On AZ-style identity

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July 8, 2011

Abstract

The AZ identity is a generalization of the LYM-inequality. In this paper, we will give a generalization of the AZ identity.

KEYWORDS: LYM-inequality, AZ-identity
MATHEMATICS SUBJECT CLASSIFICATION: 05D05

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$, Ω_n be the family of all subsets of $[n]$, and $\varnothing$ be the empty set. Let $\varnothing \neq \mathcal{F} \subseteq \Omega_n$. If $A \nsubseteq B$ for all $A, B \in \mathcal{F}$ with $A \neq B$, then $\mathcal{F}$ is called a *Sperner family* or *antichain*. For any antichain $\mathcal{F}$, the following inequality holds:

$$\sum_{X \in \mathcal{F}} \frac{1}{\binom{n}{|X|}} \leq 1. \tag{1}$$

The inequality (1) is called the LYM-inequality (Lubell, Yamamoto, Meshalkin) (see [5, Chapter 13]). Many generalizations of the LYM-inequality have been obtained (see [4, 6, 7, 9]). In particular, Ahlswede and Zhang [3] discovered an identity (see equation (2)) in which the LYM-inequality is a consequence of it.

Let $\mathbf{G}_n$ be the family of all $\mathcal{F}$ such that $\varnothing \neq \mathcal{F} \subseteq \Omega_n$. For every $\mathcal{F} \in \mathbf{G}_n$, the set

$$D_n(\mathcal{F}) = \{Y \subseteq [n] \;:\; Y \subseteq F \text{ for some } F \in \mathcal{F}\},$$

is called the *downset*, while the set

$$U_n(\mathcal{F}) = \{Y \subseteq [n] \;:\; Y \supseteq F \text{ for some } F \in \mathcal{F}\},$$

is called the *upset*. For each $X \subseteq [n]$, we set

$$Z_{\mathcal{F}}(X) = \begin{cases} \varnothing & \text{if } X \notin U_n(\mathcal{F}), \\ \bigcap_{F \in \mathcal{F}, F \subseteq X} F & \text{otherwise.} \end{cases}$$

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Theorem 1.1. [3] *For any* $\mathcal{F} \in \mathbf{G}_n$ *with* $\varnothing \notin \mathcal{F}$,

$$\sum_{X \in U_n(\mathcal{F})} \frac{|Z_{\mathcal{F}}(X)|}{|X|\binom{n}{|X|}} = 1. \tag{2}$$

□

Equation (2) is called the *AZ-identity*. Note that when $\mathcal{F}$ is an antichain, $Z_{\mathcal{F}}(F) = F$ for all $F \in \mathcal{F}$. So equation (2) becomes

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} + \sum_{X \in U_n(\mathcal{F}) \setminus \mathcal{F}} \frac{|Z_{\mathcal{F}}(X)|}{|X|\binom{n}{|X|}} = 1,$$

and as the second term on the left is non-negative, we obtain inequality (1).

Later, Ahlswede and Cai discovered an identity for two set systems.

Theorem 1.2. [1] *Let* $\mathcal{A} = \{A_1, A_2, \ldots, A_q\}$ *and* $\mathcal{B} = \{B_1, B_2, \ldots, B_q\}$ *be elements in* $\mathbf{G}_n$. *Suppose that* $A_i \neq \varnothing$ *for all* i, *and* $A_j \subseteq B_k$ *if and only if* $j = k$. *Then*

$$\sum_{i=1}^{q} \frac{1}{\binom{n-|B_i|+|A_i|}{|A_i|}} + \sum_{X \in U_n(\mathcal{A}) \setminus D_n(\mathcal{B})} \frac{|Z_{\mathcal{A}}(X)|}{|X|\binom{n}{|X|}} = 1. \tag{3}$$

□

Ahlswede and Cai [2] also discovered AZ type of identities of several other posets. For the duality of equations (2) and (3), we refer the readers to [8, 10].

Recently, Thu discovered the following generalizations of equations (2) and (3).

Theorem 1.3. [12] *Let* m *be an integer, and* $\mathcal{F} \in \mathbf{G}_n$ *with* $\varnothing \notin \mathcal{F}$. *If* $|F| + m > 0$ *for all* $F \in \mathcal{F}$, *then*

$$\sum_{X \in U_n(\mathcal{F})} \frac{|Z_{\mathcal{F}}(X)| + m}{(|X| + m)\binom{n+m}{|X|+m}} = 1. \tag{4}$$

□

Theorem 1.4. [12] *Let* m *be an integer, and* $\mathcal{A} = \{A_1, A_2, \ldots, A_q\}$ *and* $\mathcal{B} = \{B_1, B_2, \ldots, B_q\}$ *be elements in* $\mathbf{G}_n$. *Suppose that* $A_i \neq \varnothing$ *for all* i, *and* $A_j \subseteq B_k$ *if and only if* $j = k$. *If* $|A| + m > 0$ *for all* $A \in \mathcal{A}$, *then*

$$\sum_{i=1}^{q} \frac{1}{\binom{n+m-|B_i|+|A_i|}{|A_i|+m}} + \sum_{X \in U_n(\mathcal{A}) \setminus D_n(\mathcal{B})} \frac{|Z_{\mathcal{A}}(X)| + m}{(|X| + m)\binom{n+m}{|X|+m}} = 1. \tag{5}$$

□

In this paper, we will give generalizations of equations (4) and (5) (see Theorem 2.4 and Theorem 2.7).

2 Main theorems

Let us denote the set of real numbers by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. Let $a, m \in \mathbb{R}$ and $n \in \mathbb{N}$. Suppose that $ak + m \neq 0$ for $k = l, l+1, \ldots, n$. We set

$$g_{a,m}(n, l) = \frac{(n-l)!a^{n-l}}{\prod_{k=l}^{n}(ak+m)}.$$

Lemma 2.1. *Suppose $l < n$. If $ak + m \neq 0$ for $k = l, l+1, \ldots, n$, then*

$$g_{a,m}(n, l) + g_{a,m}(n, l+1) = g_{a,m}(n-1, l).$$

Proof. Note that

$$\begin{aligned} g_{a,m}(n, l) + g_{a,m}(n, l+1) &= \frac{(n-l)!a^{n-l}}{\prod_{k=l}^{n}(ak+m)} + \frac{(n-l-1)!a^{n-l-1}}{\prod_{k=l+1}^{n}(ak+m)} \\ &= \frac{(n-l)!a^{n-l} + (al+m)(n-l-1)!a^{n-l-1}}{\prod_{k=l}^{n}(ak+m)} \\ &= \frac{n(n-l-1)!a^{n-l} + m(n-l-1)!a^{n-l-1}}{\prod_{k=l}^{n}(ak+m)} \\ &= \frac{(n-l-1)!a^{n-l-1}}{\prod_{k=l}^{n-1}(ak+m)} \\ &= g_{a,m}(n-1, l). \end{aligned}$$

□

The following lemma can be verified easily.

Lemma 2.2. *Suppose that $ak + m \neq 0$ for $k = l, l+1, \ldots, n$.*

(a) *If $a = 1$ and m is an integer, then*

$$g_{1,m}(n, l) = \frac{1}{(l+m)\binom{n+m}{l+m}}.$$

(b) *If $a = 1$ and $m = 0$, then*

$$g_{1,0}(n, l) = \frac{1}{(l)\binom{n}{l}}.$$

□

We shall need the following lemma (see equation (3) of [11], or Lemma 2 of [8]).

Lemma 2.3. *Let $\varnothing \notin \mathcal{A} \in \mathbf{G}_n$ and $\varnothing \notin \mathcal{B} \in \mathbf{G}_n$. Set*

$$\mathcal{A} \vee \mathcal{B} = \{A \cup B \ : \ A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Then for each $\varnothing \neq X \subseteq [n]$,

$$|Z_{\mathcal{A}\cup\mathcal{B}}(X)| = |Z_{\mathcal{A}}(X)| + |Z_{\mathcal{B}}(X)| - |Z_{\mathcal{A}\vee\mathcal{B}}(X)|.$$

□

Theorem 2.4. *Let $a, m \in \mathbb{R}$ and $n \in \mathbb{N}$. Let $\varnothing \notin \mathcal{A} \in \mathbf{G}_n$. Suppose that $ak + m \neq 0$ for all $\min_{A\in\mathcal{A}} |A| \leq k \leq n$. Then*

$$\sum_{X\in U_n(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) = 1. \tag{6}$$

Proof. **Case 1**. Suppose $\mathcal{A} = \{A\}$. We may assume that $A = \{1, 2, \ldots, r\}$.

Note that if $r = n$, then $U_n(\mathcal{A}) = \{A\}$, $Z_{\mathcal{A}}(A) = A$, and $\sum_{X\in U_n(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) = (an + m)g_{a,m}(n, n) = 1$. Suppose $r < n$. Note that $U_n(\mathcal{A}) = U_{n-1}(\mathcal{A}) \cup \{X \cup \{n\} \;:\; X \in U_{n-1}(\mathcal{A})\}$, and $Z_{\mathcal{A}}(X) = Z_{\mathcal{A}}(X \cup \{n\})$. Therefore by Lemma 2.1,

$$\begin{aligned}
&\sum_{X\in U_n(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) \\
&\quad = \sum_{X\in U_{n-1}(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) \\
&\qquad + \sum_{X\in U_{n-1}(\mathcal{A})} (a|Z_{\mathcal{A}}(X \cup \{n\})| + m)\, g_{a,m}(n, |X| + 1) \\
&\quad = \sum_{X\in U_{n-1}(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, (g_{a,m}(n, |X|) + g_{a,m}(n, |X| + 1)) \\
&\quad = \sum_{X\in U_{n-1}(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n - 1, |X|).
\end{aligned}$$

If $r = n - 1$, then $U_{n-1}(\mathcal{A}) = \{A\}$, $Z_{\mathcal{A}}(A) = A$, and $\sum_{X\in U_{n-1}(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n - 1, |X|) = (a(n-1) + m)g_{a,m}(n-1, n-1) = 1$. So the theorem holds. Suppose $r < n - 1$. Again by Lemma 2.1,

$$\begin{aligned}
&\sum_{X\in U_{n-1}(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n - 1, |X|) \\
&\quad = \sum_{X\in U_{n-2}(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n - 2, |X|).
\end{aligned}$$

By continuing this way, we see that

$$\begin{aligned}
&\sum_{X\in U_n(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) \\
&\quad = \sum_{X\in U_r(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(r, |X|) \\
&\quad = (ar + m)g_{a,m}(r, r) \\
&\quad = 1.
\end{aligned}$$

Case 2. Suppose $\mathcal{A} = \{A_1, \ldots, A_q\}$, $q \geq 2$. Assume that the theorem holds for all q' with $1 \leq q' < q$. Let $\mathcal{B} = \{A_1, \ldots, A_{q-1}\}$ and $\mathcal{C} = \{A_q\}$. Then $\mathcal{B}\vee\mathcal{C} = \{A_1\cup A_q, \ldots, A_{q-1}\cup A_q\}$, $U_n(\mathcal{A}) = U_n(\mathcal{B})\cup U_n(\mathcal{C})$ and $U_n(\mathcal{B} \vee \mathcal{C}) = U_n(\mathcal{B}) \cap U_n(\mathcal{C})$. By Lemma 2.3,

$$|Z_{\mathcal{A}}(X)| = |Z_{\mathcal{B}}(X)| + |Z_{\mathcal{C}}(X)| - |Z_{\mathcal{B}\vee\mathcal{C}}(X)|\,.$$

So if $X \in U_n(\mathcal{B}) \setminus U_n(\mathcal{C})$, then $|Z_{\mathcal{A}}(X)| = |Z_{\mathcal{B}}(X)|$, if $X \in U_n(\mathcal{C}) \setminus U_n(\mathcal{B})$, then $|Z_{\mathcal{A}}(X)| = |Z_{\mathcal{C}}(X)|$, and if $X \in U_n(\mathcal{B}) \cap U_n(\mathcal{C})$, then $|Z_{\mathcal{A}}(X)| = |Z_{\mathcal{B}}(X)| + |Z_{\mathcal{C}}(X)| - |Z_{\mathcal{B}\vee\mathcal{C}}(X)|$.

Therefore

$$\begin{aligned}
\sum_{X\in U_n(\mathcal{A})} & (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) \\
&= \sum_{X\in U_n(\mathcal{B})\setminus U_n(\mathcal{C})} (a|Z_{\mathcal{B}}(X)| + m)\, g_{a,m}(n, |X|) \\
&\quad + \sum_{X\in U_n(\mathcal{C})\setminus U_n(\mathcal{B})} (a|Z_{\mathcal{C}}(X)| + m)\, g_{a,m}(n, |X|) \\
&\quad\quad + \sum_{X\in U_n(\mathcal{B}\vee\mathcal{C})} (a\,(|Z_{\mathcal{B}}(X)| + |Z_{\mathcal{C}}(X)| - |Z_{\mathcal{B}\vee\mathcal{C}}(X)|) + m)\, g_{a,m}(n, |X|) \\
&= \sum_{X\in U_n(\mathcal{B})} (a|Z_{\mathcal{B}}(X)| + m)\, g_{a,m}(n, |X|) \\
&\quad + \sum_{X\in U_n(\mathcal{C})} (a|Z_{\mathcal{C}}(X)| + m)\, g_{a,m}(n, |X|) \\
&\quad\quad - \sum_{X\in U_n(\mathcal{B}\vee\mathcal{C})} (a\,|Z_{\mathcal{B}\vee\mathcal{C}}(X)| + m)\, g_{a,m}(n, |X|),
\end{aligned}$$

and by induction,

$$\sum_{X\in U_n(\mathcal{A})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) = 1 + 1 - 1 = 1.$$

□

Note that by Lemma 2.2, equations (2) and (4) are consequence of Theorem 2.4.

We shall need the following lemma (see Lemma 4 of [12]).

Lemma 2.5. *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2 \in \mathbf{G}_n$ and $\varnothing \notin \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_2$. Suppose that $U_n(\mathcal{A}_1) \cap D_n(\mathcal{B}_2) = \varnothing = U_n(\mathcal{A}_2) \cap D_n(\mathcal{B}_1)$. Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. If F is a non-zero function defined on $U_n(\mathcal{A})$, then*

$$\begin{aligned}
\sum_{X\in U_n(\mathcal{A})\setminus D_n(\mathcal{B})} \frac{a|Z_{\mathcal{A}}(X)| + m}{F(X)} = & \sum_{X\in U_n(\mathcal{A}_1)\setminus D_n(\mathcal{B}_1)} \frac{a|Z_{\mathcal{A}_1}(X)| + m}{F(X)} \\
& + \sum_{X\in U_n(\mathcal{A}_2)\setminus D_n(\mathcal{B}_2)} \frac{a|Z_{\mathcal{A}_2}(X)| + m}{F(X)} \\
& \quad - \sum_{X\in U_n(\mathcal{A}_1\vee\mathcal{A}_1)} \frac{a|Z_{\mathcal{A}_1\vee\mathcal{A}_2}(X)| + m}{F(X)}.
\end{aligned}$$

□

In fact Lemma 2.5 can be proved easily by noting that

$$\begin{aligned}
U_n(\mathcal{A}) \setminus D_n(\mathcal{B}) = & (U_n(\mathcal{A}_1) \setminus (D_n(\mathcal{B}_1) \cup U_n(\mathcal{A}_2))) \\
& \cup (U_n(\mathcal{A}_2) \setminus (D_n(\mathcal{B}_2) \cup U_n(\mathcal{A}_1))) \cup U_n(\mathcal{A}_1 \vee \mathcal{A}_2),
\end{aligned}$$

and by applying Lemma 2.3.

Lemma 2.6. *Let $a, m \in \mathbb{R}$ and $n \in \mathbb{N}$. Let A, B be non-empty subsets of $[n]$. If $A \subseteq B$, and $ak+m \neq 0$ for all $|A| \leq k \leq n$, then*

$$\sum_{A \subseteq X \subseteq B} g_{a,m}(n, |X|) = g_{a,m}(n - |B| + |A|, |A|).$$

Proof. We may assume that $A = \{1, 2, \ldots, r_1\}$ and $B = \{1, 2, \ldots, r_1, r_1 + 1, \ldots, r_2\}$. We shall prove by induction on $p = r_2 - r_1$.

Suppose $p = 0$, i.e., $A = B$. Then

$$\sum_{A \subseteq X \subseteq B} g_{a,m}(n, |X|) = g_{a,m}(n, |A|).$$

Suppose $p > 1$. Assume that the lemma holds for $p' < p$.

Note that $A \subsetneq B$ and $r_2 \notin A$. Set $B' = B \setminus \{r_2\}$. Then $A \subseteq B'$, and by Lemma 2.1,

$$\begin{aligned}
\sum_{A \subseteq X \subseteq B} g_{a,m}(n, |X|) &= \sum_{A \subseteq X \subseteq B'} g_{a,m}(n, |X|) + \sum_{A \subseteq X \subseteq B'} g_{a,m}(n, |X \cup \{r_2\}|) \\
&= \sum_{A \subseteq X \subseteq B'} (g_{a,m}(n, |X|) + g_{a,m}(n, |X| + 1)) \\
&= \sum_{A \subseteq X \subseteq B'} g_{a,m}(n - 1, |X|).
\end{aligned}$$

By induction $\sum_{A \subseteq X \subseteq B'} g_{a,m}(n-1, |X|) = g_{a,m}(n-1-|B'|+|A|, |A|) = g_{a,m}(n-|B|+|A|, |A|)$. Hence $\sum_{A \subseteq X \subseteq B} g_{a,m}(n, |X|) = g_{a,m}(n - |B| + |A|, |A|)$. □

Theorem 2.7. *Let $a, m \in \mathbb{R}$ and $n \in \mathbb{N}$. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_q\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_q\}$ be elements in $\mathbf{G}_n$. Suppose that $A_i \neq \varnothing$ for all i, and $A_j \subseteq B_k$ if and only if $j = k$. If $ak + m \neq 0$ for all $\min_{A \in \mathcal{A}} |A| \leq k \leq n$, then*

$$\sum_{i=1}^{q} (a|A_i| + m) g_{a,m}(n - |B_i| + |A_i|, |A_i|) + \sum_{X \in U_n(\mathcal{A}) \setminus D_n(\mathcal{B})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) = 1. \qquad (7)$$

Proof. **Case 1**. Suppose $q = 1$. Then $\mathcal{A} = \{A_1\}$, $\mathcal{B} = \{B_1\}$, $\varnothing \neq A_1 \subseteq B_1$, and $a|A_1| + m \neq 0$. Furthermore if $X \in U_n(\mathcal{A})$, then $Z_{\mathcal{A}}(X) = A_1$. By Theorem 2.4,

$$\sum_{X \in U_n(\mathcal{A}) \cap D_n(\mathcal{B})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) + \sum_{X \in U_n(\mathcal{A}) \setminus D_n(\mathcal{B})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) = 1.$$

Note that by Lemma 2.6

$$\begin{aligned}
\sum_{X \in U_n(\mathcal{A}) \cap D_n(\mathcal{B})} (a|Z_{\mathcal{A}}(X)| + m)\, g_{a,m}(n, |X|) &= (a|A_1| + m) \sum_{A \subseteq X \subseteq B} g_{a,m}(n, |X|) \\
&= (a|A_1| + m) g_{a,m}(n - |B_1| + |A_1|, |A_1|).
\end{aligned}$$

Hence the theorem holds.

Case 2. Suppose $q > 1$. Assume that the theorem holds for all q' with $1 \leq q' < q$. Let

$$\mathcal{A}_1 = \{A_1, \ldots, A_{q-1}\}, \qquad \mathcal{A}_2 = \{A_q\},$$
$$\mathcal{B}_1 = \{B_1, \ldots, B_{q-1}\}, \qquad \mathcal{B}_2 = \{B_q\}.$$

Note that $U_n(\mathcal{A}_1) \cap D_n(\mathcal{B}_2) = \varnothing = U_n(\mathcal{A}_2) \cap D_n(\mathcal{B}_1)$. By Lemma 2.5 and induction,

$$\begin{aligned}\sum_{X \in U_n(\mathcal{A}) \setminus D_n(\mathcal{B})} \frac{a|Z_{\mathcal{A}}(X)| + m}{F(X)} &= \left(1 - \sum_{i=1}^{q-1} (a|A_i| + m) g_{a,m}(n - |B_i| + |A_i|, |A_i|)\right) \\ &\quad + (1 - (a|A_q| + m) g_{a,m}(n - |B_q| + |A_q|, |A_q|)) \\ &\quad - \sum_{X \in U_n(\mathcal{A}_1 \vee \mathcal{A}_1)} (a|Z_{\mathcal{A}_1 \vee \mathcal{A}_2}(X)| + m)\, g_{a,m}(n, |X|).\end{aligned}$$

Note that by Theorem 2.4, the $\sum_{X \in U_n(\mathcal{A}_1 \vee \mathcal{A}_1)} (a|Z_{\mathcal{A}_1 \vee \mathcal{A}_2}(X)| + m)\, g_{a,m}(n, |X|) = 1$. Hence the theorem holds. □

Note that by Lemma 2.2, equations (3) and (5) are consequence of Theorem 2.7.

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