NONLINEAR DEGENERATE ELLIPTIC PROBLEMS WITH $W_0^{1,1}(\Omega)$ SOLUTIONS

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Abstract. We study a nonlinear equation with an elliptic operator having degenerate coercivity. We prove the existence of a unique $W_0^{1,1}(\Omega)$ distributional solution under suitable summability assumptions on the source in Lebesgue spaces. Moreover, we prove that our problem has no solution if the source is a Radon measure concentrated on a set of zero harmonic capacity.

1. Introduction and statement of the results

In this paper we are going to study the nonlinear elliptic equation

$$
\begin{cases} -\mathrm{div}\left(\dfrac{a(x,\nabla u)}{(1+|u|)^\gamma}\right) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

under the following assumptions. The set Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $\gamma > 0$, f belongs to some Lebesgue space, and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e., $a(\cdot, \xi)$ is measurable on Ω for every ξ in $\mathbb{R}^N$, and $a(x, \cdot)$ is continuous on $\mathbb{R}^N$ for almost every x in Ω) such that

$$a(x,\xi)\cdot\xi \geq \alpha\,|\xi|^2\,, \tag{1.2}$$

$$|a(x,\xi)| \leq \beta\,|\xi|\,, \tag{1.3}$$

$$[a(x,\xi) - a(x,\eta)]\cdot(\xi-\eta) > 0\,, \tag{1.4}$$

for almost every x in Ω and for every ξ and η in $\mathbb{R}^N$, $\xi \neq \eta$, where α and β are positive constants. We are going to prove that, under suitable assumptions on γ and f, problem (1.1) has a unique distributional solution u obtained by approximation, with u belonging to the (non-reflexive) Sobolev space $W_0^{1,1}(\Omega)$. Furthermore, we are going to prove that problem (1.1) does not have a solution if $\gamma > 1$ and the datum f is a bounded Radon measure concentrated on a set of zero harmonic capacity.

Problems like (1.1) have been extensively studied in the past. In [7] (see also [15], [16], [19]), existence and regularity results were proved, under the assumption that $a(x,\xi) = A(x)\,\xi$, with A a uniformly elliptic bounded matrix, and $0 < \gamma \leq 1$, for the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{A(x)\,\nabla u}{(1+|u|)^\gamma}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where f belongs to $L^m(\Omega)$ for some $m \geq 1$.

The main difficulty in dealing with problem (1.5) (or (1.1)) is that the differential operator, even if well defined between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$, is not coercive on $H_0^1(\Omega)$ due to the fact that if u is large, $\frac{1}{(1+|u|)^\gamma}$ tends to zero (see [19] for an explicit example).

This lack of coercivity implies that the classical methods used in order to prove the existence of a solution for elliptic equations (see [18]) cannot be applied even if the datum f is regular. However, in [7], a whole range of existence results was proved, yielding solutions belonging to some Sobolev space $W_0^{1,q}(\Omega)$, with $q = q(\gamma, m) \leq 2$, if f is regular enough. Under weaker summability assumptions on f, the gradient of u (and even u itself) may not be in $L^1(\Omega)$: in this case, it is possible to give a meaning to solutions of problem (1.5), using the concept of *entropy solutions* which has been introduced in [3].

If $\gamma > 1$, a non existence result for problem (1.5) was proved in [1] (where the principal part is nonlinear with respect to the gradient), even for $L^\infty(\Omega)$ data f. Therefore, if the operator becomes "too degenerate", existence may be lost even for data expected to give bounded solutions. However, as proved in [5], existence of solutions can be recovered by adding a lower order term of order zero. Indeed, if we consider the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{A(x)\nabla u}{(1+|u|)^\gamma}\right) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with f in $L^m(\Omega)$, then the following results can be proved in the case $\gamma > 1$ (see [5] and [11]):

i) if $m > \gamma\frac{N}{2}$, then there exists a weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$;
ii) if $m \geq \gamma + 2$, then there exists a weak solution in $H_0^1(\Omega) \cap L^m(\Omega)$;
iii) if $\frac{\gamma+2}{2} < m < \gamma + 2$, then there exists a distributional solution in $W_0^{1,\frac{2m}{\gamma+2}}(\Omega) \cap L^m(\Omega)$;

iv) if $1 \le m \le \frac{\gamma+2}{2}$, then there exists an entropy solution in $L^m(\Omega)$ whose gradient belongs to the Marcinkiewicz space $M^{\frac{2m}{\gamma+2}}(\Omega)$.

Note that if $\gamma + 2 \le m < \gamma\frac{N}{2}$ and m tends to $\gamma\frac{N}{2}$, the summability result of ii) is not "continuous" with the boundedness result of i), according to the following example (see also Example 3.3 of [5]).

Example 1.1. If $\frac{2}{\gamma} < \sigma < N-2$, then $u(x) = \frac{1}{|x|^\sigma} - 1$ is a distributional solution of (1.6) with $A(x) \equiv I$, and $f(x) = \frac{\sigma(N-2+\sigma(\gamma-1))}{|x|^{2-\sigma(\gamma-1)}} + \frac{1}{|x|^\sigma} - 1$. Due to the assumptions on σ, both f and u belong to $L^m(\Omega)$, with $m < \gamma\frac{N}{2}$. If m tends to $\gamma\frac{N}{2}$, i.e., if σ tends to $\frac{2}{\gamma}$, the solution u does not become bounded.

As stated before, this paper is concerned with two borderline cases connected with point iv) above:

A. if $m = \frac{\gamma+2}{2}$, we will prove in Section 2 the existence of $W_0^{1,1}(\Omega)$ distributional solutions, and in Section 3 their uniqueness;
B. if f is a bounded Radon measure concentrated on a set E of zero harmonic capacity and $\gamma > 1$, we will prove in Section 4 non existence of solutions.

In the linear case, i.e., for the boundary value problem (1.6), a simple proof of the existence result is given in [6].

Remark 1.2. Let $a(x,\xi) = A(x)\,\xi$, with A a bounded and measurable uniformly elliptic matrix, and let $u \ge 0$ be a solution of

$$-\mathrm{div}\left(\frac{A(x)\nabla u}{(1+u)^\gamma}\right) + u = f\,,$$

with $\gamma > 1$ and $f \ge 0$. If we define

$$z = \frac{1}{\gamma-1}\left(1 - \frac{1}{(1+u)^{\gamma-1}}\right),$$

then z is a solution of

$$-\mathrm{div}(A(x)\nabla z) + \left(\frac{1}{(1-(\gamma-1)z)^{\frac{1}{\gamma-1}}} - 1\right) = f\,,$$

which is an equation whose lower order term becomes singular as z tends to the value $\frac{1}{\gamma-1}$. For a study of these problems, see [4] and [14].

Remark 1.3. We explicitely state that our existence results can be generalized to equations with differential operators defined on $W_0^{1,p}(\Omega)$, with $p > 1$: if $\gamma \ge \frac{(p-2)_+}{p-1}$ and if $m = \frac{\gamma(p-1)+2}{p}$, then it is possible to prove the existence of a distributional solution u in $W_0^{1,1}(\Omega) \cap L^m(\Omega)$ of the boundary value problem

$$
\begin{cases} -\mathrm{div}\left(\dfrac{a(x,\nabla u)}{(1+|u|)^{\gamma(p-1)}} \right) + u = f & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}
$$

where $a(x,\xi)$ satisfies (1.2), (1.3) and (1.4) with p instead of 2 (in (1.3), a grows as $|\xi|^{p-1}$).

2. Existence of a $W_0^{1,1}(\Omega)$ solution

In this section we prove the existence of a $W_0^{1,1}(\Omega)$ solution to problem (1.1). Our result is the following.

Theorem 2.1. *Let $\gamma > 0$, and let f be a function in $L^{\frac{\gamma+2}{2}}(\Omega)$. Then there exists a distributional solution u in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)$ of (1.1), that is,*

$$
\int_\Omega \frac{a(x,\nabla u)\cdot\nabla\varphi}{(1+|u|)^\gamma} + \int_\Omega u\,\varphi = \int_\Omega f\,\varphi\,, \quad \forall \varphi \in W_0^{1,\infty}(\Omega)\,. \tag{2.1}
$$

Remark 2.2. The previous result gives existence of a solution u in $W_0^{1,1}(\Omega)$ to (1.6) for every $\gamma > 0$. If $0 < \gamma \le 1$ existence results for (1.1) can also be proved by the same techniques of [7]. More precisely, if f belongs to $L^m(\Omega)$ with $m > \frac{N}{N(1-\gamma)+1+\gamma}$ then (1.1) has a solution in $W_0^{1,q}(\Omega)$, with $q = \frac{Nm(1-\gamma)}{N-m(1+\gamma)}$. Note that when m tends to $\frac{N}{N(1-\gamma)+1+\gamma}$, then q tends to 1. We have now two cases: if $\frac{\gamma+2}{2} > \frac{N}{N(1-\gamma)+1+\gamma}$, that is, if $0 < \gamma < \frac{2}{N-1}$, our result is weaker than the one in [7]. On the other hand, if $\frac{2}{N-1} \le \gamma \le 1$, then our result, which strongly uses the lower order term of order zero, is better.

Remark 2.3. The same existence result, with the same proof, holds for the following boundary value problem

$$
\begin{cases} -\mathrm{div}(b(x,u,\nabla u)) + u = f & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases}
$$

with $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ a Carathéodory function such that

$$
\frac{\alpha|\xi|^2}{(1+|s|)^\gamma} \le b(x,s,\xi)\cdot\xi \le \beta|\xi|^2\,,
$$

where α, β, γ are positive constants.

To prove Theorem 2.1 we will work by approximation. First of all, let g be a function in $L^\infty(\Omega)$. Then, by the results of [5], there exists a solution v in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x,\nabla v)}{(1+|v|)^\gamma}\right) + v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

In order for this paper to be self contained, we give here the easy proof of this fact. Let $M = \|g\|_{L^\infty(\Omega)} + 1$, and consider the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x,\nabla v)}{(1+|T_M(v)|)^\gamma}\right) + v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here and in the following we define $T_k(s) = \max(-k, \min(s,k))$ for $k \geq 0$ and s in $\mathbb{R}$. Since the differential operator is pseudomonotone and coercive thanks to the assumptions on a and to the truncature, by the results of [18] there exists a weak solution v in $H_0^1(\Omega)$ of (2.3). Choosing $(|v| - \|g\|_{L^\infty(\Omega)})_+ \operatorname{sgn}(v)$ as a test function we obtain, dropping the nonnegative first term,

$$\int_\Omega |v|\,(|v| - \|g\|_{L^\infty(\Omega)})_+ \leq \int_\Omega \|g\|_{L^\infty(\Omega)}\,(|v| - \|g\|_{L^\infty(\Omega)})_+ \,.$$

Thus,

$$\int_\Omega (|v| - \|g\|_{L^\infty(\Omega)})\,(|v| - \|g\|_{L^\infty(\Omega)})_+ \leq 0\,,$$

so that $|v| \leq \|g\|_{L^\infty(\Omega)} < M$. Therefore, $T_M(v) = v$, and v is a bounded weak solution of (2.2).

Let now f_n be a sequence of $L^\infty(\Omega)$ functions which converges to f in $L^{\frac{\gamma+2}{2}}(\Omega)$, and such that $|f_n| \leq |f|$ almost everywhere in Ω, and consider the approximating problems

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x,\nabla u_n)}{(1+|u_n|)^\gamma}\right) + u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

A solution u_n in $H_0^1(\Omega) \cap L^\infty(\Omega)$ exists choosing $g = f_n$ in (2.2). We begin with some *a priori* estimates on the sequence $\{u_n\}$.

LEMMA 2.4. *If u_n is a solution to problem (2.4), then, for every $k \geq 0$,*

$$\int_{\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}\,; \tag{2.5}$$

$$\int_{\{|u_n|\geq k\}} \frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}} \leq C\left[\int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}\right]^{\frac{2}{\gamma+2}}\,; \tag{2.6}$$

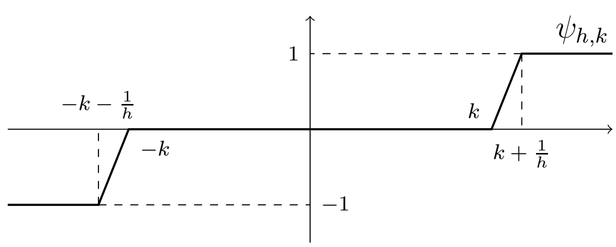
$$\int_{\{|u_n|\geq k\}} |\nabla u_n| \leq C\left[\int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}\right]^{\frac{1}{\gamma+2}}\,; \tag{2.7}$$

$$\alpha\int_\Omega |\nabla T_k(u_n)|^2 \leq k\,(1+k)^\gamma \int_\Omega |f|\,. \tag{2.8}$$

Here, and in the following, C denotes a positive constant depending on α, γ, $\text{meas}(\Omega)$, and the norm of f in $L^{\frac{\gamma+2}{2}}(\Omega)$.

Proof. Let $k \geq 0$, $h > 0$, and let $\psi_{h,k}(s)$ be the function defined by

$$\psi_{h,k}(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq k, \\ h\,(s-k) & \text{if } k < s \leq k+\frac{1}{h}, \\ 1 & \text{if } s > k+\frac{1}{h}, \\ \psi_{h,k}(s) = -\psi_{h,k}(-s) & \text{if } s < 0. \end{cases}$$



Note that

$$\lim_{h\to+\infty} \psi_{h,k}(s) = \begin{cases} 1 & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ -1 & \text{if } s < -k. \end{cases}$$

Let $\varepsilon > 0$, and choose $(\varepsilon+|u_n|)^{\frac{\gamma}{2}}\,\psi_{h,k}(u_n)$ as a test function in (2.4); such a test function is admissible since u_n belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$

and $\psi_{h,k}(0) = 0$. We obtain

$$
\begin{aligned}
&\frac{\gamma}{2}\int_\Omega a(x,\nabla u_n)\cdot\nabla u_n \frac{(\varepsilon+|u_n|)^{\frac{\gamma}{2}-1}}{(1+|u_n|)^\gamma}\,|\psi_{h,k}(u_n)| \\
&\qquad + \int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|u_n|)^\gamma}\,\psi'_{h,k}(u_n)\,(\varepsilon+|u_n|)^{\frac{\gamma}{2}} \\
&\qquad + \int_\Omega u_n\,(\varepsilon+|u_n|)^{\frac{\gamma}{2}}\,\psi_{h,k}(u_n) \\
&\quad = \int_\Omega f_n\,(\varepsilon+|u_n|)^{\frac{\gamma}{2}}\,\psi_{h,k}(u_n)\,.
\end{aligned}
\tag{2.9}
$$

By (1.2), and since $\psi'_{h,k}(s) \geq 0$, the first two terms are nonnegative, so that we obtain, recalling that $|f_n| \leq |f|$,

$$
\int_\Omega u_n\,(\varepsilon+|u_n|)^{\frac{\gamma}{2}}\,\psi_{h,k}(u_n) \leq \int_\Omega |f|\,(\varepsilon+|u_n|)^{\frac{\gamma}{2}}\,|\psi_{h,k}(u_n)|\,.
$$

Letting ε tend to zero and h tend to infinity, we obtain, by Fatou's lemma (on the left hand side) and by Lebesgue's theorem (on the right hand side, recall that u_n belongs to $L^\infty(\Omega)$),

$$
\int_{\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \int_{\{|u_n|\geq k\}} |f|\,|u_n|^{\frac{\gamma}{2}}\,.
$$

Using Hölder's inequality on the right hand side we obtain

$$
\int_{\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \left[\int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}\right]^{\frac{2}{\gamma+2}} \left[\int_{\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}}\right]^{\frac{\gamma}{\gamma+2}}.
$$

Simplifying equal terms we thus have

$$
\int_{\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}\,,
$$

which is (2.5). Note that from (2.5), written for $k = 0$, it follows

$$
\int_\Omega |u_n|^{\frac{\gamma+2}{2}} \leq \int_\Omega |f|^{\frac{\gamma+2}{2}} = \|f\|^{\frac{\gamma+2}{2}}_{L^{\frac{\gamma+2}{2}}(\Omega)}\,.
\tag{2.10}
$$

Now we consider (2.9) written for $\varepsilon = 1$. Dropping the nonnegative second and third terms, and using that $|f_n| \leq |f|$, we have

$$
\frac{\gamma}{2}\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|u_n|)^{\frac{\gamma+2}{2}}}\,|\psi_{h,k}(u_n)| \leq \int_\Omega |f|(1+|u_n|)^{\frac{\gamma+2}{2}}|\psi_{h,k}(u_n)|\,.
$$

Using (1.2), and letting h tend to infinity, we get (using again Fatou's lemma and Lebesgue's theorem)

$$
\alpha\frac{\gamma}{2}\int_{\{|u_n|\geq k\}} \frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}} \leq \int_{\{|u_n|\geq k\}} |f|(1+|u_n|)^{\frac{\gamma}{2}}\,.
$$

Hölder's inequality on the right hand side then gives

$$\begin{aligned}
&\alpha\frac{\gamma}{2}\int_{\{|u_n|\geq k\}}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}}\\
&\qquad\leq\left[\int_{\{|u_n|\geq k\}}|f|^{\frac{\gamma+2}{2}}\right]^{\frac{2}{\gamma+2}}\left[\int_{\{|u_n|\geq k\}}(1+|u_n|)^{\frac{\gamma+2}{2}}\right]^{\frac{\gamma}{\gamma+2}}\\
&\qquad\leq\left[\int_{\{|u_n|\geq k\}}|f|^{\frac{\gamma+2}{2}}\right]^{\frac{2}{\gamma+2}}\left[\int_{\Omega}(1+|u_n|)^{\frac{\gamma+2}{2}}\right]^{\frac{\gamma}{\gamma+2}},
\end{aligned}$$

so that, by (2.10),

$$\alpha\frac{\gamma}{2}\int_{\{|u_n|\geq k\}}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}}\leq C\left[\int_{\{|u_n|\geq k\}}|f|^{\frac{\gamma+2}{2}}\right]^{\frac{2}{\gamma+2}},$$

which is (2.6).

Then, again by Hölder's inequality, and by (2.6) and (2.10),

$$\begin{aligned}
&\int_{\{|u_n|\geq k\}}|\nabla u_n|=\int_{\{|u_n|\geq k\}}\frac{|\nabla u_n|}{(1+|u_n|)^{\frac{\gamma+2}{4}}}(1+|u_n|)^{\frac{\gamma+2}{4}}\\
&\qquad\leq\left[\int_{\{|u_n|\geq k\}}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}}\right]^{\frac{1}{2}}\left[\int_{\{|u_n|\geq k\}}(1+|u_n|)^{\frac{\gamma+2}{2}}\right]^{\frac{1}{2}}\\
&\qquad\leq C\left[\int_{\{|u_n|\geq k\}}|f|^{\frac{\gamma+2}{2}}\right]^{\frac{1}{\gamma+2}}\left[\int_{\Omega}(1+|u_n|)^{\frac{\gamma+2}{2}}\right]^{\frac{1}{2}}\\
&\qquad\leq C\left[\int_{\{|u_n|\geq k\}}|f|^{\frac{\gamma+2}{2}}\right]^{\frac{1}{\gamma+2}},
\end{aligned}\tag{2.11}$$

so that (2.7) is proved.

Finally, choosing $T_k(u_n)$ as a test function in (2.4) we get, dropping the nonnegative linear term, and using (1.2),

$$\alpha\int_{\Omega}|\nabla T_k(u_n)|^2\leq k(1+k)^{\gamma}\int_{\Omega}|f|,$$

which is (2.8). □

LEMMA 2.5. *If $\{u_n\}$ is the sequence of solutions to (2.4), there exists a subsequence, still denoted by $\{u_n\}$, and a function u in $L^{\frac{\gamma+2}{2}}(\Omega)$, with $T_k(u)$ belonging to $H_0^1(\Omega)$ for every $k>0$, such that u_n almost everywhere converges to u in Ω, and $T_k(u_n)$ weakly converges to $T_k(u)$ in $H_0^1(\Omega)$.*

Proof. Consider (2.6) written for $k=0$:

$$\int_{\Omega}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}}\leq C\,\|f\|_{L^{\frac{\gamma+2}{2}}(\Omega)}.\tag{2.12}$$

Since (if $\gamma \neq 2$)

$$\frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}} = \frac{16}{(2-\gamma)^2}\,|\nabla[(1+|u_n|)^{\frac{2-\gamma}{4}}-1]|^2\,,$$

the sequence $v_n = \frac{4}{2-\gamma}[(1+|u_n|)^{\frac{2-\gamma}{4}}-1]\mathrm{sgn}(u_n)$ is bounded in $H_0^1(\Omega)$ by (2.12). If $\gamma = 2$ we have

$$\frac{|\nabla u_n|^2}{(1+|u_n|)^2} = |\nabla\log(1+|u_n|)|^2\,,$$

so that $v_n = [\log(1+|u_n|)]\mathrm{sgn}(u_n)$ is bounded in $H_0^1(\Omega)$. In both cases, up to a subsequence still denoted by v_n, v_n converges to some function v weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω. If $\gamma < 2$, define

$$u(x) = \left[\left(\frac{2-\gamma}{4}|v(x)|+1\right)^{\frac{4}{2-\gamma}}-1\right]\mathrm{sgn}(v(x))\,,$$

if $\gamma > 2$ define

$$u(x) = \begin{cases} \left[\left(\frac{2-\gamma}{4}|v(x)|+1\right)^{\frac{4}{2-\gamma}}-1\right]\mathrm{sgn}(v(x)) & \text{if } |v(x)| < \frac{4}{\gamma-2},\\ +\infty & \text{if } v(x) = \frac{4}{\gamma-2},\\ -\infty & \text{if } v(x) = -\frac{4}{\gamma-2}, \end{cases}$$

while if $\gamma = 2$, define

$$u(x) = [\mathrm{e}^{|v(x)|}-1]\mathrm{sgn}(v(x))\,.$$

Thus, u_n almost everywhere converges, up to a subsequence still denoted by u_n, to u. From now on, we will consider this particular subsequence, for which it holds that u_n almost everywhere converges to u.

We use now (2.5) written for $k = 0$:

$$\int_\Omega |u_n|^{\frac{\gamma+2}{2}} \leq \int_\Omega |f|^{\frac{\gamma+2}{2}} \leq C\,.$$

Since u_n almost everywhere converges to u, we have from Fatou's lemma that

$$\int_\Omega |u|^{\frac{\gamma+2}{2}} \leq C\,.$$

Hence u belongs to $L^{\frac{\gamma+2}{2}}(\Omega)$, which implies that u is almost everywhere finite (note that if $\gamma > 2$ this fact did not follow from the definition of u, since $|v|$ could have assumed the value $\frac{4}{\gamma-2}$ on a set of positive measure).

Let now $k > 0$; since from (2.8) it follows that the sequence $\{T_k(u_n)\}$ is bounded in $H_0^1(\Omega)$, there exists a subsequence $T_k(u_{n_j})$ which weakly converges to some function v_k in $H_0^1(\Omega)$. Using the almost everywhere convergence of u_n to u, we have that $v_k = T_k(u)$. Since the limit is independent on the subsequence, then the whole sequence $\{T_k(u_n)\}$ weakly converges to $T_k(u)$, for every $k > 0$. □

REMARK 2.6. Using the fact that $T_k(u)$ is in $H_0^1(\Omega)$ for every $k > 0$, and the results of [3], we have that there exists a unique measurable function v with values in $\mathbb{R}^N$, such that

$$\nabla T_k(u) = v\,\chi_{\{|u|\leq k\}} \qquad \text{almost everywhere in } \Omega, \text{ for every } k > 0.$$

Following again [3], we will *define* $\nabla u = v$, the approximate gradient of u.

REMARK 2.7. We emphasize that if $\gamma = 2$, then (2.11), written for $k = 0$, becomes

$$\int_\Omega |\nabla u_n| \leq \left[\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^2}\right]^{\frac{1}{2}} \left[\int_\Omega (1+|u_n|)^2\right]^{\frac{1}{2}}.$$

Since

$$\frac{|\nabla u_n|^2}{(1+|u_n|)^2} = |\nabla \log(1+|u_n|)|^2,$$

a nonlinear interpolation result follows: let A be in $\mathbb{R}^+$ and let v in $L^2(\Omega)$ be such that $\log(A+|v|)$ belongs to $H_0^1(\Omega)$. Then v belongs to $W_0^{1,1}(\Omega)$, and

$$\int_\Omega |\nabla v| \leq \|\log(A+|v|)\|_{H_0^1(\Omega)} \left[\int_\Omega (A+|v|)^2\right]^{\frac{1}{2}}.$$

Our next result deals with the strong convergence of $T_k(u_n)$ in $H_0^1(\Omega)$.

PROPOSITION 2.8. *Let u_n and u be the sequence of solutions to problems (2.4) and the function in $L^{\frac{\gamma+2}{2}}(\Omega)$ given by Lemma 2.5. Then, for every fixed $k > 0$, $T_k(u_n)$ strongly converges to $T_k(u)$ in $H_0^1(\Omega)$, as n tends to infinity.*

Proof. We follow the proof of [17].

Let $h > k$ and choose $T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]$ as a test function in (2.4). We have

$$\begin{aligned}(2.13)\qquad &\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]}{(1+|u_n|)^\gamma} \\ &\qquad = -\int_\Omega (u_n - f_n)\, T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)].\end{aligned}$$

We observe that the right hand side converges to zero as first n and then h tend to infinity, since u_n converges to u almost everywhere in Ω and u_n and f_n are bounded in $L^{\frac{\gamma+2}{2}}(\Omega)$. Thus, if we define $\varepsilon(n,h)$ as any quantity such that

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\varepsilon(n,h)=0\,,$$

then

$$\int_\Omega (u_n-f_n)\,T_{2k}[u_n-T_h(u_n)+T_k(u_n)-T_k(u)]=\varepsilon(n,h)\,.$$

Let $M=4k+h$. Observing that $\nabla T_{2k}[u_n-T_h(u_n)+T_k(u_n)-T_k(u)]=0$ if $|u_n|\geq M$, by (2.13) we have

$$\begin{aligned}\varepsilon(n,h)=&\int_{\{|u_n|<k\}}\frac{a(x,\nabla T_M(u_n))\cdot\nabla[u_n-T_h(u_n)+T_k(u_n)-T_k(u)]}{(1+|u_n|)^\gamma}\\ &+\int_{\{|u_n|\geq k\}}\frac{a(x,\nabla T_M(u_n))\cdot\nabla[u_n-T_h(u_n)+T_k(u_n)-T_k(u)]}{(1+|u_n|)^\gamma}\,.\end{aligned}$$

Since $u_n-T_h(u_n)=0$ in $\{|u_n|\leq k\}$ and $\nabla T_k(u_n)=0$ in $\{|u_n|\geq k\}$, we have, using that $a(x,0)=0$,

$$\begin{aligned}\varepsilon(n,h)=&\int_\Omega\frac{a(x,\nabla T_k(u_n))\cdot\nabla[T_k(u_n)-T_k(u)]}{(1+|u_n|)^\gamma}\\ &+\int_{\{|u_n|\geq k\}}\frac{a(x,\nabla T_M(u_n))\cdot\nabla[u_n-T_h(u_n)]}{(1+|u_n|)^\gamma}\\ &-\int_{\{|u_n|\geq k\}}\frac{a(x,\nabla T_M(u_n))\cdot\nabla T_k(u)}{(1+|u_n|)^\gamma}\,.\end{aligned}$$

The second term of the right hand side is positive, so that

$$\begin{aligned}\varepsilon(n,h)\geq&\int_\Omega\frac{[a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u))]\cdot\nabla[T_k(u_n)-T_k(u)]}{(1+k)^\gamma}\\ &+\int_\Omega\frac{a(x,\nabla T_k(u))\cdot\nabla[T_k(u_n)-T_k(u)]}{(1+|u_n|)^\gamma}\\ &-\int_{\{|u_n|\geq k\}}\frac{a(x,\nabla T_M(u_n))\cdot\nabla T_k(u)}{(1+|u_n|)^\gamma}=I_n+J_n-K_n\,.\end{aligned}$$

The last two terms tend to zero as n tends to infinity. Indeed

$$\lim_{n\to+\infty}J_n=\lim_{n\to+\infty}\int_\Omega\frac{a(x,\nabla T_k(u))\cdot\nabla[T_k(u_n)-T_k(u)]}{(1+|u_n|)^\gamma}=0\,,$$

since $T_k(u_n)$ converges to $T_k(u)$ weakly in $H_0^1(\Omega)$ and $\frac{a(x,\nabla T_k(u))}{(1+|u_n|)^\gamma}$ is strongly compact in $(L^2(\Omega))^N$ by the growth assumption (1.3) on a.

The last term can be rewritten as

$$K_n = \int_\Omega \frac{a(x,\nabla T_M(u_n))\cdot\nabla T_k(u)\chi_{\{|u_n|\geq k\}}}{(1+|u_n|)^\gamma}\,.$$

Since M is fixed with respect to n, then the sequence $\{a(x,\nabla T_M(u_n))\}$ is bounded in $(L^2(\Omega))^N$. Hence, there exists σ in $(L^2(\Omega))^N$, and a subsequence $\{a(x,\nabla T_M(u_{n_j}))\}$, such that

$$\lim_{j\to+\infty} a(x,\nabla T_M(u_{n_j})) = \sigma\,,$$

weakly in $(L^2(\Omega))^N$. On the other hand,

$$\lim_{n\to+\infty} \frac{\nabla T_k(u)\chi_{\{k\leq|u_n|\}}}{(1+|u_n|)^\gamma} = \frac{\nabla T_k(u)\chi_{\{k\leq|u|\}}}{(1+|u|)^\gamma} = 0\,,$$

strongly in $(L^2(\Omega))^N$, and so

$$\lim_{j\to+\infty} K_{n_j} = \lim_{j\to+\infty}\int_{\{|u_{n_j}|\geq k\}} \frac{a(x,\nabla T_M(u_{n_j}))\cdot\nabla T_k(u)}{(1+|u_{n_j}|)^\gamma} = 0\,.$$

Since the limit does not depend on the subsequence, we have

$$\lim_{n\to+\infty} K_n = \lim_{n\to+\infty}\int_{\{|u_n|\geq k\}} \frac{a(x,\nabla T_M(u_n))\cdot\nabla T_k(u)}{(1+|u_n|)^\gamma} = 0\,,$$

as desired. Therefore,

$$\varepsilon(n,h) \geq I_n = \int_\Omega \frac{[a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))]\cdot\nabla[T_k(u_n)-T_k(u)]}{(1+k)^\gamma}\,,$$

so that, thanks to (1.4),

$$\lim_{n\to+\infty}\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))]\cdot\nabla[T_k(u_n)-T_k(u)] = 0\,.$$

Using this formula, (1.4) and the results of [10] and [8], we then conclude that $T_k(u_n)$ strongly converges to $T_k(u)$ in $H_0^1(\Omega)$, as desired. $\square$

COROLLARY 2.9. *Let u_n and u be as in Proposition 2.8. Then ∇u_n converges to ∇u almost everywhere in Ω, where ∇u has been defined in Remark 2.6.*

LEMMA 2.10. *Let u_n and u be as in Proposition 2.8. Then ∇u_n strongly converges to ∇u in $(L^1(\Omega))^N$. Moreover u_n strongly converges to u in $L^{\frac{\gamma+2}{2}}(\Omega)$.*

Proof. We begin by proving the convergence of ∇u_n to ∇u. Let $\varepsilon > 0$, and let $k > 0$ be sufficiently large such that

$$\left[\int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}\right]^{\frac{1}{\gamma+2}} < \varepsilon\,, \tag{2.14}$$

uniformly with respect to n. This can be done thanks to (2.10) and to the absolute continuity of the integral. Let E be a measurable set. Writing

$$\int_E |\nabla u_n| = \int_E |\nabla T_k(u_n)| + \int_{E\cap\{|u_n|\geq k\}} |\nabla u_n|$$

we have, by (2.7), and by (2.14),

$$\int_E |\nabla u_n| \leq \int_E |\nabla T_k(u_n)| + C\varepsilon\,.$$

Using Hölder's inequality and (2.8), we obtain

$$\int_E |\nabla u_n| \leq C\,\mathrm{meas}(E)^{\frac{1}{2}} k^{\frac{1}{2}}\,(1+k)^{\frac{\gamma}{2}} \left(\int_\Omega |f|\right)^{\frac{1}{2}} + C\varepsilon\,.$$

Choosing $\mathrm{meas}(E)$ small enough (recall that k is now fixed) we have

$$\int_E |\nabla u_n| \leq C\varepsilon\,,$$

uniformly with respect to n, where C does not depend on n or ε. Since ∇u_n almost everywhere converges to ∇u by Corollary 2.9, we can apply Vitali's theorem to obtain the strong convergence of ∇u_n to ∇u in $(L^1(\Omega))^N$.

As for the second convergence, by (2.5) we have

$$\begin{aligned}\int_E |u_n|^{\frac{\gamma+2}{2}} &\leq \int_{E\cap\{|u_n|\leq k\}} |u_n|^{\frac{\gamma+2}{2}} + \int_{E\cap\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}} \\ &\leq k^{\frac{\gamma+2}{2}}\mathrm{meas}(E) + \int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}\,.\end{aligned}$$

As before, we first choose k such that the second integral is small, uniformly with respect to n, and then the measure of E small enough such that the first term is small. The almost everywhere convergence of u_n to u, and Vitali's theorem, then imply that u_n strongly converges to u in $L^{\frac{\gamma+2}{2}}(\Omega)$. □

Remark 2.11. Since we have proved that ∇u_n strongly converges to ∇u in $(L^1(\Omega))^N$, so that u belongs to $W_0^{1,1}(\Omega)$, then the approximate gradient ∇u of u is nothing but the distributional gradient of u (see [3]).

Proof of Theorem 2.1. Using the previous results, we pass to the limit, as n tends to infinity, in the weak formulation of (2.4). Starting from

$$\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\varphi}{(1+|u_n|)^\gamma} + \int_\Omega u_n\,\varphi = \int_\Omega f_n\,\varphi\,, \qquad \varphi\in W_0^{1,\infty}(\Omega)\,,$$

the limit of the second and the last integral is easy to compute; indeed, recall that by Lemma 2.10, and by definition of f_n, the sequences $\{u_n\}$ and $\{f_n\}$ strongly converge to u and f respectively in $L^{\frac{\gamma+2}{2}}(\Omega)$, hence in $L^1(\Omega)$. For the first integral, we have that $a(x, \nabla u_n)$ converges almost everywhere in Ω to $a(x, \nabla u)$ thanks to Corollary 2.9, and the continuity assumption on $a(x, \cdot)$; furthermore, (1.3) implies that

$$|a(x, \nabla u_n)| \leq \beta |\nabla u_n|\,,$$

and the right hand side is compact in $L^1(\Omega)$ by Lemma 2.10. Thus, by Vitali's theorem $a(x, \nabla u_n)$ strongly converges to $a(x, \nabla u)$ in $(L^1(\Omega))^N$, so that

$$\lim_{n\to+\infty} \int_\Omega \frac{a(x, \nabla u_n)\cdot\nabla\varphi}{(1+|u_n|)^\gamma} = \int_\Omega \frac{a(x, \nabla u)\cdot\nabla\varphi}{(1+|u|)^\gamma}\,,$$

where we have also used that u_n almost everywhere converges to u, and Lebesgue's theorem. Thus, we have that

$$\int_\Omega \frac{a(x, \nabla u)\cdot\nabla\varphi}{(1+|u|)^\gamma} + \int_\Omega u\,\varphi = \int_\Omega f\,\varphi\,, \qquad \forall \varphi \in W_0^{1,\infty}(\Omega)\,,$$

i.e., u satisfies (2.1). □

3. Uniqueness of the solution obtained by approximation

Let $f \in L^{\frac{\gamma+2}{2}}(\Omega)$, let f_n be a sequence of $L^\infty(\Omega)$ functions converging to f in $L^{\frac{\gamma+2}{2}}(\Omega)$, with $|f_n| \leq |f|$, and let u_n be a solution of (2.4). In Section 2 we proved the existence of a distributional solution u in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)$ to (1.1), such that, up to a subsequence,

$$\lim_{n\to+\infty} \|u_n - u\|_{W_0^{1,1}(\Omega)\cap L^{\frac{\gamma+2}{2}}(\Omega)} = 0\,. \tag{3.1}$$

Now, let $g \in L^{\frac{\gamma+2}{2}}(\Omega)$, let g_n be a sequence of $L^\infty(\Omega)$ functions converging to g in $L^{\frac{\gamma+2}{2}}(\Omega)$, with $|g_n| \leq |g|$, and let z_n in $H_0^1(\Omega) \cap L^\infty(\Omega)$ be a weak solution of

$$\begin{cases} -\mathrm{div}\left(\dfrac{a(x, \nabla z_n)}{(1+|z_n|)^\gamma}\right) + z_n = g_n & \text{in } \Omega, \\ z_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Then, up to a subsequence, we can assume that

$$\lim_{n\to+\infty} \|z_n - z\|_{W_0^{1,1}(\Omega)\cap L^{\frac{\gamma+2}{2}}(\Omega)} = 0\,, \tag{3.3}$$

where z in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)$ is a distributional solution of

$$\begin{cases} -\mathrm{div}\left(\dfrac{a(x,\nabla z)}{(1+|z|)^\gamma}\right) + z = f & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Our result, which will imply the uniqueness of the solution by approximation (see [12]) of (1.1), is the following.

THEOREM 3.1. *Assume that u_n and z_n are solutions of (2.4) and (3.2) respectively, and that (3.1) and (3.3) hold true, with u and z solutions of (1.1) and (3.4) respectively. Then*

$$\int_\Omega |u - z| \le \int_\Omega |f - g| \,. \tag{3.5}$$

Moreover,

$$f \le g \text{ a.e. in } \Omega \quad \text{implies} \quad u \le z \text{ a.e. in } \Omega. \tag{3.6}$$

Proof. Substracting (3.2) from (2.4) we obtain

$$-\mathrm{div}\left(\left[\frac{a(x,\nabla u_n)}{(1+|u_n|)^\gamma} - \frac{a(x,\nabla z_n)}{(1+|z_n|)^\gamma}\right]\right) + u_n - z_n = f_n - g_n \,.$$

Choosing $T_h(u_n - z_n)$ as a test function we have

$$\begin{aligned} &\int_\Omega \left[\frac{a(x,\nabla u_n)}{(1+|u_n|)^\gamma} - \frac{a(x,\nabla z_n)}{(1+|z_n|)^\gamma}\right] \cdot \nabla T_h(u_n - z_n) \\ &\quad + \int_\Omega (u_n - z_n) T_h(u_n - z_n) = \int_\Omega (f_n - g_n) T_h(u_n - z_n) \,. \end{aligned}$$

This equality can be written in an equivalent way as

$$\begin{aligned} &\int_\Omega \frac{[a(x,\nabla u_n) - a(x,\nabla z_n)] \cdot \nabla T_h(u_n - z_n)}{(1+|u_n|)^\gamma} \\ &\quad + \int_\Omega (u_n - z_n) T_h(u_n - z_n) = \int_\Omega (f_n - g_n) T_h(u_n - z_n) \\ &\quad - \int_\Omega \left[\frac{1}{(1+|u_n|)^\gamma} - \frac{1}{(1+|z_n|)^\gamma}\right] a(x,\nabla z_n) \cdot \nabla T_h(u_n - z_n) \,. \end{aligned}$$

By (1.4), the first term of the left hand side is nonnegative, so that it can be dropped; using Lagrange's theorem on the last term of the right hand side, we therefore have, since the absolute value of the derivative of the function $s \mapsto \frac{1}{(1+|s|)^\gamma}$ is bounded by γ,

$$\begin{aligned}\int_\Omega (u_n - z_n) T_h(u_n - z_n) &\le \int_\Omega (f_n - g_n) T_h(u_n - z_n) \\ &+ \gamma h \int_\Omega |a(x, \nabla z_n)| |\nabla T_h(u_n - z_n)| \,.\end{aligned}$$

Dividing by h we obtain

$$\begin{aligned}\int_\Omega (u_n - z_n) \frac{T_h(u_n - z_n)}{h} &\le \int_\Omega |f_n - g_n| \frac{|T_h(u_n - z_n)|}{h} \\ &+ \gamma \int_\Omega |a(x, \nabla z_n)| |\nabla T_h(u_n - z_n)| \,.\end{aligned}$$

Since, for every fixed n, u_n and z_n belong to $H_0^1(\Omega)$, and $a(x,\xi)$ satisfies (1.3), the limit as h tends to zero gives

$$\int_\Omega |u_n - z_n| \le \int_\Omega |f_n - g_n| \,, \tag{3.7}$$

which then yields (3.5) passing to the limit and using the second part of Lemma 2.10.

The use of $T_h(u_n - z_n)^+$ as a test function and the same technique as above imply that

$$\int_\Omega (u_n - z_n)^+ \le \int_{\{u_n \ge z_n\}} (f_n - g_n) \,.$$

Hence, passing to the limit as n tends to infinity, we obtain, if we suppose that $f \le g$ almost everywhere in Ω,

$$\int_\Omega (u - z)^+ \le \int_{\{u \ge z\}} (f - g) \le 0 \,,$$

so that (3.6) is proved. $\square$

Thanks to (3.5), we can prove that problem (1.1) has a unique solution obtained by approximation.

Corollary 3.2. *There exists a unique solution obtained by approximation of (1.1), in the sense that the solution u in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)$ obtained as limit of the sequence u_n of solutions of (2.4) does not depend on the sequence f_n chosen to approximate the datum f in $L^{\frac{\gamma+2}{2}}(\Omega)$.*

Remark 3.3. Note that (3.7) implies the uniqueness of the solution of (2.2), while (3.6) implies that if $f \ge 0$, then the solution u of (1.1) is nonnegative.

Remark 3.4. Corollary 3.2, together with estimates (3.5) and (2.5), implies that the map S from $L^{\frac{\gamma+2}{2}}(\Omega)$ into itself defined by $S(f) = u$, where u is the solution of (1.1) with datum f, is well defined and satisfies

$$\|S(f) - S(g)\|_{L^1(\Omega)} \leq \|f - g\|_{L^1(\Omega)}, \quad \|S(f)\|_{L^{\frac{\gamma+2}{2}}(\Omega)} \leq \|f\|_{L^{\frac{\gamma+2}{2}}(\Omega)}.$$

4. A non existence result

As stated in the Introduction, we prove here a non existence result for solutions of (1.1) if the datum is a bounded Radon measure concentrated on a set E of zero harmonic capacity.

Theorem 4.1. *Assume that $\gamma > 1$, and let μ be a nonnegative Radon measure, concentrated on a set E of zero harmonic capacity. Then there is no solution to*

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x,\nabla u)}{(1+u)^\gamma}\right) + u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

More precisely, if $\{f_n\}$ is a sequence of nonnegative $L^\infty(\Omega)$ functions which converges to μ in the tight sense of measures, and if u_n is the sequence of solutions to (2.4), then u_n tends to zero almost everyhwere in Ω and

$$\lim_{n\to+\infty} \int_\Omega u_n\,\varphi = \int_\Omega \varphi\, d\mu \qquad \forall\, \varphi \in W_0^{1,\infty}(\Omega).$$

Remark 4.2. A similar non existence result for the case $\gamma \leq 1$ is much more complicated to obtain. Indeed, if for example $a(x,\xi) = \xi$, and $\gamma = 1$, the change of variables $v = \log(1+u)$ yields that v is a solution to

$$\begin{cases} -\Delta v + \mathrm{e}^v - 1 = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Existence and non existence of solutions for such a problem has been studied in [9] (where the concept of "good measure" was introduced) and in [21] (if $N = 2$) and [2] (if $N \geq 3$).

Proof. Let μ be as in the statement. Then (see [13]) for every $\delta > 0$ there exists a function ψ_δ in $C_0^\infty(\Omega)$ such that

$$0 \leq \psi_\delta \leq 1, \qquad \int_\Omega |\nabla\psi_\delta|^2 \leq \delta, \qquad \int_\Omega (1-\psi_\delta)d\mu \leq \delta.$$

Note that, as a consequence of the estimate on ψ_δ in $H_0^1(\Omega)$, and of the fact that $0 \le \psi_\delta \le 1$, ψ_δ tends to zero in the weak* topology of $L^\infty(\Omega)$ as δ tends to zero.

If f_n is a sequence of nonnegative functions which converges to μ in the tight convergence of measures, that is, if

$$\lim_{n\to+\infty} \int_\Omega f_n\, \varphi = \int_\Omega \varphi\, d\mu\,, \qquad \forall \varphi \in C^0(\overline{\Omega})\,,$$

then

$$0 \le \lim_{n\to+\infty} \int_\Omega f_n\,(1-\psi_\delta) = \int_\Omega (1-\psi_\delta)\,d\mu \le \delta\,. \tag{4.1}$$

Let u_n be the nonnegative solution to the approximating problem (2.4). If we choose $1-(1+u_n)^{1-\gamma}$ as a test function in (2.4), we have, by (1.2), and dropping the nonnegative lower order term,

$$\alpha(\gamma-1)\int_\Omega \left|\frac{\nabla u_n}{(1+u_n)^\gamma}\right|^2 \le (\gamma-1)\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+u_n)^{2\gamma}} \le \int_\Omega f_n\,.$$

Recalling (1.3), we thus have

$$\int_\Omega \left|\frac{a(x,\nabla u_n)}{(1+u_n)^\gamma}\right|^2 \le \beta\int_\Omega \left|\frac{\nabla u_n}{(1+u_n)^\gamma}\right|^2 \le C\int_\Omega f_n\,,$$

with C depending on α, β and γ. Therefore, up to a subsequence, there exist σ in $(L^2(\Omega))^N$ and ρ in $L^2(\Omega)$ such that

$$\lim_{n\to+\infty} \frac{a(x,\nabla u_n)}{(1+u_n)^\gamma} = \sigma\,, \quad \lim_{n\to+\infty}\left|\frac{\nabla u_n}{(1+u_n)^\gamma}\right| = \rho\,, \tag{4.2}$$

weakly in $(L^2(\Omega))^N$ and $L^2(\Omega)$ respectively.

The choice of $[1-(1+u_n)^{1-\gamma}](1-\psi_\delta)$ as a test function in (2.4) gives

$$\begin{aligned}
&(\gamma-1)\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+u_n)^{2\gamma}}(1-\psi_\delta)\\
&\qquad + \int_\Omega u_n[1-(1+u_n)^{1-\gamma}](1-\psi_\delta)\\
&\quad = \int_\Omega f_n[1-(1+u_n)^{1-\gamma}](1-\psi_\delta)\\
&\qquad + \int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\psi_\delta}{(1+u_n)^\gamma}[1-(1+u_n)^{1-\gamma}]\\
&\quad \le \int_\Omega f_n(1-\psi_\delta)\\
&\qquad + \int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\psi_\delta}{(1+u_n)^\gamma}[1-(1+u_n)^{1-\gamma}]\,.
\end{aligned} \tag{4.3}$$

We study the right hand side. For the first term, (4.1) implies that

$$\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_\Omega f_n\,(1-\psi_\delta)=0\,,$$

while for the second one, we have, using (4.2), and the boundedness of $[1-(1+u_n)^{1-\gamma}]$,

$$\lim_{n\to+\infty}\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\psi_\delta}{(1+u_n)^\gamma}[1-(1+u_n)^{1-\gamma}]=\int_\Omega \sigma\cdot\nabla\psi_\delta[1-(1+u)^{1-\gamma}]\,.$$

Recalling that σ is in $(L^2(\Omega))^N$, that ψ_δ tends to zero in $H_0^1(\Omega)$, and using the boundedness $[1-(1+u)^{1-\gamma}]$, we have

$$\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\psi_\delta}{(1+u_n)^\gamma}[1-(1+u_n)^{1-\gamma}]=0\,.$$

Therefore, since both terms of the left hand side of (4.3) are nonnegative, we obtain

$$\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+u_n)^{2\gamma}}(1-\psi_\delta)=0\,.$$

Assumption (1.2) then gives

$$\begin{aligned}\lim_{\delta\to 0^+}\lim_{n\to+\infty}\alpha\int_\Omega\left|\frac{\nabla u_n}{(1+u_n)^\gamma}\right|^2(1-\psi_\delta)&\\ \le\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+u_n)^{2\gamma}}(1-\psi_\delta)&=0\,.\end{aligned}$$

Since the functional

$$v\in L^2(\Omega)\mapsto\int_\Omega |v|^2(1-\psi_\delta)$$

is weakly lower semicontinuous on $L^2(\Omega)$, we have

$$\int_\Omega|\rho|^2=\lim_{\delta\to 0^+}\int_\Omega|\rho|^2(1-\psi_\delta)\le\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_\Omega\left|\frac{\nabla u_n}{(1+u_n)^\gamma}\right|^2(1-\psi_\delta)=0\,,$$

which implies that $\rho=0$. Thus, since

$$\frac{\nabla u_n}{(1+u_n)^\gamma}=\frac{1}{\gamma-1}\nabla\left(1-(1+u_n)^{1-\gamma}\right),$$

by the second limit of (4.2) the sequence $1-(1+u_n)^{1-\gamma}$ weakly converges to zero in $H_0^1(\Omega)$, and so (up to subsequences) it strongly converges to zero in $L^2(\Omega)$. Therefore u_n (up to subsequences) tends to zero almost everywhere in Ω. Since the limit does not depend on the subsequence, the whole sequence u_n tends to zero almost everywhere in Ω.

We now have, for Φ in $(L^2(\Omega))^N$, and by (1.3),

$$\left|\int_\Omega \frac{a(x,\nabla u_n)}{(1+|u_n|)^\gamma}\cdot\Phi\right| \le \int_\Omega \left|\frac{a(x,\nabla u_n)}{(1+|u_n|)^\gamma}\right||\Phi| \le \beta\int_\Omega \frac{|\nabla u_n|}{(1+|u_n|)^\gamma}|\Phi|\,.$$

Thus, by (4.2),

$$\left|\int_\Omega \sigma\cdot\Phi\right| = \lim_{n\to+\infty}\left|\int_\Omega \frac{a(x,\nabla u_n)}{(1+|u_n|)^\gamma}\cdot\Phi\right| \le \beta\int_\Omega \rho\,|\Phi| = 0\,,$$

which implies that $\sigma = 0$. Therefore, passing to the limit in (2.4), that is, in

$$\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\varphi}{(1+u_n)^\gamma} + \int_\Omega u_n\,\varphi = \int_\Omega f_n\,\varphi\,, \qquad \varphi\in W_0^{1,\infty}(\Omega)\,,$$

we get, since the first term tends to zero,

$$\lim_{n\to+\infty}\int_\Omega u_n\,\varphi = \int_\Omega \varphi\,d\mu\,,$$

for every φ in $W_0^{1,\infty}(\Omega)$, as desired. □

Remark 4.3. With minor technical changes (see [13]) one can prove the same result if μ is a signed Radon measure concentrated on a set E of zero harmonic capacity.

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