# A NOTE ON THE COMPUTATION OF THE FROBENIUS NUMBER OF A NUMERICAL SEMIGROUP

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ABSTRACT. In this article we present a formula for the computation of the Frobenius number and the conductor of a numerical semigroup from the sockel of a quotient of certain semigroup ring.

## 1. Introduction and Review

For further details and as a general reference on numerical semigroups, the reader should refer to the works of Rosales and García Sánchez [4], and Ramírez Alfonsín [5].

Let $k$ be a field. Let $n_1, \ldots, n_d$ be positive integer numbers with $\gcd(n_1, \ldots, n_d) = 1$. Consider the numerical semigroup

$$H := \mathbb{N}n_1 + \ldots + \mathbb{N}n_d$$

minimally generated by $n_1, \ldots, n_d$. It is well-known the existence of an element $c \in \mathbb{N}_0$ minimal such that $c + \mathbb{N}_0 \subseteq H$. This number is called the conductor of $H$. The number $f := c - 1$ is then the biggest integer not belonging to $H$, and it is called the Frobenius number of $H$.

Let $n$ be a nonzero element of $H$. The set

$$\mathrm{Ap}(H, n) := \{h \in H \mid h - n \notin H\}$$

is called the Apéry set of $n$ in $H$. It is easily checked that (cf. [6])

$$f = \max \mathrm{Ap}(H, n) - n. \qquad (\dagger)$$

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Let $I \neq \varnothing$ be a subset of $\mathbb{Z}$ satisfying $I \neq \mathbb{Z}$ and $I + H \subseteq I$. Such an $I$ is said to be a fractional $H$-ideal. The $H$-ideal $M := \{s \in H \mid s \neq 0\}$ is the (uniquely determined) maximal ideal of $H$. It will be important in the sequel to consider also the $H$-ideal

$$M^- := \{z \in \mathbb{Z} \mid z + M \subseteq H\}.$$

Note that $\mathbb{N}_0 \supseteq M^- \supseteq H$, and since $f \in M^-$ one has indeed $M^- \supsetneq H$. The cardinality of the set of elements in $M^- \setminus H$ will be denoted by $r(H)$. Note also that $f = \max\{m \mid m \in M^- \setminus H\}$.

Let $R := k[X_1, \ldots, X_d]$ (resp. $k[t]$) be the polynomial ring over $k$ graded by $\deg(X_i) = n_i$ for every $i \in \{1, \ldots, d\}$ (resp. $\deg(t) = 1$). Let $\phi$ be the graded homomorphism of $k$-algebras $\phi : R \to k[t]$ given by $X_i \mapsto t^{n_1}$ for every $i \in \{1, \ldots, d\}$. The image of $\phi$ is the semigroup ring associated with $H$, and it is denoted by $k[H]$. The homogeneous prime ideal $\mathfrak{p} := \ker \phi$ is said to be the presentation ideal of $k[H]$.

Let us consider $\mathfrak{p}' := \phi(\mathfrak{p})$ the image in $k[X_1, \ldots, X_{d-1}]$ by the epimorphism mapping $X_d$ onto 0, and define the quotient ring

$$R' := k[H]/(t^{n_d}).$$

The following ring isomorphisms are easily checked:

$$R' \cong k[X_1, \ldots, X_{d-1}]/\mathfrak{p}' \cong k[\overline{X}_1, \ldots, \overline{X}_{d-1}],$$

where $\overline{X}_i$ denotes the class of $X_i$ modulo $\mathfrak{p}'$ for every $i \in \{1, \ldots, d-1\}$. Furthermore, the ring $R'$ is $*$–local, i.e., it has a unique maximal graded ideal $\mathfrak{m}_{R'}$.

## 2. The Main Result

Let us define the *trivial submodule* (or sockel) of $R'$ as the set of elements in $R'$ which are killed by the homogeneous maximal ideal $\mathfrak{m}_{R'}$ of $R'$, namely

$$\operatorname{Triv}(R') := \{x \in R' \mid x \cdot \mathfrak{m}_{R'} = (0)\}.$$

It can be identified as the image of $\operatorname{Hom}(k, R')$ via the map $f \to f(1)$.

Note that, for $\Lambda := \{\lambda \in \mathbb{N}_0 \mid \lambda + M \in \mathbb{N}_0 n_d\} \setminus \mathbb{N}_0 n_d$ one has an isomorphism between the trivial submodule $\operatorname{Triv}(R')$ and the set

$$\Big\{ \sum_{\lambda \in \Lambda} \alpha_\lambda t^\lambda \mid \alpha_\lambda \in k \Big\}.$$

Furthermore, we have a bijection between $\Lambda$ and $M^- \setminus H$ given by mapping every $\lambda \in \Lambda$ to $\lambda - n_d \in M^- \setminus H$. This together with ($\star$) leads to the equality

$$r(H) = \dim \mathrm{Triv}(R').$$

This means in particular that the trivial submodule $\mathrm{Triv}(R')$ is a *finite* dimensional vector space over the field $k$: Let us then choose a basis $\mathcal{B} := \{b_1, \ldots, b_{r(H)}\}$ and take the element $b \in \mathcal{B}$ such that

$$\deg(b) = \max\{\deg(b_i) \mid i = 1, 2, \ldots, r(H)\}.$$

Now it is easily checked that:

**Lemma 2.1.** $\deg(b)$ *is independent of the choice of* $\mathcal{B}$.

A consequence of the previous reasonings is the following

**Theorem 2.2.** *We have:*

$$f = c - 1 = \deg(b) - n_d.$$

**Corollary 2.3.** *We have:*

$$\max\ \mathrm{Ap}(H, n_d) = \deg(b).$$

*Proof.* The result follows straightforward from the equation (†) at the beginning of the paper. □

**Example 2.4.** Let us take the monomial curve $C := (t^6, t^8, t^9)$. The corresponding semigroup is $H_C = \mathbb{N}_0 \cdot 6 + \mathbb{N}_0 \cdot 8 + \mathbb{N}_0 \cdot 9$. The presentation ideal associated with $C$ is $\mathfrak{p} = (X_1^3 - X_3^2, X_2^3 - X_1X_3^2)$, so $R' = k[X_1, X_2]/\mathfrak{p}' \cong k[x_1, x_2]$ with $\mathfrak{p}' = (X_1^3, X_2^3)$ and $x_i = X_i \bmod \mathfrak{p}'$ for $i = 1, 2$. Then we have $\mathrm{Triv}(R') = k \cdot x_1^2x_2^2$. In this case is $f = \deg(x_1^2x_2^2) - 9 = 12 + 16 - 9 = 19$, as one can easily check directly from $H_C$.

**Example 2.5.** Let us take now $C := (t^7, t^8, t^9, t^{11})$. The corresponding semigroup is $H_C = \mathbb{N}_0 \cdot 7 + \mathbb{N}_0 \cdot 8 + \mathbb{N}_0 \cdot 9 + \mathbb{N} \cdot 11$. The presentation ideal associated with $C$ in this case is

$$\mathfrak{p} = (x_1^4 - X_2X_3X_4, X_2^2 - X_1X_3, X_3^2 - X_1X_4, X_4^2 - X_1^2X_2),$$

so $R' = k[X_1, X_2, X_3]/\mathfrak{p}' \cong k[x_1, x_2, x_3]$ with

$$\mathfrak{p}' = (X_1^4, X_2^2 - X_1X_3, X_3^2, -X_1^2X_2)$$

and $x_i = X_i \bmod \mathfrak{p}'$ for $i = 1, 2, 3$. Then the corresponding sockel is $\mathrm{Triv}(R') = k \cdot x_1^3 + k \cdot x_2^3 + k \cdot x_1x_2x_3$. In this case the Frobenius number is $f = \max\{21, 24\} - 11 = 13$ and the conductor $c = f + 1 = 14$.

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