# **A New Viewpoint to the Discrete Approximation: Discrete Yang-Fourier Transforms of Discrete-Time Fractal Signal**

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#### **Abstract**

It is suggest that a new fractal model for the Yang-Fourier transforms of discrete approximation based on local fractional calculus and the Discrete Yang-Fourier transforms are investigated in detail.

*Key words: local fractional calculus, fractal, Yang Fourier transforms, discrete approximation, discrete Yang-Fourier transforms*

MSC2010: 28A80, 26A99, 26A15, 41A35

### **1 Introduction**

Fractional Fourier transform becomes a hot topic in both mathematics and engineering. There are many definitions of fractional Fourier transforms [1-5]. Hereby we write down the Yang-Fourier transforms [3-5]

$$
F_{\alpha}\left\{f(x)\right\} = f_{\omega}^{F,\alpha}(\omega) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} E_{\alpha}\left(-i^{\alpha} \omega^{\alpha} x^{\alpha}\right) f(x) \left(dx\right)^{\alpha}, \tag{1.1}
$$

and its inverse, denoted by [3,4]

$$
f(x) = F_{\alpha}^{-1}\left(f_{\omega}^{F,\alpha}(\omega)\right) := \frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{\infty} E_{\alpha}\left(i^{\alpha}\omega^{c}x^{\alpha}\right) f_{\omega}^{F,\alpha}(\omega)(d\omega)^{\alpha}, \tag{1.2}
$$

where local fractional integral of  $f(t)$  is denoted by [3-8]

$$
{}_{a}I_{b}^{(\alpha)}f(x)
$$
  
=
$$
\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}f(t)(dt)^{\alpha}
$$
  
=
$$
\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}f(t_{j})(\Delta t_{j})^{\alpha}
$$
 (1.3)

with  $\Delta t_i = t_{i+1} - t_i$  and  $\Delta t = \max\{\Delta t_i, \Delta t_i, \Delta t_i, \ldots\}$ , where for  $j = 0, \ldots, N-1$ ,  $[t_i, t_{i+1}]$  is a partition of the interval  $[a,b]$  and  $t_0 = a,t_N = b$ . Here, for  $|x-x_0| < \delta$  with  $\delta > 0$ , there exists any *x* such that

$$
\left|f\left(x\right)-f\left(x_0\right)\right|<\varepsilon^\alpha.\tag{1.4}
$$

Now  $f(x)$  is called local fractional continuous at  $x = x_0$  and we have [5]

$$
\lim_{x \to x_0} f\left(x\right) = f\left(x_0\right). \tag{1.5}
$$

Suppose that  $\{f_0, f_1, \dots, f_{N-1}\}\$ is an  $N_{th}$  order regular sampling with spacing  $\Delta x$  some piecewise local fractional continuous function over mammal window  $[0, L]$ . In the present paper, our arms are to get some assurance that local fractional integral of *f* can be reasonably

approximated by the corresponding integration of  $\tilde{f}$  and we will get the discrete Yang-Fourier transforms.

# **2 A fractal model for the Yang-Fourier transforms of discrete approximation**

Now we determine from our data,

$$
\frac{1}{\Gamma(1+\alpha)}\int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \tilde{f}(t)\phi(t)(dt)^{\alpha} \approx \frac{1}{\Gamma(1+\alpha)}\int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} f(t)\phi(t)(dt)^{\alpha}
$$
\n(2.1)

for any local fractional continuous function on the natural widow. This sampling can be used to complete a corresponding sum approximation for the integration,

$$
\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} f(t) \phi(t) (dt)^{\alpha}
$$
\n
$$
\approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f(k\Delta t) \phi(k\Delta t) (\Delta t)^{\alpha}
$$
\n
$$
= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \phi(k\Delta t) (\Delta t)^{\alpha} .
$$
\n(2.2)

Notice, however, that

$$
\frac{1}{\Gamma(1+\alpha)}\sum_{k=0}^{N-1}f_k\phi(k\Delta t)(\Delta t)^{\alpha} \n= \frac{1}{\Gamma(1+\alpha)}\sum_{k=0}^{N-1}f_k\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t}\phi(t)\delta_{k\Delta t}(t)(dt)^{\alpha}\right](\Delta t)^{\alpha} \n= \frac{1}{\Gamma(1+\alpha)}\int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t}\left[\frac{1}{\Gamma(1+\alpha)}\sum_{k=0}^{N-1}f_k\delta_{k\Delta t}(t)(\Delta t)^{\alpha}\right]\phi(t)(dt)^{\alpha}
$$
\n(2.3)

where

$$
\frac{1}{\Gamma(1+\alpha)}\int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t}\phi(t)\delta_{k\Delta t}(t)(dt)^{\alpha}=\phi(k\Delta t), \text{ for } k=0,1,\cdots,N-1
$$

So,

$$
\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}^{\alpha}}^{\frac{2N-1}{2}^{\alpha}} f(t) \phi(t) (dt)^{\alpha}
$$
\n
$$
= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}^{\alpha}}^{\frac{2N-1}{2}^{\alpha}} \left[ \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \delta_{k\alpha}(t) (\Delta t)^{\alpha} \right] \phi(t) (dt)^{\alpha}
$$
\n(2.4)

Suggesting that, with the natural window, we use

$$
\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} \tilde{f}_k \delta_{k\Delta t}(t),
$$
\n(2.5)

where  $\tilde{f}_k = f_k (\Delta t)^{\alpha}$  for  $k = 0, 1, \dots, N - 1$ .

Now there are two natural choices: Either  $\tilde{f}$  define to be 0 outside the nature window, or define  $\tilde{f}$  to be periodic with period *T* equalling the length of the natural window,

$$
T = N\Delta t \,. \tag{2.7}
$$

Combing with our definition of  $\tilde{f}$  on the natural window, the first choice would be give

$$
\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} \tilde{f}_k \delta_{k\Delta t}(t),
$$
\n(2.8)

while the second choice would be give

$$
\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=-\infty}^{\infty} \tilde{f}_k \delta_{k\Delta t}(t)
$$
\n(2.9)

with  $\tilde{f}_{k+N} = \tilde{f}_k$ .

Clearly, the latter is the more clear choice. That is to say, suppose that  ${f_0, f_1, \dots, f_{N-1}}$  is the  $N_{th}$  order regular sampling with spacing  $\Delta t$  of some function  $f$ . The corresponding discrete approximation of  $f$  is the periodic, regular array

$$
\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=-\infty}^{\infty} \tilde{f}_k \delta_{k\Delta t}(t)
$$
\n(2.10)

with spacing  $\Delta t$  index period  $N$ , and its coefficients

$$
\tilde{f}_k = \begin{cases} f_k (\Delta x)^{\alpha}, & \text{if } k = 0, 1, \dots, N - 1, \\ f_{k+N}, & \text{in general.} \end{cases}
$$
\n(2.11)

From the Yang-Fourier transform theory, we then know

$$
F_{\alpha}\left\{f\left(x\right)\right\}=f_{\omega}^{F,\alpha}\left(\omega\right)
$$

is a local fractional continuous and is given by

$$
f_{\omega}^{F,\alpha}(\omega)
$$
\n
$$
= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) E_{\alpha}(-i^{\alpha} \omega^{\alpha} t^{\alpha}) (dt)^{\alpha}
$$
\n
$$
= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} f(t) E_{\alpha}(-i^{\alpha} \omega^{\alpha} t^{\alpha}) (dt)^{\alpha}
$$
\n
$$
\approx \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \tilde{f}(t) E_{\alpha}(-i^{\alpha} \omega^{\alpha} t^{\alpha}) (dt)^{\alpha}
$$
\n
$$
= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \left( \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_{k} \delta_{k\Delta t}(t) (\Delta t)^{\alpha} \right) E_{\alpha}(-i^{\alpha} \omega^{\alpha} t^{\alpha}) (dt)^{\alpha}
$$
\n
$$
= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_{k} (\Delta t)^{\alpha} \left( \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \delta_{k\Delta t}(t) E_{\alpha}(-i^{\alpha} \omega^{\alpha} t^{\alpha}) (dt)^{\alpha} \right)
$$
\n
$$
= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_{k} (\Delta t)^{\alpha} E_{\alpha}(-i^{\alpha} \omega^{\alpha} k^{\alpha} (\Delta t)^{\alpha})
$$
\n(2.12)

So, approximation of the formula

$$
\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}f(t)E_{\alpha}\left(-i^{\alpha}\omega^{\alpha}t^{\alpha}\right)(dt)^{\alpha}
$$

reduces to

$$
f_{\omega}^{F,\alpha}(\omega) \approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k(\Delta t)^{\alpha} E_{\alpha}(-i^{\alpha} \omega^{\alpha} k^{\alpha} (\Delta t)^{\alpha}). \tag{2.13}
$$

with  $T = N\Delta t$ 

Taking  $\omega = n\Delta\omega$  and  $\frac{2}{\epsilon}$ *T*  $\frac{\pi}{\pi} = \Delta \omega$  in (2.13) implies that

$$
\phi(n) \n= f_{\omega}^{F,\alpha}(\omega) \n\approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k(\Delta t)^{\alpha} E_{\alpha}(-i^{\alpha} \omega^{\alpha} k^{\alpha} (\Delta t)^{\alpha}) \n= \frac{1}{\Gamma(1+\alpha)} \frac{T^{\alpha}}{N^{\alpha}} \sum_{k=0}^{N-1} f_k E_{\alpha}(-i^{\alpha} (2\pi)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha}) \n= \frac{1}{\Gamma(1+\alpha)} \frac{T^{\alpha}}{N^{\alpha}} \sum_{k=0}^{N-1} \phi(k) E_{\alpha}(-i^{\alpha} (2\pi)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha})
$$
\n
$$
\text{where if } \quad
$$
\n(2.14)

In the same manner, if

$$
f(t) = \frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{\infty} E_{\alpha} (i^{\alpha} \omega^{\alpha} t^{\alpha}) f_{\omega}^{F,\alpha} (\omega) (d\omega)^{\alpha},
$$

then we can write

$$
f_k(k\Delta t) \approx \frac{1}{(2\pi)^{\alpha}} \sum_{n=0}^{N-1} f_{\omega}^{F,\alpha} (n\Delta \omega) (\Delta \omega)^{\alpha} E_{\alpha} (i^{\alpha} t^{\alpha} n^{\alpha} (\Delta \omega)^{\alpha}) \qquad (2.15)
$$

with  $\omega = N \Delta \omega$ .

Taking 
$$
t = k\Delta t
$$
 and  $\frac{2\pi}{T} = \Delta \omega$  in (2.15) implies that  
\n
$$
\varphi(k)
$$
\n
$$
= f_k (k\Delta t)
$$
\n
$$
\approx \frac{1}{(2\pi)^{\alpha}} \sum_{n=0}^{N-1} f_{\omega}^{F,\alpha} (n\Delta \omega) (\Delta \omega)^{\alpha} E_{\alpha} (i^{\alpha} n^{\alpha} (\Delta t)^{\alpha} k^{\alpha} (\Delta \omega)^{\alpha})
$$
\n
$$
= \frac{1}{T^{\alpha}} \sum_{n=0}^{N-1} \phi(n) E_{\alpha} (i^{\alpha} n^{\alpha} k^{\alpha} (2\pi)^{\alpha} / N^{\alpha}).
$$
\n(2.16)

Combing the formulas (2.14) and (2.16), we have the following results:

$$
\phi(n) = \frac{1}{\Gamma(1+\alpha)} \frac{T^{\alpha}}{N^{\alpha}} \sum_{k=0}^{N-1} \phi(k) E_{\alpha} \left( -i^{\alpha} \left( 2\pi \right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha} \right)
$$
(2.17)

and

$$
\varphi(k) = \frac{1}{T^{\alpha}} \sum_{n=0}^{N-1} \phi(n) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right). \tag{2.18}
$$

Setting  $F(n) = \frac{1}{T^{\alpha}} \phi(n)$  and interchanging *k* and *n*, we get

$$
\varphi(n) = \sum_{k=0}^{N-1} F(k) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right)
$$
\n(2.19)

and

$$
F(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{n=0}^{N-1} \varphi(n) E_{\alpha} \left(-i^{\alpha} \left(2\pi\right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha}\right).
$$
 (2.20)

# **3 Discrete Yang-Fourier transforms of discrete-time fractal signal**

#### *Definition 1*

Suppose that  $F(k)$  be a periodic discrete-time fractal signal with period N. From (2.20) the sequence  $f(n)$  is defined by

$$
F(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{n=0}^{N-1} f(n) E_{\alpha} \left( -i^{\alpha} \left( 2\pi \right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha} \right), \tag{3.1}
$$

which is called *N* -point discrete Yang-Fourier transform of  $F(n)$ , denoted by

$$
f(n) \leftrightarrow F(k).
$$

#### *Definition 2*

Inverse discrete Yang-Fourier transform

From (2.19), the transform assigning the signal  $F(k)$  to  $f(n)$  is called the inverse discrete Yang-Fourier transform, which is rewritten as

$$
f(n) = \sum_{k=0}^{N-1} F(k) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right)
$$
\n
$$
F(k) = f(n) \leftrightarrow F(k) = f(n) \leftrightarrow F(k) = 0. \tag{3.2}
$$

Suppose that  $f(n) \leftrightarrow F(k)$ ,  $f_1(n) \leftrightarrow F_1(k)$  and  $f_2(n) \leftrightarrow F_2(k)$ , the following relations are valid:

#### *Property 1*

$$
af_1(n) + bf_2(n) \leftrightarrow aF_1(k) + bF_2(k).
$$
 (3.3)

Proof. Taking into account the linear transform of discrete Yang-Fourier transform, we directly deduce the result.

#### *Property 2*

Let  $f(k)$  be a periodic discrete fractal signal with period N. Then we have

$$
\sum_{n=j}^{j+N-1} f(n) = \sum_{n=0}^{N-1} f(n).
$$
 (3.4)

Proof. We directly deduce the result when  $j = mN + l$  with  $0 \le l \le N - 1$ .

#### *Theorem 3*

Suppose that

$$
F(n) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{k=0}^{N-1} f(k) E_{\alpha} \left(-i^{\alpha} (2\pi)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha}\right),
$$

then we have

$$
f(k) = \sum_{n=0}^{N-1} F(n) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right)
$$
\n(3.5)

*Proof.* From the formulas (2.11)-(2.20) we deduce to the results.

### **4 Conclusions**

In the present paper we discuss a model for the Yang-Fourier transforms of discrete approximation. As well, we give the discrete Yang-Fourier transforms of fractal signal as follows:

$$
F(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{n=0}^{N-1} f(n) E_{\alpha} \left( -i^{\alpha} \left( 2\pi \right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha} \right)
$$

and

$$
f(n) = \sum_{k=0}^{N-1} F(k) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right).
$$

Furthermore, some results are discussed.

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