (5.1) reads

$$\begin{cases} z_n^* &= z_n + h[az_n^* + bz_{n-\kappa+1}], \\ z_{n+1} &= z_n^* + [cz_n^* + dz_{n-\kappa+1}]\Delta w_n. \end{cases} \tag{5.4}$$

The stability results there [23, Theorem 4.1] indicate that under (5.2) the method (5.4) can only preserve mean-square stability of (5.1) with stepsize restrictions, but the new scheme (5.3) exhibits a better stability property.

Corollary 5.1 *For the linear equation (5.1), if (5.2) holds, then the SSBE (5.3) is mean-square stable for any stepsize $h = \tau/\kappa, 1 \le \kappa \in \mathbb{Z}^+$.*

Proof. The assertion readily follows from Theorem 4.2.

Apparently, the SSBE (5.3) achieves an advantage over (5.4) in stability property that the SSBE (5.3) is able to inherit stability of (5.1) for any stepsize $h = \tau/\kappa, 1 \le \kappa \in \mathbb{Z}^+$. If one drops the stepsize restriction $h = \frac{\tau}{\kappa}, \kappa \in \mathbb{Z}^+$ and allow for arbitrary stepsize $h > 0$, one can arrive at a sharper stability result from Theorem 4.2.

Corollary 5.2 *For the linear equation (5.1), if (5.2) holds, then the SSBE(1.2a)-(1.2b) is mean-square stable for any stepsize $h > 0$.*

6 Numerical experiments

In this section we give several numerical examples to illustrate intuitively the strong convergence and the mean-square stability obtained in previous sections.

6.1 A linear example

The first test equation is a linear Itô SDDE

$$\begin{cases} \mathrm{d}x(t) = (ax(t) + bx(t-1))\mathrm{d}t + (cx(t) + dx(t-1))\mathrm{d}w(t), \\ x(t) = 0.5, \quad t \in [-1, 0]. \end{cases} \tag{6.1}$$

Denoting $y_N^{(i)}$ as the numerical approximation to $x^{(i)}(t_N)$ at end point t_N in the i-th simulation of all M simulations, we approximate means of absolute errors ϵ as

$$\epsilon = \frac{1}{M}\sum_{i=1}^{M} |y_N^{(i)} - y^{(i)}(t_N)|.$$

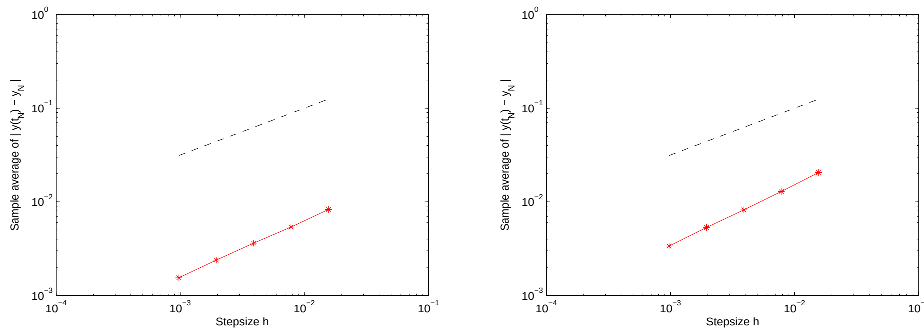

Figure 1: $\log \epsilon$ with $t_N = 1$ versus $\log h$ for Example I (left) and Example II (right).

Table 1: Numerical results for Example II and III with $t_N = 8$.

	Example II			Example III		
h	EM	SSBE (5.4)	SSBE (5.3)	EM	SSBE (5.4)	SSBE (5.3)
2^{-7}	0.0008	0.0011	0.0008	0.0014	0.0020	0.0014
2^{-6}	0.0013	0.0016	0.0013	0.0025	0.0036	0.0023
2^{-5}	0.0021	0.0029	0.0019	0.0058	0.0070	0.0035
2^{-4}	0.0034	0.0058	0.0027	0.2744	0.0157	0.0053
2^{-3}	0.0086	0.0148	0.0038	6.1598e+010	0.0628	0.0078

In our experiments, we use the SSBE (5.3) to compute an "exact solution" with small stepsize $h = 2^{-12}$ and $M = 5000$. We choose two sets of parameters as follows

- Example I: $a = -2, b = 1, c = d = 0.5$;
- Example II: $a = -6, b = 3, c = d = 1$.
- Example III: $a = -20, b = 12, c = 2, d = 1$.

In Figure 1, computational errors ϵ versus stepsize h on a log-log scale are plotted and dashed lines of slope one half are added. One can clearly see that SSBE (5.3) for linear test equation (6.1) is convergent and has strong order of 1/2. In Table 1, computational errors ϵ with $t_N = 8$ are presented for the well-known Euler-Maruyama method [18], the SSBE method (5.4) and the improved SSBE method (5.3) in this paper. There one can find that the improved SSBE method (5.3) has the best accuracy among the three methods. In particular, for Example III with stiffness in drift term (i.e., $a = -20$), when the moderate stepsize $h = 1/8$ was used, the Euler-Maruyama method becomes unstable and the two SSBE methods still remain stable, but with the improved SSBE (5.3) producing better result.

To compare stability property of the improved SSBE and SSBE in [23], simulations by SSBE (5.3) and (5.4) are both depicted in Figure 2, 3. There solutions produced by (5.3) and (5.4) are plotted in solid line and dashed line, respectively. As is shown in the figures, methods (5.3) and (5.4) exhibit different stability behavior. One can observe from Figure 2 that (5.3) for Example II is mean-square stable for $h = 1, 1/2, 1/3, 1/4$. But (5.4) is unstable for $h = 1, 1/2$. For Example III, the improved SSBE (5.3) is always stable for $h = 1, 1/4, 1/6, 1/10$, but (5.4) becomes stable when the stepsize h decreases to $h = 1/10$.

The numerical results demonstrate that the scheme (5.3) has a greater advantage in mean-square stability than (5.4).

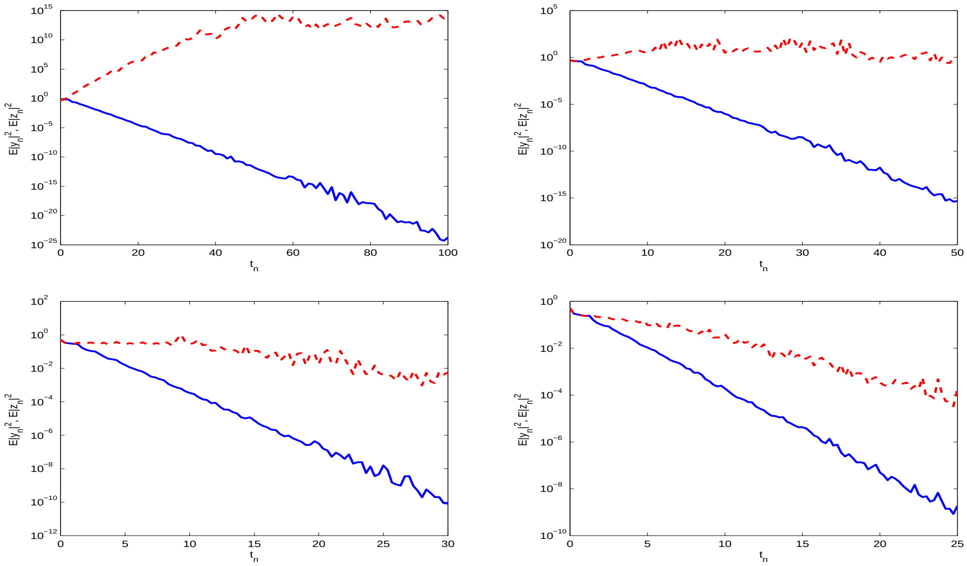

Figure 2: Simulations for (6.1) with $a = -6, b = 3, c = d = 1$. Upper left: $h = 1$, upper right: $h = 1/2$, lower left: $h = 1/3$, lower right: $h = 1/4$.

6.2 A nonlinear example

Consider a nonlinear SDDE with a time-varying delay as follows

$$\begin{cases} dx(t) = [-4x(t) - 3x^3(t) + x(t-\tau(t))]\, dt + [x(t) + x(t-\tau(t))]\, dw(t), t > 0, \\ x(t) = 1, \quad t \in [-1, 0], \end{cases} \tag{6.2}$$

where $\tau(t) = \frac{1}{1+t^2}$. Obviously, equation (6.2) satisfies conditions (3.1)-(3.3) in Assumption 3.1, with $\gamma_1 = -4, \gamma_2 = 1, \gamma_3 = \gamma_4 = 2$. Thus $2\gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4 = -2 < 0$ and the problem is exponentially mean-square stable. As is shown in Figure 4, the SSBE (5.3) can well reproduce stability for quite large stepsize $h = 1, 2, 5$. This is consistent with our result established in Theorem 4.2.

Appendix

Proof of Theorem 3.2. Since both f and g are locally Lipschitz continuous, Theorem 3.2.2 of [16] shows that there is a unique maximal local solution $x(t)$ on $t \in [[0, \rho_\infty[[$, where the stopping time $\rho_R = \inf\{t \geq 0 : |x(t)| \geq R\}$. By Itô's formula we obtain that for $t \geq 0$

$$|x(t \wedge \rho_R)|^2 = |\psi(0)|^2 + 2\int_0^{t\wedge\rho_R} x(s)^T f(x(s), x(s-\tau(s)))\mathrm{d}s$$
$$+2\int_0^{t\wedge\rho_R} x(s)^T g(x(s), x(s-\tau(s)))\mathrm{d}w_s + \int_0^{t\wedge\rho_R} |g(x(s), x(s-\tau(s)))|^2\mathrm{d}s$$

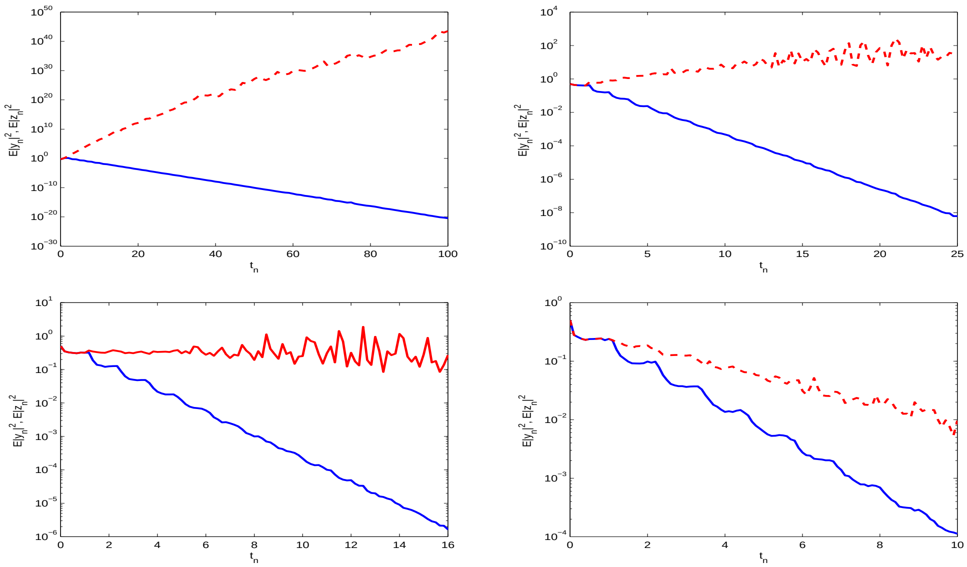

Figure 3: Simulations for (6.1) with $a = -20, b = 12, c = 2, d = 1$. Upper left: $h = 1$, upper right: $h = 1/4$, lower left: $h = 1/6$, lower right: $h = 1/10$.

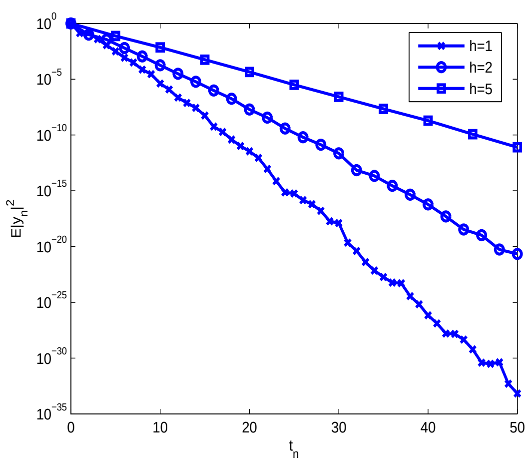

Figure 4: Simulations for (6.2) by SSBE (5.3) using various stepsizes.

$$
\le |\psi(0)|^2 + 3K\int_0^{t\wedge\rho_R}(1+|x(s)|^2+|x(s-\tau(s))|^2)\mathrm{d}s
$$

$$
+2\int_0^{t\wedge\rho_R} x(s)^T g(x(s),x(s-\tau(s)))\mathrm{d}w_s, \tag{6.3}
$$

where the condition (3.6) was used. Thus

$$
\begin{aligned}
\sup_{0\le s\le t}|x(s\wedge\rho_R)|^2 \;\; &\le \;\; |\psi(0)|^2 + 3K\int_0^t(1+2\sup_{0\le r\le s}|x(r\wedge\rho_R)|^2+\|\psi\|^2)\mathrm{d}s \\
&\quad +2\sup_{0\le s\le t}\int_0^{s\wedge\rho_R} x(r)^T g(x(r),x(r-\tau(r)))\mathrm{d}w_r. \qquad (6.4)
\end{aligned}
$$

Now, raising both sides of (6.4) to the power $p/2$ and using Hölder's inequality yield

$$
\begin{aligned}
&\sup_{0\le s\le t}|x(s\wedge\rho_R)|^p \le 3^{p/2-1}\{|\psi(0)|^p \\
&+(3K)^{p/2}(3T)^{p/2-1}\int_0^t(1+2^{p/2}\sup_{0\le r\le s}|x(r\wedge\rho_R)|^p+\|\psi\|^p)\mathrm{d}s \\
&+2^{p/2}\sup_{0\le s\le t}\left|\int_0^{s\wedge\rho_R} x(r)^T g(x(r),x(r-\tau(r)))\mathrm{d}w_r\right|^{p/2}\Bigg\}. \qquad (6.5)
\end{aligned}
$$

By the Burkholder-Davis-Gundy inequality [17], one computes that, with $c_1 = c_1(p,T)$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le t}|x(s\wedge\rho_R)|^p\right] \quad &\le c_1\Bigg\{1+\mathbb{E}\|\psi\|^p+\int_0^t\mathbb{E}\sup_{0\le r\le s}|x(r\wedge\rho_R)|^p\mathrm{d}s \\
&+\mathbb{E}\left[\int_0^{t\wedge\rho_R}|x(s)|^2|g(x(s),x(s-\tau(s)))|^2\mathrm{d}s\right]^{p/4}\Bigg\}. \qquad (6.6)
\end{aligned}
$$

Next, by an elementary inequality,

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{t\wedge\rho_R}|x(s)|^2|g(x(s),x(s-\tau(s)))|^2\mathrm{d}s\right]^{p/4} \\
\le\;\; &\mathbb{E}\left[\sup_{0\le s\le t}|x(s\wedge\rho_R)|^2\int_0^{t\wedge\rho_R}|g(x(s),x(s-\tau(s)))|^2\mathrm{d}s\right]^{p/4} \\
\le\;\; &\frac{1}{2c_1}\mathbb{E}\left[\sup_{0\le s\le t}|x(s\wedge\rho_R)|^p\right]+\frac{c_1}{2}T^{p/2-1}\mathbb{E}\int_0^{t\wedge\rho_R}|g(x(s),x(s-\tau(s)))|^p\mathrm{d}s \\
\le\;\; &\frac{1}{2c_1}\mathbb{E}\left[\sup_{0\le s\le t}|x(s\wedge\rho_R)|^p\right]+\frac{c_1}{2}(3T)^{p/2-1}K^{p/2}\int_0^t(1+\mathbb{E}\sup_{0\le r\le s}|x(r\wedge\rho_R)|^p+\mathbb{E}\|\psi\|^p)\mathrm{d}s.
\end{aligned}
$$

Inserting it into (6.6), for proper constants c_2, c_3 we have that

$$
\mathbb{E}\sup_{0\le s\le t}|x(s\wedge\rho_R)|^p \le c_2(1+\mathbb{E}\|\psi\|^p)+c_3\int_0^t\mathbb{E}\sup_{0\le r\le s}|x(r\wedge\rho_R)|^p\mathrm{d}s.
$$

The Gronwall inequality gives

$$\mathbb{E} \sup_{0 \le s \le T} |x(s \wedge \rho_R)|^p \le c_2(1 + \mathbb{E}\|\psi\|^p)\mathrm{e}^{c_3 T}. \tag{6.7}$$

This implies

$$R^p \mathbb{P}\{\rho_R \le T\} \le c_2(1 + \mathbb{E}\|\psi\|^p)\mathrm{e}^{c_3 T}.$$

Letting $R \to \infty$ leads to

$$\lim_{R \to \infty} \mathbb{P}\{\rho_R \le T\} = 0.$$

Since $T > 0$ is arbitrary, we must have $\rho_R \to \infty$ a.s. and hence $\rho_\infty = \infty$ a.s. The existence and uniqueness of the global solution is justified. Finally, the desired moment bound follows from (6.7) by letting $R \to \infty$ and setting $C = c_2 e^{c_3 T}$.

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