$W_0^{1,1}$ MINIMA OF NON COERCIVE FUNCTIONALS

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Abstract. We study an integral non coercive functional defined on $H_0^1(\Omega)$, proving the existence of a minimum in $W_0^{1,1}(\Omega)$.

In this paper we study a class of integral functionals defined on $H_0^1(\Omega)$, but non coercive on the same space, so that the standard approach of the Calculus of Variations does not work. However, the functionals are coercive on $W_0^{1,1}(\Omega)$ and we will prove the existence of minima, despite the non reflexivity of $W_0^{1,1}(\Omega)$, which implies that, in general, the Direct Methods fail due to lack of compactness.

Let J be the functional defined as

$$J(v) = \int_\Omega \frac{j(x,\nabla v)}{[1+b(x)|v|]^2} + \frac{1}{2}\int_\Omega |v|^2 - \int_\Omega f\, v\,, \quad v \in H_0^1(\Omega)\,.$$

We assume that Ω is a bounded open set of $\mathbb{R}^N$, $N > 2$, that $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is such that $j(\cdot,\xi)$ is measurable on Ω for every ξ in $\mathbb{R}^N$, $j(x,\cdot)$ is convex and belongs to $C^1(\mathbb{R}^N)$ for almost every x in Ω, and

$$\alpha|\xi|^2 \le j(x,\xi) \le \beta|\xi|^2\,, \tag{1}$$

$$|j_\xi(x,\xi)| \le \gamma|\xi|\,, \tag{2}$$

for some positive α, β and γ, for almost every x in Ω, and for every ξ in $\mathbb{R}^N$. We assume that b is a measurable function on Ω such that

$$0 \le b(x) \le B\,, \quad \text{for almost every } x \text{ in } \Omega, \tag{3}$$

where $B > 0$, while f belongs to some Lebesgue space. For $k > 0$ and $s \in \mathbb{R}$, we define the truncature function as $T_k(s) = \max(-k, \min(s,k))$.

In [3] the minimization in $H_0^1(\Omega)$ of the functional

$$I(v) = \int_\Omega \frac{j(x,\nabla v)}{[1+|v|]^\theta} - \int_\Omega f\, v\,, \quad 0<\theta<1,\ f\in L^m(\Omega)\,,$$

was studied. It was proved that $I(v)$ is coercive on the Sobolev space $W_0^{1,q}(\Omega)$, for some $q = q(\theta,m)$ in $(1,2)$, and that $I(v)$ achieves its minimum on $W_0^{1,q}(\Omega)$. This approach does not work for $\theta > 1$ (see Remark 7 below). Here we will able to overcome this difficulty thanks to the presence of the lower order term $\int_\Omega |v|^2$, which will yield the coercivity of J on $W_0^{1,1}(\Omega)$; then we will prove the existence of minima in $W_0^{1,1}(\Omega)$, even if it is a non reflexive space.

Integral functionals like J or I are studied in [1], in the context of the Thomas-Fermi-von Weizsäcker theory.

We are going to prove the following result.

THEOREM 1. *Let $f \in L^2(\Omega)$. Then there exists u in $W_0^{1,1}(\Omega) \cap L^2(\Omega)$ minimum of J, that is,*

(4)
$$\int_\Omega \frac{j(x,\nabla u)}{[1+b(x)|u|]^2} + \frac{1}{2}\int_\Omega |u|^2 - \int_\Omega f u \le \int_\Omega \frac{j(x,\nabla v)}{[1+b(x)|v|]^2} + \frac{1}{2}\int_\Omega |v|^2 - \int_\Omega f v,$$

for every v in $H_0^1(\Omega)$. Moreover $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$.

In [2] we studied the following elliptic boundary problem:

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x)\,\nabla u}{(1+b(x)|u|)^2}\right) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

under the same assumptions on Ω, b and f, with $0 < \alpha \le a(x) \le \beta$. It is easy to see that the Euler equation of J, with $j(x,\xi) = \frac{1}{2}a(x)|\xi|^2$, is not equation (5). Therefore Theorem 1 cannot be deduced from [2]. Nevertheless some technical steps of the two papers (for example, the a priori estimates) are similar.

We will prove Theorem 1 by approximation. Therefore, we begin with the case of bounded data.

LEMMA 2. *If g belongs to $L^\infty(\Omega)$, then there exists a minimum w belonging to $H_0^1(\Omega) \cap L^\infty(\Omega)$ of the functional*

$$v \in H_0^1(\Omega) \mapsto \int_\Omega \frac{j(x,\nabla v)}{[1+b(x)|v|]^2} + \frac{1}{2}\int_\Omega |v|^2 - \int_\Omega g\, v\,.$$

Proof. Since the functional is not coercive on $H_0^1(\Omega)$, we cannot directly apply the standard techniques of the Calculus of Variations. Therefore, we begin by approximating it. Let $M > 0$, and let J_M be the functional defined as

$$J_M(v) = \int_\Omega \frac{j(x,\nabla v)}{[1+b(x)|T_M(v)|]^2} + \frac{1}{2}\int_\Omega |v|^2 - \int_\Omega g\, v\,, \quad v \in H_0^1(\Omega)\,.$$

Since J_M is both weakly lower semicontinuous (due to the convexity of j and to De Giorgi's theorem, see [4]) and coercive on $H_0^1(\Omega)$, for every $M > 0$ there exists a minimum w_M of J_M on $H_0^1(\Omega)$. Let $A = \|g\|_{L^\infty(\Omega)}$, let $M > A$, and consider the inequality $J_M(w_M) \le J_M(T_A(w_M))$, which holds true since w_M is a minimum of J_M. We have

$$\begin{aligned} &\int_\Omega \frac{j(x,\nabla w_M)}{[1+b(x)|T_M(w_M)|]^2} + \frac{1}{2}\int_\Omega |w_M|^2 - \int_\Omega g\, w_M \\ &\quad \le \int_\Omega \frac{j(x,\nabla T_A(w_M))}{[1+b(x)|T_M(T_A(w_M))|]^2} + \frac{1}{2}\int_\Omega |T_A(w_M)|^2 - \int_\Omega g\, T_A(w_M) \\ &\quad = \int_{\{|w_M|\le A\}} \frac{j(x,\nabla w_M)}{[1+b(x)|T_M(w_M)|]^2} + \frac{1}{2}\int_\Omega |T_A(w_M)|^2 - \int_\Omega g T_A(w_M)\,, \end{aligned}$$

where, in the last passage, we have used that $T_M(T_A(w_M)) = T_M(w_M)$ on the set $\{|w_M| \leq A\}$, and that $j(x,0) = 0$. Simplifying equal terms, we thus get

$$\int_{\{|w_M|\geq M\}} \frac{j(x,\nabla w_M)}{[1+b(x)|T_M(w_M)|]^2} + \frac{1}{2}\int_\Omega [|w_M|^2 - |T_A(w_M)|^2] \leq \int_\Omega g\,[w_M - T_A(w_M)]\,.$$

Dropping the first term, which is nonnegative, we obtain

$$\frac{1}{2}\int_\Omega [w_M - T_A(w_M)]\,[w_M + T_A(w_M)] \leq \int_\Omega g\,[w_M - T_A(w_M)]\,,$$

which can be rewritten as

$$\frac{1}{2}\int_\Omega [w_M - T_A(w_M)]\,[w_M + T_A(w_M) - 2g] \leq 0\,.$$

We then have, since $w_M = T_A(w_M)$ on the set $\{|w_M| \leq A\}$,

$$\frac{1}{2}\int_{\{w_M > A\}} [w_M - A][w_M + A - 2g] + \frac{1}{2}\int_{\{w_M < -A\}} [w_M + A][w_M - A - 2g] \leq 0\,.$$

Since $|g| \leq A$, we have $A - 2g \geq -A$, and $-A - 2g < A$, so that

$$0 \leq \frac{1}{2}\int_{\{w_M > A\}} [w_M - A]^2 + \frac{1}{2}\int_{\{w_M < -A\}} [w_M + A]^2 \leq 0\,,$$

which then implies that $\text{meas}(\{|w_M| \geq A\}) = 0$, and so $|w_M| \leq A$ almost everywhere in Ω. Recalling the definition of A, we thus have

$$\|w_M\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}\,. \tag{6}$$

Since $M > \|g\|_{L^\infty(\Omega)}$, we thus have $T_M(w_M) = w_M$. Starting now from $J_M(w_M) \leq J_M(0) = 0$ we obtain, by (6),

$$\int_\Omega \frac{j(x,\nabla w_M)}{[1+b(x)|w_M|]^2} + \frac{1}{2}\int_\Omega |w_M|^2 \leq \int_\Omega g\,w_M \leq \text{meas}(\Omega)\,\|g\|^2_{L^\infty(\Omega)}\,,$$

which then implies, by (1) and (3), and dropping the nonnegative second term,

$$\frac{\alpha}{[1+B\|g\|_{L^\infty(\Omega)}]^2}\int_\Omega |\nabla w_M|^2 \leq \text{meas}(\Omega)\,\|g\|^2_{L^\infty(\Omega)}\,.$$

Thus, $\{w_M\}$ is bounded in $H_0^1(\Omega)\cap L^\infty(\Omega)$, and so, up to subsequences, it converges to some function w in $H_0^1(\Omega)\cap L^\infty(\Omega)$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω. We prove now that

$$\int_\Omega \frac{j(x,\nabla w)}{[1+b(x)|w|]^2} \leq \liminf_{M\to+\infty} \int_\Omega \frac{j(x,\nabla w_M)}{[1+b(x)|w_M|]^2}\,. \tag{7}$$

Indeed, since j is convex, we have

$$\int_\Omega \frac{j(x,\nabla w_M)}{[1+b(x)|w_M|]^2} \geq \int_\Omega \frac{j(x,\nabla w)}{[1+b(x)|w_M|]^2} - \int_\Omega \frac{j_\xi(x,\nabla w)}{[1+b(x)|w_M|]^2} \cdot \nabla[w_M - w]\,.$$

Using assumption (1), the fact that w belongs to $H_0^1(\Omega)$, the almost everywhere convergence of w_M to w and Lebesgue's theorem, we have

$$\lim_{M\to+\infty} \int_\Omega \frac{j(x,\nabla w)}{[1+b(x)|w_M|]^2} = \int_\Omega \frac{j(x,\nabla w)}{[1+b(x)|w|]^2}\,. \tag{8}$$

Using assumption (2), the fact that w belongs to $H_0^1(\Omega)$, and the almost everywhere convergence of w_M to w, we have by Lebesgue's theorem that

$$\lim_{M\to+\infty} \frac{j_\xi(x,\nabla w)}{[1+b(x)|w_M|]^2} = \frac{j_\xi(x,\nabla w)}{[1+b(x)|w|]^2}\,, \quad \text{strongly in } (L^2(\Omega))^N.$$

Since ∇w_M tends to ∇w weakly in the same space, we thus have that

$$\lim_{M\to+\infty} \int_\Omega \frac{j_\xi(x,\nabla w)}{[1+b(x)|w_M|]^2} \cdot \nabla[w_M - w] = 0\,. \tag{9}$$

Using (8) and (9), we have that (7) holds true. On the other hand, using (1) and Lebesgue's theorem again, it is easy to see that

$$\lim_{M\to+\infty} \int_\Omega \frac{j(x,\nabla v)}{[1+b(x)|T_M(v)|]^2} = \int_\Omega \frac{j(x,\nabla v)}{[1+b(x)|v|]^2}\,, \quad \forall v \in H_0^1(\Omega)\,.$$

Thus, starting from $J_M(w_M) \leq J_M(v)$, we can pass to the limit as M tends to infinity (using also the strong convergence of w_M to w in $L^2(\Omega)$), to have that w is a minimum. □

As stated before, we prove Theorem 1 by approximation. More in detail, if $f_n = T_n(f)$ then Lemma 2 with $g = f_n$ implies that there exists a minimum u_n in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of the functional

$$J_n(v) = \int_\Omega \frac{j(x,\nabla v)}{[1+b(x)|v|]^2} + \frac{1}{2}\int_\Omega |v|^2 - \int_\Omega f_n\, v\,, \quad v \in H_0^1(\Omega)\,.$$

In the following lemma we prove some uniform estimates on u_n.

Lemma 3. *Let u_n in $H_0^1(\Omega) \cap L^\infty(\Omega)$ be a minimum of J_n. Then*

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+b(x)|u_n|)^2} \leq \frac{1}{2\alpha}\int_\Omega |f|^2\,; \tag{10}$$

$$\int_\Omega |\nabla T_k(u_n)|^2 \leq \frac{(1+B\,k)^2}{2\alpha}\int_\Omega |f|^2\,; \tag{11}$$

$$\int_\Omega |u_n|^2 \leq 4\int_\Omega |f|^2\,; \tag{12}$$

(13) $$\int_\Omega |\nabla u_n| \le \left[\frac{1}{2\alpha}\int_\Omega |f|^2\right]^{\frac12}\left(\text{meas}(\Omega)^{\frac12} + 2B\left[\int_\Omega |f|^2\right]^{\frac12}\right);$$

(14) $$\int_\Omega |G_k(u_n)|^2 \le 4\int_{\{|u_n|\ge k\}} |f|^2\,,$$

where $G_k(s) = s - T_k(s)$ *for* $k \ge 0$ *and* s *in* $\mathbb{R}$.

Proof. The minimality of u_n implies that $J_n(u_n) \le J_n(0)$, that is,

(15) $$\int_\Omega \frac{j(x,\nabla u_n)}{[1+b(x)|u_n|]^2} + \frac12\int_\Omega u_n^2 \le \int_\Omega f_n\, u_n\,.$$

Using (1) on the left hand side, and Young's inequality on the right hand side gives

$$\alpha\int_\Omega \frac{|\nabla u_n|^2}{[1+b(x)|u_n|]^2} + \frac12\int_\Omega u_n^2 \le \frac12\int_\Omega u_n^2 + \frac12\int_\Omega f_n^2\,,$$

which then implies (10). Let now $k \ge 0$. The above estimate, and (3), give

$$\frac{1}{(1+Bk)^2}\int_\Omega |\nabla T_k(u_n)|^2 \le \int_{\{|u_n|\le k\}} \frac{|\nabla u_n|^2}{[1+b(x)|u_n|]^2} \le \frac{1}{2\alpha}\int_\Omega |f|^2\,,$$

and therefore (11) is proved. On the other hand, dropping the first positive term in (15) and using Hölder's inequality on the right hand side, we have

$$\frac12\int_\Omega |u_n|^2 \le \int_\Omega |f_n u_n| \le \left[\int_\Omega |f_n|^2\right]^{\frac12}\left[\int_\Omega |u_n|^2\right]^{\frac12},$$

that is, (12) holds. Hölder's inequality, assumption (3), and estimates (10) and (12) give (13):

(16) $$\begin{aligned}\int_\Omega |\nabla u_n| &\le \left[\int_\Omega \frac{|\nabla u_n|^2}{[1+b(x)|u_n|]^2}\right]^{\frac12}\left[\int_\Omega [1+b(x)|u_n|]^2\right]^{\frac12}\\ &\le \left[\frac{1}{2\alpha}\int_\Omega |f|^2\right]^{\frac12}\left(\text{meas}(\Omega)^{\frac12} + 2B\left[\int_\Omega |f|^2\right]^{\frac12}\right).\end{aligned}$$

We are left with estimate (14). Since $J_n(u_n) \le J_n(T_k(u_n))$ we have

$$\begin{aligned}&\frac12\int_\Omega \frac{j(x,\nabla u_n)}{[1+b(x)|u_n|]^2} + \frac12\int_\Omega |u_n|^2 - \int_\Omega f_n u_n\\ &\le \frac12\int_\Omega \frac{j(x,\nabla T_k(u_n))}{[1+b(x)|T_k(u_n)|]^2} + \frac12\int_\Omega |T_k(u_n)|^2 - \int_\Omega f_n T_k(u_n)\,.\end{aligned}$$

Recalling the definition of $G_k(s)$, and using that $|s|^2 - |T_k(s)|^2 \ge |G_k(s)|^2$, the last inequality implies

$$\frac12\int_\Omega \frac{j(x,\nabla G_k(u_n))}{[1+b(x)|u_n|]^2} + \frac12\int_\Omega |G_k(u_n)|^2 \le \int_\Omega f_n G_k(u_n)\,.$$

Dropping the first term of the left hand side and using Hölder's inequality on the right one, we obtain

$$\frac{1}{2}\int_\Omega |G_k(u_n)|^2 \leq \left[\int_{\{|u_n|\geq k\}} |f|^2\right]^{\frac{1}{2}} \left[\int_\Omega |G_k(u_n)|^2\right]^{\frac{1}{2}},$$

that is, (14) holds. □

LEMMA 4. *Let u_n in $H_0^1(\Omega)\cap L^\infty(\Omega)$ be a minimum of J_n. Then there exists a subsequence, still denoted by $\{u_n\}$, and a function u in $W_0^{1,1}(\Omega)\cap L^2(\Omega)$, with $T_k(u)$ in $H_0^1(\Omega)$ for every $k>0$, such that u_n converges to u almost everywhere in Ω, strongly in $L^2(\Omega)$ and weakly in $W_0^{1,1}(\Omega)$, and $T_k(u_n)$ converges to $T_k(u)$ weakly in $H_0^1(\Omega)$. Moreover,*

$$\lim_{n\to+\infty} \frac{\nabla u_n}{1+b(x)|u_n|} = \frac{\nabla u}{1+b(x)|u|} \quad \textit{weakly in } (L^2(\Omega))^N. \tag{17}$$

Proof. By (13), the sequence u_n is bounded in $W_0^{1,1}(\Omega)$. Therefore, it is relatively compact in $L^1(\Omega)$. Hence, up to subsequences still denoted by u_n, there exists u in $L^1(\Omega)$ such that u_n almost everywhere converges to u. From Fatou's lemma applied to (12) we then deduce that u belongs to $L^2(\Omega)$.

We are going to prove that u_n strongly converges to u in $L^2(\Omega)$. Let E be a measurable subset of Ω; then by (14) we have

$$\begin{aligned}\int_E |u_n|^2 &\leq 2\int_E |T_k(u_n)|^2 + 2\int_E |G_k(u_n)|^2 \\ &\leq 2k^2\text{meas}(E) + 2\int_\Omega |G_k(u_n)|^2 \\ &\leq 2k^2\text{meas}(E) + 8\int_{\{|u_n|\geq k\}} |f|^2 .\end{aligned}$$

Since u_n is bounded in $L^2(\Omega)$ by (12), we can choose k large enough so that the second integral is small, uniformly with respect to n; once k is chosen, we can choose the measure of E small enough such that the first term is small. Thus, the sequence $\{u_n^2\}$ is equiintegrable and so, by Vitali's theorem, u_n strongly converges to u in $L^2(\Omega)$.

Now we to prove that u_n weakly converges to u in $W_0^{1,1}(\Omega)$. Let E be a measurable subset of Ω. By Hölder's inequality, assumption (3), and (10), one has, for $i\in\{1,\ldots,N\}$,

$$\begin{aligned}\int_E \left|\frac{\partial u_n}{\partial x_i}\right| \leq \int_E |\nabla u_n| &\leq \left[\int_E \frac{|\nabla u_n|^2}{[1+b(x)|u_n|]^2}\right]^{\frac{1}{2}} \left[\int_E [1+b(x)|u_n|]^2\right]^{\frac{1}{2}} \\ &\leq \left[\frac{1}{2\alpha}\int_\Omega |f|^2\right]^{\frac{1}{2}} \left[\int_E [1+B|u_n|]^2\right]^{\frac{1}{2}}.\end{aligned}$$

Since the sequence $\{u_n\}$ is compact in $L^2(\Omega)$, this estimate implies that the sequence $\{\frac{\partial u_n}{\partial x_i}\}$ is equiintegrable. Thus, by Dunford-Pettis theorem, and up to subsequences, there exists Y_i in $L^1(\Omega)$ such that $\frac{\partial u_n}{\partial x_i}$ weakly converges to Y_i in $L^1(\Omega)$. Since $\frac{\partial u_n}{\partial x_i}$ is the distributional partial derivative of u_n, we have, for every n in $\mathbb{N}$,

$$\int_\Omega \frac{\partial u_n}{\partial x_i}\,\varphi = -\int_\Omega u_n\,\frac{\partial \varphi}{\partial x_i}\,, \quad \forall \varphi \in C_0^\infty(\Omega)\,.$$

We now pass to the limit in the above identities, using that $\partial_i u_n$ weakly converges to Y_i in $L^1(\Omega)$, and that u_n strongly converges to u in $L^2(\Omega)$: we obtain

$$\int_\Omega Y_i\,\varphi = -\int_\Omega u\,\frac{\partial \varphi}{\partial x_i}\,, \quad \forall \varphi \in C_0^\infty(\Omega)\,.$$

This implies that $Y_i = \frac{\partial u}{\partial x_i}$, and this result is true for every i. Since Y_i belongs to $L^1(\Omega)$ for every i, u belongs to $W_0^{1,1}(\Omega)$, as desired.

Since by (11) it follows that the sequence $\{T_k(u_n)\}$ is bounded in $H_0^1(\Omega)$, and since u_n tends to u almost everywhere in Ω, then $T_k(u_n)$ weakly converges to $T_k(u)$ in $H_0^1(\Omega)$, and $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k \geq 0$.

Finally, we prove (17). Let Φ be a fixed function in $(L^\infty(\Omega))^N$. Since u_n almost everywhere converges to u in Ω, we have

$$\lim_{n\to+\infty} \frac{\Phi}{1+b(x)|u_n|} = \frac{\Phi}{1+b(x)|u|} \quad \text{almost everywhere in } \Omega.$$

By Egorov's theorem, the convergence is therefore quasi uniform; i.e., for every $\delta > 0$ there exists a subset E_δ of Ω, with $\text{meas}(E_\delta) < \delta$, such that

$$\lim_{n\to+\infty} \frac{\Phi}{1+b(x)|u_n|} = \frac{\Phi}{1+b(x)|u|} \quad \text{uniformly in } \Omega \setminus E_\delta. \tag{18}$$

We now have

$$\begin{aligned}
&\left|\int_\Omega \frac{\nabla u_n}{1+b(x)|u_n|}\cdot\Phi - \int_\Omega \frac{\nabla u}{1+b(x)|u|}\cdot\Phi\right| \\
&\quad\leq \left|\int_{\Omega\setminus E_\delta} \nabla u_n \cdot \frac{\Phi}{1+b(x)|u_n|} - \int_{\Omega\setminus E_\delta} \nabla u \cdot \frac{\Phi}{1+b(x)|u|}\right| \\
&\qquad + \|\Phi\|_{L^\infty(\Omega)} \int_{E_\delta} [|\nabla u_n| + |\nabla u|]\,.
\end{aligned}$$

Using the equiintegrability of $|\nabla u_n|$ proved above, and the fact that $|\nabla u|$ belongs to $L^1(\Omega)$, we can choose δ such that the second term of the right hand side is arbitrarily small, uniformly with respect to n, and then use (18) to choose n large enough so that the first term is arbitrarily small. Hence, we have proved that

$$\lim_{n\to+\infty} \frac{\nabla u_n}{1+b(x)|u_n|} = \frac{\nabla u}{1+b(x)|u|} \quad \text{weakly in } (L^1(\Omega))^N. \tag{19}$$

On the other hand, from (10) it follows that the sequence $\frac{\nabla u_n}{1+b(x)|u_n|}$ is bounded in $(L^2(\Omega))^N$, so that it weakly converges to some function σ

in the same space. Since (19) holds, we have that $\sigma = \frac{\nabla u}{1+b(x)|u|}$, and (17) is proved. $\square$

REMARK 5. The fact that we need to prove (17) is one of the main differences with the paper [2].

Proof of Theorem 1. Let u_n be as in Lemma 4. The minimality of u_n implies that

$$\begin{aligned}\int_\Omega \frac{j(x,\nabla u_n)}{[1+b(x)|u_n|]^2} + \frac{1}{2}\int_\Omega |u_n|^2 - \int_\Omega f_n u_n \\ \le \int_\Omega \frac{j(x,\nabla v)}{[1+b(x)|v|]^2} + \frac{1}{2}\int_\Omega |v|^2 - \int_\Omega f_n v\end{aligned} \tag{20}$$

for every v in $H_0^1(\Omega)$. The result will then follow by passing to the limit in the previous inequality. The right hand side of (20) is easy to handle since f_n converges to f in $L^2(\Omega)$. Let us study the limit of the left hand side of (20). The convexity of j implies that

$$\begin{aligned}\int_\Omega \frac{j(x,\nabla u_n)}{[1+b(x)|u_n|]^2} \ge \int_\Omega \frac{j(x,\nabla T_k(u))}{[1+b(x)|u_n|]^2} \\ - \int_\Omega \frac{j_\xi(x,\nabla T_k(u))}{[1+b(x)|u_n|]} \cdot \left(\frac{\nabla u_n}{[1+b(x)|u_n|]} - \frac{\nabla T_k(u)}{[1+b(x)|u_n|]}\right).\end{aligned}$$

By (17), assumptions (1) and (2), and Lebesgue's theorem, we have

$$\begin{aligned}\liminf_{n\to+\infty} \int_\Omega \frac{j(x,\nabla u_n)}{[1+b(x)|u_n|]^2} \ge \int_\Omega \frac{j(x,\nabla T_k(u))}{[1+b(x)|u|]^2} \\ - \int_\Omega \frac{j_\xi(x,\nabla T_k(u))}{[1+b(x)|u|]} \cdot \frac{\nabla[u-T_k(u)]}{[1+b(x)|u|]},\end{aligned}$$

that is, since $j_\xi(x,\nabla T_k(u)) \cdot \nabla(u - T_k(u)) = 0$,

$$\int_\Omega \frac{j(x,\nabla T_k(u))}{[1+b(x)|u|]^2} \le \liminf_{n\to+\infty} \int_\Omega \frac{j(x,\nabla u_n)}{[1+b(x)|u_n|]^2}.$$

Letting k tend to infinity, and using Levi's theorem, we obtain

$$\int_\Omega \frac{j(x,\nabla u)}{[1+b(x)|u|]^2} \le \liminf_{n\to+\infty} \int_\Omega \frac{j(x,\nabla u_n)}{[1+b(x)|u_n|]^2}. \tag{21}$$

Inequality (21) and Lemma 4 imply that

$$\begin{aligned}\liminf_{n\to+\infty} \int_\Omega \frac{j(x,\nabla u_n)}{[1+b(x)|u_n|]^2} + \frac{1}{2}\int_\Omega |u_n|^2 - \int_\Omega f_n u_n \\ \ge \int_\Omega \frac{j(x,\nabla u)}{[1+b(x)|u|]^2} + \frac{1}{2}\int_\Omega |u|^2 - \int_\Omega f u.\end{aligned}$$

Thus, for every v in $H_0^1(\Omega)$,

$$\int_\Omega \frac{j(x,\nabla u)}{[1+b(x)|u|]^2} + \frac{1}{2}\int_\Omega |u|^2 - \int_\Omega f u \le \int_\Omega \frac{j(x,\nabla v)}{[1+b(x)|v|]^2} + \frac{1}{2}\int_\Omega |v|^2 - \int_\Omega f v,$$

so that u is a minimum of J; its regularity has been proved in Lemma 4. ☐

Remark 6. If we suppose that the coefficient $b(x)$ satisfies the stronger assumption

$$0 < A \leq b(x) \leq B\,, \quad \text{almost everywhere in } \Omega,$$

it is possible to prove that $J(u) \leq J(w)$ not only for every w in $H_0^1(\Omega)$, but also for the test functions w such that

$$\begin{cases} T_k(w) \text{ belongs to } H_0^1(\Omega) \text{ for every } k > 0, \\ \log(1 + A|w|) \text{ belongs to } H_0^1(\Omega), \\ w \text{ belongs to } L^2(\Omega). \end{cases} \tag{22}$$

Indeed, if w is as in (22), we can use $T_k(w)$ as test function in (4) and we have

$$J(u) \leq J(T_k(w)) = \int_\Omega \frac{j(x, \nabla T_k(w))}{[1 + b(x)|T_k(w)|]^2} + \frac{1}{2}\int_\Omega |T_k(w)|^2 - \int_\Omega f T_k(w).$$

In the right hand side is possible to pass to the limit, as k tends to infinity, so that we have $J(u) \leq J(w)$, for every test function w as in (22).

Remark 7. We explicitly point out the differences, concerning the coercivity, between the functionals studied in [3] and the functionals studied in this paper. Indeed, let $0 < \rho < \frac{N-2}{2}$, and consider the sequence of functions

$$v_n = \exp\left[T_n\left(\frac{1}{|x|^\rho} - 1\right)\right] - 1\,,$$

defined in $\Omega = B_1(0)$. Then

$$\log(1 + |v_n|) = T_n\left(\frac{1}{|x|^\rho} - 1\right)\,,$$

is bounded in $H_0^1(\Omega)$ (since the function $v(x) = \frac{1}{|x|^\rho} - 1$ belongs to $H_0^1(\Omega)$ by the assumptions on ρ), but, by Levi's theorem,

$$\lim_{n\to+\infty} \int_\Omega |\nabla v_n| = \rho \int_\Omega \frac{\exp\left[\frac{1}{|x|^\rho} - 1\right]}{|x|^{\rho+1}} = +\infty\,.$$

Hence, the functional

$$v \in H_0^1(\Omega) \mapsto \int_\Omega \frac{|\nabla v|^2}{(1 + |v|)^2} = \int_\Omega |\nabla \log(1 + |v|)|^2\,,$$

which is of the type studied in [3], is non coercive on $W_0^{1,1}(\Omega)$. On the other hand, recalling (16), we have

$$\int_\Omega |\nabla v| = \int_\Omega \frac{|\nabla v|}{1 + |v|}(1 + |v|) \leq \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{(1 + |v|)^2} + \frac{1}{2}\int_\Omega (1 + |v|)^2\,.$$

Thus, the functional

$$v \in H_0^1(\Omega) \mapsto \int_\Omega \frac{|\nabla v|^2}{(1+|v|)^2} + \int_\Omega |v|^2 \,,$$

which is of the type studied here, is coercive on $W_0^{1,1}(\Omega)$.

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