Hausdorff limits of Rolle leaves

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Let $\mathcal{R}$ be an o-minimal expansion of the real field. We introduce a class of Hausdorff limits, the T^∞-limits over $\mathcal{R}$, that do not in general fall under the scope of Marker and Steinhorn's definability-of-types theorem. We prove that if $\mathcal{R}$ admits analytic cell decomposition, then every T^∞-limit over $\mathcal{R}$ is definable in the pfaffian closure of $\mathcal{R}$.

Introduction

We fix an o-minimal expansion $\mathcal{R}$ of the real field. In this paper, we study T^∞-limits over $\mathcal{R}$ as defined in Section 1 below; they generalize the pfaffian limits over $\mathcal{R}$ introduced in [5, Section 4]. Pfaffian limits over $\mathcal{R}$ are definable in the pfaffian closure $\mathcal{P}(\mathcal{R})$ of $\mathcal{R}$ [7], by the variant of Marker and Steinhorn's definability-of-types theorem [6] found in van den Dries [1, Theorem 3.1] and [4, Theorem 1]. The T^∞-limits over $\mathcal{R}$ considered here do not seem to fall under the scope of these theorems, as explained in Section 1 below. Nevertheless, T^∞-limits were used by Lion and Rolin [3] to establish the o-minimality of the expansion of $\mathbb{R}_{\text{an}}$ by all Rolle leaves over $\mathbb{R}_{\text{an}}$ of codimension one.

To state our results, we work in the setting of [5, Introduction]; in particular, recall that a set $W \subseteq \mathbb{R}^n$ is a **Rolle leaf over** $\mathcal{R}$ if there exists a nested Rolle leaf $(W_0, \dots, W_k)$ over $\mathcal{R}$ such that $W = W_k$.

First, we obtain the following generalization of [3, Théorème 1].

Proclaim (Theorem A) *Let $\mathcal{N}(\mathcal{R})$ be the expansion of $\mathcal{R}$ by all Rolle leaves over $\mathcal{R}$.*

(1) *There is an o-minimal expansion $T^\infty(\mathcal{R})$ of $\mathcal{N}(\mathcal{R})$ in which every T^∞-limit over $\mathcal{R}$ is definable.*

(2) *Let $M \subseteq \mathbb{R}^n$ be a bounded, definable C^2-manifold and d be a definable and integrable nested distribution on M. Let $K \subseteq \mathbb{R}^n$ be a T^∞-limit obtained from d. Then* $\dim K \leq \dim d$.

The question then arises how $T^\infty(\mathcal{R})$ relates to the pfaffian closure $\mathcal{P}(\mathcal{R})$ of $\mathcal{R}$. Indeed, we do not know in general if $T^\infty(\mathcal{R})$ is interdefinable with $\mathcal{N}(\mathcal{R})$ or $\mathcal{P}(\mathcal{R})$, or if $T^\infty(T^\infty(\mathcal{R}))$ is interdefinable with $T^\infty(\mathcal{R})$. Based on [5], we can answer such questions under an additional hypothesis:

Proclaim (Theorem B) *Assume that $\mathcal{R}$ admits analytic cell decomposition.*

(1) *Every T^∞-limit over $\mathcal{P}(\mathcal{R})$ is definable in $\mathcal{P}(\mathcal{R})$.*

(2) *The structures $T^\infty(\mathcal{R})$ and $\mathcal{P}(\mathcal{R})$ are interdefinable; in particular, $T^\infty(\mathcal{R})$ and $T^\infty(T^\infty(\mathcal{R}))$ are interdefinable.*

We view the combination of Theorems A(2) and B(1) as a non-first order extension of [1, Theorem 3.1] and [4, Theorem 1].

Our proofs of these theorems rely heavily on terminology and notation introduced in [5, Introduction and Section 2]; we do not repeat the respective definitions here. We prove Theorem A in Section 3 below using the approach of [7], but based on a straightforward adaptation of some results of [5, Section 4] to T^∞-limits carried out in Section 2 below. Theorem B then follows by adapting [5, Proposition 7.1] to T^∞-limits and using [5, Proposition 10.4]; the details are given in Section 4.

1 The definitions

Let $M \subseteq \mathbb{R}^n$ be a bounded, definable C^2-manifold of dimension m. We adopt the terminology and results found in [5, Introduction and Section 2], and we let $d = (d_0, \dots, d_k)$ be a definable and integrable nested distribution on M.

A sequence $(V_\iota)_{\iota\in\mathbb{N}}$ of integral manifolds of d_k is a **T^∞-sequence of integral manifolds of** d if there are a core distribution $e = (e_0, \dots, e_l)$ of d, a sequence (W_ι) of Rolle leaves of e and a definable family $\mathcal{B}$ of closed integral manifolds of d_{k-l} such that each V_ι is an admissible integral manifold of d with core W_ι corresponding to e and definable part in $\mathcal{B}$ corresponding to W_ι, as defined in [5, Definition 4.1].

In this situation, we call (W_ι) the **core sequence** of the sequence (V_ι) **corresponding to** e and $\mathcal{B}$ a **definable part** of the sequence (V_ι) **corresponding to** (W_ι).

Remarks (1) We think of the core sequence of (V_ι) as representing the "non-definable part" of (V_ι). If $W_\iota = W_1$ for all ι, then (V_ι) is an admissible sequence of integral manifolds of d as defined in [5, Definition 4.3].

(2) Let (V_ι) be a T^∞-sequence of integral manifolds of d. Then there is a T^∞-sequence (U_ι) of integral manifolds of $(d_0,\dots,d_{k-1})$ such that $V_\iota \subseteq U_\iota$ for $\iota \in \mathbb{N}$.

Let (V_ι) be a T^∞-sequence of integral manifolds of d. If (V_ι) converges to K in $\mathcal{K}_n$ (the space of all compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric), we call K a T^∞**-limit over** $\mathcal{R}$. In this situation, we say that K **is obtained from** d, and we put

$$\deg K := \min\{\deg f :\ K \text{ is obtained from } f\}.$$

Remarks (3) It is unknown whether the family of all Rolle leaves of e is definable in $\mathcal{P}(\mathcal{R})$ [1]. As a consequence, contrary to the situation described by [5, Lemma 4.5] for pfaffian limits over $\mathcal{R}$, the variant of Marker and Steinhorn's definability-of-types theorem [6] found in [1, Theorem 3.1] and [4, Theorem 1] does not apply; in particular, we do not know in general whether a T^∞-limit over $\mathcal{R}$ is definable in $\mathcal{P}(\mathcal{R})$.

(4) If $W_\iota = W_1$ for all ι, then K is a pfaffian limit over $\mathcal{R}$ as introduced in [5, Definition 4.4].

2 Towards the proof of Theorem A

Let $M \subseteq \mathbb{R}^n$ be a definable C^2-manifold of dimension m.

Pfaffian fiber cutting

We fix a finite family $\Delta = \{d^1,\dots,d^q\}$ of definable nested distributions on M; we write $d^p = (d^p_0,\dots,d^p_{k(p)})$ for $p = 1,\dots,q$. As in [5, Section 3], we associate to Δ the following set of distributions on M:

$$\mathcal{D}_\Delta := \left\{d^0_0 \cap d^1_{k(1)} \cap \cdots \cap d^{p-1}_{k(p-1)} \cap d^p_j :\ p = 1,\dots,q \text{ and } j = 0,\dots,k(p)\right\},$$

where we put $d^0_0 := g_M$. If N is a C^2-submanifold of M compatible with $\mathcal{D}_\Delta$, we let $d^{\Delta,N} = \left(d^{\Delta,N}_0,\dots,d^{\Delta,N}_{k(\Delta,N)}\right)$ be the nested distribution on N obtained by listing the set $\left\{g^N :\ g \in \mathcal{D}_\Delta\right\}$ in order of decreasing dimension. In this situation, if V_p is an integral manifold of $d^p_{k(p)}$, for $p = 1,\dots,q$, then the set $N \cap V_1 \cap \cdots \cap V_q$ is an integral manifold of $d^{\Delta,N}_{k(\Delta,N)}$.

Let $A \subseteq M$ be definable. For $I \subseteq \{1,\dots,q\}$ we put $\Delta(I) := \{d^p :\ p \in I\}$.

[1] For instance, a positive answer to this question for all e definable in $\mathcal{P}(\mathcal{R})$ would imply the second part of Hilbert's 16th problem.

Lemma 2.1 *Let $I \subseteq \{1,\dots,q\}$. Then there is a finite partition $\mathcal{P}$ of definable C^2-cells contained in A such that $\mathcal{P}$ is compatible with $\mathcal{D}_{\Delta(J)}$ for every $J \subseteq \{1,\dots,q\}$ and*

(i) $\dim d^{\Delta(I),N}_{k(\Delta(I),N)} = 0$ *for every $N \in \mathcal{P}$;*

(ii) *whenever V_p is a Rolle leaf of d^p for $p \in I$, every component of $A \cap \bigcap_{p\in I} V_p$ intersects some cell in $\mathcal{P}$.*

Proof By induction on $\dim A$; if $\dim A = 0$, there is nothing to do, so we assume $\dim A > 0$ and the corollary is true for lower values of $\dim A$. By [5, Proposition 2.2] and the inductive hypothesis, we may assume that A is a C^2-cell compatible with $\mathcal{D}_{\Delta(J)}$ for $J \subseteq \{1,\dots,q\}$. Thus, if $\dim d^{\Delta(I),A}_{k(\Delta(I),A)} = 0$, we are done; otherwise, we let ϕ and B be as in [5, Lemma 3.1] with $\Delta(I)$ in place of Δ.

Let V_p be a Rolle leaf of d^p for each p; it suffices to show that every component of $X := A \cap \bigcap_{p\in I} V_p$ intersects B. However, since $d^{\Delta(I),A}_{k(\Delta(I),A)}$ has dimension, X is a closed, embedded submanifold of A. Thus, ϕ attains a maximum on every component of X, and any point in X where ϕ attains a local maximum belongs to B. ◻

Corollary 2.2 *Let d be a definable nested distribution on M and $m \le n$. Then there is a finite partition $\mathcal{P}$ of C^2-cells contained in A such that for every Rolle leaf V of d, we have*

$$\Pi_m(A \cap V) = \bigcup_{N\in\mathcal{P}} \Pi_m(N \cap V)$$

and for every $N \in \mathcal{P}$, the set $N \cap V$ is a submanifold of U, $\Pi_m\restriction_{(N\cap V)}$ is an immersion and for every $n' \le n$ and every strictly increasing $\lambda : \{1,\dots,n'\} \longrightarrow \{1,\dots,n\}$, the projection $\Pi_\lambda\restriction_{(N\cap V)}$ has constant rank.

Proof Apply Lemma 2.1 with $q := n+1$, $d^p := \ker dx_p$ for $p = 1,\dots,n$, $d^q := d$ and $I := \{1,\dots,m,n+1\}$. ◻

T^∞-limits

We assume that M has a definable C^2-carpeting function ϕ, and we let $d = (d_0,\dots,d_k)$ be a definable distribution on M with core distribution $e = (e_0,\dots,e_l)$.

First, we reformulate [5, Proposition 4.7]. We adopt the notation introduced before [5, Proposition 4.6] and note that the q in [5, Remark 4.2] can be chosen independent of the particular W.

Proposition 2.3 *Let (V_ι) be a T^∞-sequence of integral manifolds of d with core sequence (W_ι), and assume that $K' := \lim_\iota \operatorname{fr} V_\iota$ exists. Then K' is a finite union of T^∞-limits obtained from $d^{M'}$ with core sequences among $\big((W_\iota)_1^{M'}\big)_\iota, \dots, \big((W_\iota)_q^{M'}\big)_\iota$.*

Proof Exactly as for [5, Proposition 4.7], except for replacing "core W" by "core sequence (W_ι)" and "core $W_p^{M'}$" by "core sequence $\big((W_\iota)_p^{M'}\big)$". □

Second, as we do not know yet whether T^∞-limits are definable in an o-minimal structure, we work with the following notion of dimension (see also van den Dries and Speissegger [2, Section 8.2]): we call $N \subseteq \mathbb{R}^n$ a C^0**-manifold of dimension** p if $N \neq \emptyset$ and each point of N has an open neighbourhood in N homeomorphic to $\mathbb{R}^p$; in this case p is uniquely determined (by a theorem of Brouwer), and we write $p = \dim(N)$. Correspondingly, a set $S \subseteq \mathbb{R}^n$ **has dimension** if S is a countable union of C^0-manifolds, and in this case put

$$\dim(S) := \begin{cases} \max\{\dim(N) : N \subseteq S \text{ is a } C^0\text{-manifold}\} & \text{if } S \neq \emptyset \\ -\infty & \text{otherwise.} \end{cases}$$

It follows (by a Baire category argument) that, if $S = \bigcup_{i\in\mathbb{N}} S_i$ and each S_i has dimension, then S has dimension and $\dim(S) = \max\{\dim(S_i) : i \in \mathbb{N}\}$. Thus, if N is a C^1-manifold of dimension p, then N has dimension in the sense of this definition and the two dimensions of N agree.

Corollary 2.4 *In the situation of [5, Lemma 1.5], the set $\lim_\iota V_\iota \setminus \lim_\iota \operatorname{fr} V_\iota$ is either empty or has dimension p.* □

Therefore, we replace [5, Lemma 4.5] by

Proposition 2.5 *Let K be a T^∞-limit obtained from d. Then K has dimension and satisfies $\dim K \leq \dim d$.*

Proof Let (V_ι) be a T^∞-sequence of integral manifolds of d such that $K = \lim_\iota V_\iota$. We proceed by induction on $\dim d$. If $\dim d = 0$, then [5, Corollary 3.3(2)] gives a uniform bound on the cardinality of V_ι, so K is finite. So assume $\dim d > 0$ and the corollary holds for lower values of $\dim d$.

By [5, Proposition 2.2 and Remark 4.2], we may assume that M is a definable C^2-cell; in particular, there is a definable C^2-carpeting function ϕ on M. For each $\sigma \in \Sigma_n$, let $M_{\sigma,2n}$ be as before [5, Lemma 1.3] with d_k in place of d. Then by that lemma, $M = \bigcup_{\sigma\in\Sigma} M_{\sigma,2n}$ and each $M_{\sigma,2n}$ is an open subset of M. Hence d is compatible with each $M_{\sigma,2n}$, and after passing to a subsequence if necessary, we may assume that $K_\sigma = \lim_\iota (V_\iota \cap M_{\sigma,2n})$ exists for each σ. It follows that $K = \bigcup_{\sigma \in \Sigma_n} K_\sigma$, so by [5, Lemma 1.3(2)], after replacing M with each $\sigma^{-1}(M_{\sigma,2n})$, we may assume that d_k is $2n$-bounded. Passing to a subsequence again, we may assume that $K' := \lim_\iota \operatorname{fr} V_\iota$ exists as well. Then by Corollary 2.4, the set $K \setminus K'$ is either empty or has dimension $\dim d$. By Proposition 2.3 and the discussion before [5, Proposition 4.6], the set K' is a finite union of T^∞-limits obtained from a definable nested distribution d' on a definable manifold M' that satisfies $\deg d' \le \deg d$ and $\dim d' < \dim d$. So K' has dimension with $\dim K' < \dim d$ by the inductive hypothesis, and the proposition is proved. □

Definition 2.6 A T^∞-limit $K \subseteq \mathbb{R}^n$ obtained from d is **proper** if $\dim K = \dim d$.

Corollary 2.7 *Let $K \subseteq \mathbb{R}^n$ be a T^∞-limit obtained from d. Then K is a finite union of proper T^∞-limits over $\mathcal{R}$ of degree at most* $\deg d$.

Proof We proceed by induction on $\dim d$; as in the previous proof, we assume $\dim d > 0$ and the corollary holds for lower values of $\dim d$. If $\dim K = \dim d$, we are done, so assume that $\dim K < \dim d$. Also as in the previous proof, we now reduce to the case where d_k is $2n$-bounded and $K' := \lim_\iota \operatorname{fr} V_\iota$ exists. Then Corollary 2.4 implies that $K = K'$, so the corollary follows from Proposition 2.3 and the inductive hypothesis. □

Finally, T^∞-limits over $\mathcal{R}$ are well behaved with respect to intersecting with closed definable sets. To see this, define $\mathbf{M} := M \times (0,1)$ and write (x,ϵ) for the typical element of $\mathbf{M}$ with $x \in M$ and $\epsilon \in (0,1)$. We consider the components of d as distributions on $\mathbf{M}$ in the obvious way, and we set $\mathbf{d}_0 := g_{\mathbf{M}}$, $\mathbf{d}_1 := d\epsilon\restriction_{\mathbf{M}}$ and $\mathbf{d}_{1+i} := d_i \cap \mathbf{d}_1$ for $i = 1, \dots, k$ and put $\mathbf{d} := (\mathbf{d}_0, \dots, \mathbf{d}_{1+k})$. Moreover, whenever e is a core distribution of d, we similarly define a corresponding core distribution $\mathbf{e} = (\mathbf{e}_0, \dots, \mathbf{e}_{1+l})$ of $\mathbf{d}$. In this situation, for every Rolle leaf W of e and every $\epsilon \in (0,1)$, the set $\mathbf{W} := W \times \{\epsilon\}$ is a Rolle leaf of $\mathbf{e}$.

Proposition 2.8 *Let K be a T^∞-limit obtained from d, and let $C \subseteq \mathbb{R}^n$ be a definable closed set. Then there is a definable open subset* $\mathbf{N}$ *of* $\mathbf{M}$ *and there are $q \in \mathbb{N}$ and T^∞-limits $K_1, \dots, K_q \subseteq \mathbb{R}^{n+1}$ obtained from $\mathbf{d}^{\mathbf{N}}$ such that $K \cap C = \Pi_n(K_1) \cup \cdots \cup \Pi_n(K_q)$.*

Sketch of proof For $\epsilon > 0$ put $T(C,\epsilon) := \{x \in \mathbb{R}^n : d(x,C) < \epsilon\}$. Note first that $K \cap C = \bigcap_{\epsilon>0} \left(K \cap T(C,\epsilon)\right)$, and the latter is equal to $\lim_{\epsilon \to 0} \left(K \cap T(C,\epsilon)\right)$ in the sense of [5, Definition 1.7]. Next, let (V_ι) be a T^∞-sequence of integral manifolds of d such that $K = \lim_\iota V_\iota$. Then for every $\epsilon > 0$, there is a subsequence $(\iota(\kappa))$ of (ι) such that the sequence $(V_{\iota(\kappa)} \cap T(C,\epsilon))$ converges to some compact set K_ϵ. Note that $K_\epsilon \cap T(C,\epsilon) = K \cap T(C,\epsilon)$, since $T(C,\epsilon)$ is an open set.

Fix a sequence (ϵ_κ) of positive real numbers approaching 0, and for each κ, choose $\iota(\kappa)$ such that $d(V_{\iota(\kappa)} \cap T(C,\epsilon_\kappa), K_{\epsilon_\kappa}) < \epsilon_\kappa$. Passing to a subsequence if necessary, we may assume that $\lim_\kappa K_{\epsilon_\kappa}$ and $\lim_\kappa \left(V_{\iota(\kappa)} \cap T(C,\epsilon_\kappa)\right)$ exist; note that these limits are then equal. Hence by the above,

$$\begin{aligned} K \cap C &= \lim_\kappa \left(K \cap T(C,\epsilon_\kappa)\right) = \lim_\kappa \left(K_{\epsilon_\kappa} \cap T(C,\epsilon_\kappa)\right) \\ &\subseteq \lim_\kappa K_{\epsilon_\kappa} = \lim_\kappa \left(V_{\iota(\kappa)} \cap T(C,\epsilon_\kappa)\right). \end{aligned}$$

The reverse inclusion is obvious, so $K \cap C = \lim_\kappa \left(V_{\iota(\kappa)} \cap T(C,\epsilon_\kappa)\right)$. Therefore, put $\mathbf{N} := \{(x,\epsilon) \in \mathbf{M} :\ d(x,C) < \epsilon\}$; then $\mathbf{N}$ is an open, definable subset of $\mathbf{M}$ and by the above $K \cap C = \lim_\kappa (V_{\iota(\kappa)} \cap \mathbf{N}^{\epsilon_\kappa})$, where $\mathbf{N}^\epsilon := \{x \in M :\ (x,\epsilon) \in \mathbf{N}\}$. Hence $K \cap C = \lim_\kappa \Pi_n\big((V_{\iota(\kappa)} \times \{\epsilon_\kappa\}) \cap \mathbf{N}\big)$. Since $\lim_\kappa \epsilon_\kappa = 0$, it follows that $K \cap C = \Pi_n\big(\lim_\kappa \big((V_{\iota(\kappa)} \times \{\epsilon_\kappa\}) \cap \mathbf{N}\big)\big)$. Since the sequence $\left(V_{\iota(\kappa)} \times \{\epsilon_\kappa\}\right)$ is a T^∞-sequence of integral manifolds of $\mathbf{d}$, the proposition now follows from [5, Remark 4.2]. □

Remark 2.9 Let $\mathcal{B}$ and $\mathcal{C}$ be two definable families of closed subsets of $\mathbb{R}^n$. Then the T^∞-limits in the previous proposition depend uniformly on $C \in \mathcal{C}$, for all T^∞-limits obtained from d with definable part $\mathcal{B}$. That is, there are $\mu, q \in \mathbb{N}$, a bounded, definable manifold $\mathbf{M} \subseteq \mathbb{R}^{n+\mu+1}$, a definable nested distribution $\mathbf{d}$ on $\mathbf{M}$ and a definable family $\mathbf{B}$ of subsets of $\mathbb{R}^{n+\nu+1}$ such that whenever K is a T^∞-limit obtained from d with definable part $\mathcal{B}$ and $C \in \mathcal{C}$, there are T^∞-limits $K_1, \dots, K_q \subseteq \mathbb{R}^{n+\nu+1}$ obtained from $\mathbf{d}$ with definable part $\mathbf{B}$ such that $K \cap C = \Pi_n(K_1) \cup \cdots \cup \Pi_n(K_q)$.

3 O-minimality and proof of Theorem A

Similar to [3, 7], we show that all sets definable in $T^\infty(\mathcal{R})$ are of the following form:

Definition 3.1 A set $X \subseteq \mathbb{R}^m$ is a **basic T^∞-set** if there exist $n \geq m$, a definable, bounded C^2-manifold $M \subseteq \mathbb{R}^n$, a definable nested distribution d on M with core distribution e and, for $\kappa \in \mathbb{N}$, a T^∞-sequence $(V_{\kappa,\iota})_\iota$ of integral manifolds of d with core sequence $(W_{\kappa,\iota})_\iota$ corresponding to e and definable part $\mathcal{B}$ independent of κ, such that:

(i) for each κ, the limit $K_\kappa := \lim_\iota V_{\kappa,\iota}$ exists in $\mathcal{K}_n$;

(ii) the sequence $(\Pi_m(K_\kappa))_\kappa$ is increasing and has union X.

In this situation, we say that X is **obtained from** d with **core distribution** e and **definable part** $\mathcal{B}$. A **T^∞-set** is a finite union of basic T^∞-sets. We denote by T^∞_m the collection of all T^∞-sets in $\mathbb{R}^m$ and put $T^\infty := (T^\infty_m)_{m \in \mathbb{N}}$.

Proposition 3.2 *In the situation of Definition 3.1, there is an $N \in \mathbb{N}$ such that every basic T^∞-set obtained from d with core distribution e and definable part $\mathcal{B}$ has at most N components. In particular, if $X \subseteq \mathbb{R}^m$ is a T^∞-set and $l \leq m$, there is an $N \in \mathbb{N}$ such that for every $a \in \mathbb{R}^l$ the fiber X_a has at most N components.*

Proof Let N be a bound on the number of components of the sets $W \cap B$ as W ranges over all Rolle leaves of e and B ranges over $\mathcal{B}$. Let X be a basic T^∞-set as in Definition 3.1. Then each $V_{\kappa,\iota}$ has at most N components, so each K_κ has at most N components, and hence X has at most N components. Combining this observation with Remark 2.9 yields, for every T^∞-set $X \subseteq \mathbb{R}^m$, a uniform bound on the number of connected components of the fibers of X. □

Proposition 3.3 (1) *Any coordinate projection of a T^∞-limit over $\mathcal{R}$ is a T^∞-set.*

(2) *Every bounded definable set is a T^∞-set.*

(3) *Let d be a definable nested distribution on $M := (-1,1)^n$ and L be a Rolle leaf of d. Then L is a T^∞-set.*

Proof (1) is obvious. For (2), let $C \subseteq \mathbb{R}^n$ be a bounded, definable cell. By cell decomposition, it suffices to show that C is a T^∞-set. Let ϕ be a definable carpeting function on C. Then $C = \bigcup_{i=1}^\infty \mathrm{cl}\big(\phi^{-1}((1/i,\infty))\big)$, so let $\mathbf{C} := \{(x,r) \in C \times (0,1) : \phi(x) > r\}$ and put $\mathbf{d}_1 := \ker dr\restriction_{\mathbf{C}}$ and $\mathbf{d} := (g_{\mathbf{C}}, \mathbf{d}_1)$. Then for $r > 0$, the set $\mathbf{C}^r = \phi^{-1}((r,\infty)) \times \{r\}$ is an admissible integral manifold of $\mathbf{d}$ with core $\mathbf{C}$ and definable part $\mathbf{C}^r$, so $\mathrm{cl}(\mathbf{C}^r)$ is a T^∞-limit obtained from $\mathbf{d}$.

(3) Let ϕ be a carpeting function on M. Then

$$L = \bigcup_{i=1}^\infty \mathrm{cl}\big(L \cap \phi^{-1}((1/i,\infty))\big),$$

so we let $\mathbf{M} := \{(x,r) \in M \times (0,1) : \phi(x) > r\}$ and put $\mathbf{d}_0 := g_{\mathbf{M}}$, $\mathbf{d}_1 := \ker dr\restriction_{\mathbf{M}}$, $\mathbf{d}_{1+i} := \mathbf{d}_1 \cap d_i$ for $i = 1,\dots,k$ and $\mathbf{d} := (\mathbf{d}_0,\dots,\mathbf{d}_{1+k})$. Let $L_1,\dots,L_q$ be the components of $(L \times (0,1)) \cap \mathbf{M}$; note that each L_p is a Rolle leaf of $\mathbf{d}$. Thus for $r > 0$ and each p, the set $L_p \cap \phi^{-1}((r,\infty))$ is an admissible integral manifold of $\mathbf{d}$ with core L_p and definable part $\mathbf{M}^r = \phi^{-1}((r,\infty)) \times \{r\}$. □

Proposition 3.4 *The collection of all T^∞-sets is closed under taking finite unions, finite intersections, coordinate projections, cartesian products, permutations of coordinates and topological closure.*

Proof Closure under taking finite unions, coordinate projections and permutations of coordinates is obvious from the definition and the properties of nested pfaffian sets over $\mathcal{R}$.

For topological closure, let $X \subseteq \mathbb{R}^m$ be a basic T^∞-set with associated data as in Definition 3.1. Then

$$\mathrm{cl}(X) = \lim_\kappa \Pi_m(K_\kappa) = \Pi_m(\lim_\kappa \lim_\iota V_{\kappa,\iota}) = \Pi_m(\lim_\kappa V_{\kappa,\iota(\kappa)})$$

for some subsequence $(\iota(\kappa))_\kappa$, so $\mathrm{cl}(X)$ is a T^∞-set by Proposition 3.3(1).

For cartesian products, let $X_1 \subseteq \mathbb{R}^{m_1}$ and $X_2 \subseteq \mathbb{R}^{m_2}$ be basic T^∞-sets, and let $M^i \subseteq \mathbb{R}^{n_i}$, $d^i = (d^i_0, \dots, d^i_{k^i})$, $e^i = (e^i_0, \dots, e^i_{l^i})$ and $\left(V^i_{\iota,\kappa}\right)$ be the data associated to X_i as in Definition 3.1, for $i = 1,2$. We assume that both M^1 and M^2 are connected; the general case is easily reduced to this situation. Define

$$\mathbf{M} := \left\{(x,y,u,v) :\ (x,u) \in M^1 \text{ and } (y,v) \in M^2\right\},$$

where x ranges over $\mathbb{R}^{m_1}$, y over $\mathbb{R}^{m_2}$, u over $\mathbb{R}^{n_1-m_1}$ and v over $\mathbb{R}^{n_2-m_2}$. We interpret d^i and e^i as sets of distributions on $\mathbf{M}$ correspondingly, for $i = 1,2$, and we define $\mathbf{d} := (d^1_0, \dots, d^1_{k^1}, d^1_{k^1} \cap d^2_1, \dots, d^1_{k^1} \cap d^2_{k^2})$ and $\mathbf{e} := (e^1_0, \dots, e^1_{l^1}, e^1_{l^1} \cap e^2_1, \dots, e^1_{l^1} \cap e^2_{l^2})$. Since M^1 and M^2 are connected, each set

$$V_{\kappa,\iota} := \left\{(x,y,u,v) :\ (x,u) \in V^1_{\kappa,\iota} \text{ and } (y,v) \in V^2_{\kappa,\iota}\right\}$$

is an admissible integral manifold of $\mathbf{d}$ with core distribution $\mathbf{e}$. It is now easy to see that for each κ, the limit $K_\kappa := \lim_\iota V_{\kappa,\iota}$ exists in $\mathcal{K}_{n_1+n_2}$, and that the sequence $\left(\Pi_{k_1+k_2}(K_\kappa)\right)$ is increasing and has union $X_1 \times X_2$.

For intersections, let $X_1, X_2 \subseteq \mathbb{R}^m$ be basic T^∞-sets. Then $X_1 \cap X_2 = \Pi_k((X_1 \times X_2) \cap \Delta)$, where $\Delta := \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^m :\ x_i = y_i \text{ for } i = 1,\dots,m\}$. Therefore, we let $X \subseteq \mathbb{R}^m$ be a basic T^∞-set and $C \subseteq \mathbb{R}^m$ be closed and definable, and we show that $X \cap C$ is a T^∞-set. Let the data associated to X be as in Definition 3.1, and let $\mathbf{M}$, $\mathbf{d}$ and $\mathbf{e}$ be associated to that data as before Proposition 2.8. Let also $\mathbf{N}$ be the open subset of $\mathbf{M}$ given by that proposition with $C' := C \times \mathbb{R}^{n-m}$ in place of C. Then by that proposition, there is a $q \in \mathbb{N}$ such that for every κ the set $K_\kappa \cap C'$ is the union of the projections of T^∞-limits $K^1_\kappa, \dots, K^q_\kappa$ obtained from $\mathbf{d}^{\mathbf{N}}$. Note that each K^j_κ is the limit of a T^∞-sequence of integral manifolds of $\mathbf{d}^{\mathbf{N}}$ with core distribution $\mathbf{e}^{\mathbf{N}}$. Replacing each sequence $\left(K^j_\kappa\right)$ by a (possibly finite) subsequence if necessary, we may assume that each sequence $\left(\Pi_m(K^j_\kappa)\right)$ is increasing. Then each $X_j := \bigcup_\kappa K^j_\kappa$ is a basic T^∞-set and $X \cap C = X_1 \cup \dots \cup X_q$. ◻

Proposition 3.5 *Let $X \subseteq \mathbb{R}^m$ be a T^∞-set. Then $\mathrm{bd}(X)$ is contained in a closed T^∞-set with empty interior.*

Proof Let the data associated to X be given as in Definition 3.1 and write $d = (d_0, \dots, d_k)$. Note that

$$\mathrm{bd}(X) \subseteq \lim_{\kappa} \mathrm{bd}(\Pi_m(K_\kappa)).$$

Fix an arbitrary κ; since $\Pi_m(K_\kappa) = \lim_\iota \Pi_m(V_{\kappa,\iota})$ we may assume, by Corollary 2.2, [5, Remark 4.2] and after replacing M if necessary, that $\Pi_k\restriction_{d_k}$ is an immersion and has constant rank $r \le m$; in particular, $\dim(V_{\kappa,\iota}) \le m$. If $r < m$, then each $\Pi_m(K_\kappa)$ has empty interior by Proposition 2.4, so

$$\lim_{\kappa} \mathrm{bd}(\Pi_m(K_\kappa)) = \lim_{\kappa} \Pi_m(K_\kappa) = \Pi_m(\lim_{\kappa} K_\kappa) = \Pi_m(\lim_{\kappa} V_{\kappa,\iota(\kappa)})$$

for some subsequence $(\iota(\kappa))$, and we conclude by Propositions 2.5 and 3.3(1) in this case. So assume that $r = m$; in particular, $\Pi_m(V_{\kappa,\iota})$ is open for every κ and ι. In this case, since M is bounded, we have $\mathrm{bd}(\Pi_m(K_\kappa)) \subseteq \Pi_m(\lim_\iota \mathrm{fr}\, V_{\kappa,\iota})$ for each κ. Hence

$$\lim_{\kappa} \mathrm{bd}(\Pi_m(K_\kappa)) \subseteq \Pi_m(\lim_{\kappa} \lim_{\iota} \mathrm{fr}\, V_{\kappa,\iota}) = \Pi_m(\lim_{\kappa} \mathrm{fr}\, V_{\kappa,\iota(\kappa)})$$

for some subsequence $(\iota(\kappa))$. Now use Propositions 2.3 and 3.3(1). □

Following [8] and [3], and proceeding exactly as in [7, Corollary 3.11 and Proposition 3.12] using the previous propositions, we obtain:

Proposition 3.6 (1) *Let $X \subseteq \mathbb{R}^m$ be a T^∞-set, and let $1 \le l \le m$. Then the set $B := \left\{a \in \mathbb{R}^l : \mathrm{cl}(X_a) \ne \mathrm{cl}(X)_a\right\}$ has empty interior.*

(2) *Let $X \subseteq [-1,1]^m$ be a T^∞-set. Then $[-1,1]^m \setminus X$ is also a T^∞-set.* □

For $m \in \mathbb{N}$, let $\mathcal{T}_m$ be the collection of all T^∞-sets $X \subseteq I^m$.

Corollary 3.7 *The collection $\mathcal{T} := (\mathcal{T}_m)_{m\in\mathbb{N}}$ forms an o-minimal structure on I.* □

Proof of Theorem A For each m, let $\tau_m : \mathbb{R}^m \longrightarrow (-1,1)^m$ be the (definable) homeomorphism given by

$$\tau_m(x_1, \dots, x_m) := \left(\frac{x_1}{1+x_1^2}, \dots, \frac{x_m}{1+x_m^2}\right),$$

and let $\mathcal{S}_m$ be the collection of sets $\tau_m^{-1}(X)$ with $X \in \mathcal{T}_m$. By Corollary 3.7, the collection $\mathcal{S} = \mathcal{S} := (\mathcal{S}_m)_m$ gives rise to an o-minimal expansion $T^\infty(\mathcal{R})$ of $\mathcal{R}$. By Proposition 3.3(2), every definable set is definable in $T^\infty(\mathcal{R})$. But if L is a Rolle leaf of a definable nested distribution d on $\mathbb{R}^n$, then $\tau_n(L)$ is a Rolle leaf of the pullback $(\tau_n^{-1})^* d$. It follows from Proposition 3.3(3) that $\tau_n(L) \in \mathcal{T}_n$, so L is definable in $T^\infty(\mathcal{R})$. Therefore, $\mathcal{N}(\mathcal{R})$ is a reduct of $T^\infty(\mathcal{R})$ in the sense of definability. □

4 Proof of Theorem B

First, we establish [5, Proposition 7.1] with "T^∞-limit" and "$T^\infty(\mathcal{R})$" in place of "pfaffian limit" and "$\mathcal{P}(\mathcal{R})$". To do so, we proceed exactly as in [5], making the following additional changes.

(B1) Replacing "admissible sequence" with "T^∞-sequence", we obtain corresponding versions of Lemma 4.8, Remark 4.9, Proposition 4.11, Corollary 4.13 and Proposition 5.3 in [5].

(B2) Using (B1), we obtain the corresponding version of [5, Proposition 7.1].

Second, assuming that $\mathcal{R}$ admits analytic cell decomposition, (B2) and [5, Proposition 10.4] imply that every T^∞-limit over $\mathcal{R}$ is definable in $\mathcal{N}(\mathcal{R})$; in particular, $T^\infty(\mathcal{R})$ and $\mathcal{N}(\mathcal{R})$ are interdefinable. Hence, by [5, Corollary 1], $T^\infty(\mathcal{R})$ and $\mathcal{P}(\mathcal{R})$ are interdefinable. Replacing once more $\mathcal{R}$ by $\mathcal{P}(\mathcal{R})$, Theorem B is now proved.

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