# A New Viewpoint to the Discrete Approximation: Discrete Yang-Fourier Transforms of Discrete-Time Fractal Signal

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#### Abstract

It is suggest that a new fractal model for the Yang-Fourier transforms of discrete approximation based on local fractional calculus and the Discrete Yang-Fourier transforms are investigated in detail.

*Key words: local fractional calculus, fractal, Yang Fourier transforms, discrete approximation, discrete Yang-Fourier transforms* 

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## **1** Introduction

Fractional Fourier transform becomes a hot topic in both mathematics and engineering. There are many definitions of fractional Fourier transforms [1-5]. Hereby we write down the Yang-Fourier transforms [3-5]

$$F_{\alpha}\left\{f\left(x\right)\right\} = f_{\omega}^{F,\alpha}\left(\omega\right) \coloneqq \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} E_{\alpha}\left(-i^{\alpha}\omega^{\alpha}x^{\alpha}\right) f\left(x\right) \left(dx\right)^{\alpha}, \tag{1.1}$$

and its inverse, denoted by [3,4]

$$f(x) = F_{\alpha}^{-1} \left( f_{\omega}^{F,\alpha}(\omega) \right) \coloneqq \frac{1}{\left(2\pi\right)^{\alpha}} \int_{-\infty}^{\infty} E_{\alpha} \left( i^{\alpha} \omega^{\alpha} x^{\alpha} \right) f_{\omega}^{F,\alpha}(\omega) \left( d\omega \right)^{\alpha}, \qquad (1.2)$$

where local fractional integral of f(t) is denoted by [3-8]

$${}_{a}I_{b}^{(\alpha)}f(x)$$

$$=\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}f(t)(dt)^{\alpha}$$

$$=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}f(t_{j})(\Delta t_{j})^{\alpha}$$
(1.3)

with  $\Delta t_j = t_{j+1} - t_j$  and  $\Delta t = \max \{ \Delta t_1, \Delta t_2, \Delta t_j, ... \}$ , where for j = 0, ..., N-1,  $\begin{bmatrix} t_j, t_{j+1} \end{bmatrix}$  is a partition of the interval  $\begin{bmatrix} a, b \end{bmatrix}$  and  $t_0 = a, t_N = b$ . Here, for  $|x - x_0| < \delta$  with  $\delta > 0$ , there exists any x such that

$$\left|f\left(x\right) - f\left(x_{0}\right)\right| < \varepsilon^{\alpha}.$$
(1.4)

Now f(x) is called local fractional continuous at  $x = x_0$  and we have [5]

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{1.5}$$

Suppose that  $\{f_0, f_1, \dots, f_{N-1}\}\$  is an  $N_{th}$  order regular sampling with spacing  $\Delta x$  some piecewise local fractional continuous function over mammal window [0, L]. In the present paper, our arms are to get some assurance that local fractional integral of f can be reasonably

approximated by the corresponding integration of  $\tilde{f}$  and we will get the discrete Yang-Fourier transforms.

# 2 A fractal model for the Yang-Fourier transforms of discrete approximation

Now we determine from our data,

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \tilde{f}(t)\phi(t)(dt)^{\alpha} \approx \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} f(t)\phi(t)(dt)^{\alpha}$$
(2.1)

for any local fractional continuous function on the natural widow. This sampling can be used to complete a corresponding sum approximation for the integration,

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} f(t)\phi(t)(dt)^{\alpha}$$

$$\approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f(k\Delta t)\phi(k\Delta t)(\Delta t)^{\alpha}$$

$$= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \phi(k\Delta t)(\Delta t)^{\alpha}.$$
(2.2)

Notice, however, that

$$\frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \phi(k\Delta t) (\Delta t)^{\alpha}$$

$$= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \left[ \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \phi(t) \delta_{k\Delta t}(t) (dt)^{\alpha} \right] (\Delta t)^{\alpha}$$

$$= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \left[ \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \delta_{k\Delta t}(t) (\Delta t)^{\alpha} \right] \phi(t) (dt)^{\alpha}$$
(2.3)

where

$$\frac{1}{\Gamma(1+\alpha)}\int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t}\phi(t)\delta_{k\Delta t}(t)(dt)^{\alpha}=\phi(k\Delta t), \text{ for } k=0,1,\cdots,N-1$$

So,

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta}^{\frac{2N-1}{2}} f(t)\phi(t)(dt)^{\alpha}$$

$$= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta}^{\frac{2N-1}{2}} \left[ \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \delta_{k\Delta t}(t)(\Delta t)^{\alpha} \right] \phi(t)(dt)^{\alpha}$$
(2.4)

Suggesting that, with the natural window, we use

$$\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} \tilde{f}_k \delta_{k\Delta t}(t), \qquad (2.5)$$

where  $\tilde{f}_k = f_k (\Delta t)^{\alpha}$  for  $k = 0, 1, \dots, N-1$ .

Now there are two natural choices: Either  $\tilde{f}$  define to be 0 outside the nature window, or define  $\tilde{f}$  to be periodic with period T equalling the length of the natural window,

$$T = N\Delta t . \tag{2.7}$$

Combing with our definition of  $\,\widetilde{f}\,$  on the natural window, the first choice would be give

$$\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} \tilde{f}_k \delta_{k\Delta t}(t), \qquad (2.8)$$

while the second choice would be give

$$\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=-\infty}^{\infty} \tilde{f}_k \delta_{k\Delta t}(t)$$
(2.9)

with  $\tilde{f}_{\boldsymbol{k}+\boldsymbol{N}}=\tilde{f}_{\boldsymbol{k}}$  .

Clearly, the latter is the more clear choice. That is to say, suppose that  $\{f_0, f_1, \dots, f_{N-1}\}$  is the  $N_{th}$  order regular sampling with spacing  $\Delta t$  of some function f. The corresponding discrete approximation of f is the periodic, regular array

$$\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=-\infty}^{\infty} \tilde{f}_k \delta_{k\Delta t}(t)$$
(2.10)

with spacing  $\Delta t$  index period N , and its coefficients

$$\tilde{f}_{k} = \begin{cases} f_{k} \left( \Delta x \right)^{\alpha}, & if \quad k = 0, 1, \cdots, N-1. \\ f_{k+N}, & in \quad general. \end{cases}$$
(2.11)

From the Yang-Fourier transform theory, we then know  $\sum_{i=1}^{n} \left( c_{i}(x_{i}) \right) = c_{i}^{F} c_{i}^{T} (c_{i}^{T})$ 

$$F_{\alpha}\left\{f\left(x\right)\right\} = f_{\omega}^{F,\alpha}\left(\omega\right)$$

is a local fractional continuous and is given by

$$\begin{aligned} f_{\omega}^{F,\alpha}(\omega) \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) E_{\alpha} \left( -i^{\alpha} \omega^{\alpha} t^{\alpha} \right) (dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} f(t) E_{\alpha} \left( -i^{\alpha} \omega^{\alpha} t^{\alpha} \right) (dt)^{\alpha} \\ &\approx \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \tilde{f}(t) E_{\alpha} \left( -i^{\alpha} \omega^{\alpha} t^{\alpha} \right) (dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \left( \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_{k} \delta_{k\Delta t}(t) (\Delta t)^{\alpha} \right) E_{\alpha} \left( -i^{\alpha} \omega^{\alpha} t^{\alpha} \right) (dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_{k} \left( \Delta t \right)^{\alpha} \left( \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2}\Delta t}^{\frac{2N-1}{2}\Delta t} \delta_{k\Delta t}(t) E_{\alpha} \left( -i^{\alpha} \omega^{\alpha} t^{\alpha} \right) (dt)^{\alpha} \right) \\ &= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_{k} \left( \Delta t \right)^{\alpha} E_{\alpha} \left( -i^{\alpha} \omega^{\alpha} k^{\alpha} \left( \Delta t \right)^{\alpha} \right) \end{aligned}$$
(2.12)

So, approximation of the formula

$$\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}f(t)E_{\alpha}(-i^{\alpha}\omega^{\alpha}t^{\alpha})(dt)^{\alpha}$$

reduces to

$$f_{\omega}^{F,\alpha}(\omega) \approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k(\Delta t)^{\alpha} E_{\alpha} \left( -i^{\alpha} \omega^{\alpha} k^{\alpha} \left( \Delta t \right)^{\alpha} \right).$$
(2.13)

with  $T = N\Delta t$ .

Taking  $\omega = n\Delta\omega$  and  $\frac{2\pi}{T} = \Delta\omega$  in (2.13) implies that

$$\begin{split} \phi(n) \\ &= f_{\omega}^{F,\alpha}(\omega) \\ &\approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \left(\Delta t\right)^{\alpha} E_{\alpha} \left(-i^{\alpha} \omega^{\alpha} k^{\alpha} \left(\Delta t\right)^{\alpha}\right) \\ &= \frac{1}{\Gamma(1+\alpha)} \frac{T^{\alpha}}{N^{\alpha}} \sum_{k=0}^{N-1} f_k E_{\alpha} \left(-i^{\alpha} \left(2\pi\right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha}\right) \\ &= \frac{1}{\Gamma(1+\alpha)} \frac{T^{\alpha}}{N^{\alpha}} \sum_{k=0}^{N-1} \varphi(k) E_{\alpha} \left(-i^{\alpha} \left(2\pi\right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha}\right) \end{split}$$
prove if

In the same manner, if

$$f(t) = \frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{\infty} E_{\alpha} (i^{\alpha} \omega^{\alpha} t^{\alpha}) f_{\omega}^{F,\alpha} (\omega) (d\omega)^{\alpha},$$

then we can write

$$f_{k}\left(k\Delta t\right) \approx \frac{1}{\left(2\pi\right)^{\alpha}} \sum_{n=0}^{N-1} f_{\omega}^{F,\alpha}\left(n\Delta\omega\right) \left(\Delta\omega\right)^{\alpha} E_{\alpha}\left(i^{\alpha}t^{\alpha}n^{\alpha}\left(\Delta\omega\right)^{\alpha}\right)$$
(2.15)

with  $\omega = N\Delta\omega$ .

Taking 
$$t = k\Delta t$$
 and  $\frac{2\pi}{T} = \Delta \omega$  in (2.15) implies that  
 $\varphi(k)$   
 $= f_k (k\Delta t)$   
 $\approx \frac{1}{(2\pi)^{\alpha}} \sum_{n=0}^{N-1} f_{\omega}^{F,\alpha} (n\Delta\omega) (\Delta\omega)^{\alpha} E_{\alpha} (i^{\alpha} n^{\alpha} (\Delta t)^{\alpha} k^{\alpha} (\Delta\omega)^{\alpha})$   
 $= \frac{1}{T^{\alpha}} \sum_{n=0}^{N-1} \phi(n) E_{\alpha} (i^{\alpha} n^{\alpha} k^{\alpha} (2\pi)^{\alpha} / N^{\alpha}).$  (2.16)

Combing the formulas (2.14) and (2.16), we have the following results:

$$\phi(n) = \frac{1}{\Gamma(1+\alpha)} \frac{T^{\alpha}}{N^{\alpha}} \sum_{k=0}^{N-1} \varphi(k) E_{\alpha} \left( -i^{\alpha} \left( 2\pi \right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha} \right)$$
(2.17)

and

$$\varphi(k) = \frac{1}{T^{\alpha}} \sum_{n=0}^{N-1} \phi(n) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right).$$
(2.18)

Setting  $F(n) = \frac{1}{T^{\alpha}} \phi(n)$  and interchanging k and n, we get

$$\varphi(n) = \sum_{k=0}^{N-1} F(k) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right)$$
(2.19)

and

$$F(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{n=0}^{N-1} \varphi(n) E_{\alpha} \left( -i^{\alpha} \left( 2\pi \right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha} \right).$$
(2.20)

# **3 Discrete Yang-Fourier transforms of discrete-time fractal signal**

#### Definition 1

Suppose that F(k) be a periodic discrete-time fractal signal with period N. From (2.20) the sequence f(n) is defined by

$$F(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{n=0}^{N-1} f(n) E_{\alpha} \left( -i^{\alpha} \left( 2\pi \right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha} \right), \tag{3.1}$$

which is called N-point discrete Yang-Fourier transform of F(n), denoted by

$$f(n) \leftrightarrow F(k).$$

#### Definition 2

Inverse discrete Yang-Fourier transform

From (2.19), the transform assigning the signal F(k) to f(n) is called the inverse discrete Yang-Fourier transform, which is rewritten as

$$f(n) = \sum_{k=0}^{N-1} F(k) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right)$$

$$F(k) = f(n) \langle \rangle F(k) \text{ and } f(n) \langle \rangle F(k) \text{ the following}$$

$$(3.2)$$

Suppose that  $f(n) \leftrightarrow F(k)$ ,  $f_1(n) \leftrightarrow F_1(k)$  and  $f_2(n) \leftrightarrow F_2(k)$ , the following relations are valid:

#### Property 1

$$af_1(n) + bf_2(n) \leftrightarrow aF_1(k) + bF_2(k). \tag{3.3}$$

Proof. Taking into account the linear transform of discrete Yang-Fourier transform, we directly deduce the result.

#### Property 2

Let f(k) be a periodic discrete fractal signal with period N. Then we have

$$\sum_{n=j}^{j+N-1} f(n) = \sum_{n=0}^{N-1} f(n).$$
(3.4)

Proof. We directly deduce the result when j = mN + l with  $0 \le l \le N - 1$ .

#### Theorem 3

Suppose that

$$F(n) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{k=0}^{N-1} f(k) E_{\alpha} \left(-i^{\alpha} \left(2\pi\right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha}\right)$$

then we have

$$f(k) = \sum_{n=0}^{N-1} F(n) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right)$$
(3.5)

*Proof.* From the formulas (2.11)-(2.20) we deduce to the results.

# **4** Conclusions

In the present paper we discuss a model for the Yang-Fourier transforms of discrete approximation. As well, we give the discrete Yang-Fourier transforms of fractal signal as follows:

$$F(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{n=0}^{N-1} f(n) E_{\alpha} \left(-i^{\alpha} \left(2\pi\right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha}\right)$$

and

$$f(n) = \sum_{k=0}^{N-1} F(k) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right).$$

Furthermore, some results are discussed.

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