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Enumeration of 0/1-matrices avoiding some 2×2 matrices

Hyeong-Kwan Ju*
Department of Mathematics
Chonnam National University,
Kwangju 500-757, South Korea
hkju@chonnam.ac.kr

Seunghyun Seo†
Department of Mathematics Education
Kangwon National University
Chuncheon 200-701, South Korea
shyunseo@kangwon.ac.kr

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Abstract

We enumerate the number of 0/1-matrices avoiding 2×2 submatrices satisfying certain conditions. We also provide corresponding exponential generating functions.

1 Introduction

Let $M(k, n)$ be the set of $k \times n$ matrices with entries 0 and 1. It is obvious that the number of elements in the set $M(k, n)$ is 2^{kn}. It would be interesting to consider the number of elements in $M(k, n)$ with certain conditions. For example, how many matrices of $M(k, n)$ do not have 2×2 submatrices of the form $\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)$? In this article we will give answers to the previous question and other questions.

Consider $M(2, 2)$, the set of all possible 2×2 submatrices. For two elements P and Q in $M(2, 2)$, we denote $P \sim Q$ if Q can be obtained from P by row or column exchanges. It is obvious that $\sim$ is an equivalence relation on $M(2, 2)$. With this equivalence relation, we have seven equivalent classes in Table 1. We let $\phi(k, n; \alpha)$ be the number of $k \times n$ 0/1-matrices which do not have 2×2 submatrices in α, where α is a subset of $M(2, 2)$.

Our goal is to express $\phi(k, n; \alpha)$ in terms of k and n explicitly for each α in the set $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$. We can easily notice that $\phi(k, n; A_2) = \phi(k, n; A_3)$ and $\phi(k, n; A_6) = \phi(k, n; A_7)$ by swapping 0 and 1. We also notice $\phi(k, n; A_4) = \phi(n, k; A_5)$ by transposing the matrices. Moreover, given the equivalence relation $\sim$, if we define the new equivalent relation $P \sim' Q$ by $P \sim Q$ or $P = Q^{t}$, then $A_4 \cup A_5$ becomes a single

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$$A_1 := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \qquad A_4 := \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

$$A_2 := \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\} \qquad A_5 := \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

$$A_3 := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \qquad A_6 := \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \qquad A_7 := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

Table 1: from A_1 to A_7

equivalent class. Also if we define another new equivalent relation $P \sim'' Q$ by $P \sim Q$ or $P = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) - Q$, then $A_2 \cup A_3$ and $A_6 \cup A_7$ become a single equivalent class respectively.

In fact, $\phi(k, n; A_1)$ is well known (see [2, 6, 7]) and 0/1-matrices avoiding type A_1 are called 0/1-lonesum matrices (we will define and discuss it in 2.2). Lonesum matrices are the primary motivation of this article and its corresponding work. Except for the aforementioned case, we have not found any literature concerning other cases.

In this paper we calculate $\phi(k, n; \alpha)$, where α's are A_2 (equivalently A_3), A_4 (equivalently A_5), $A_2 \cup A_3$, $A_4 \cup A_5$ and $A_6 \cup A_7$. Finding a closed form of $\phi(k, n; A_6) = \phi(k, n; A_7)$ is still open to us.

2 Preliminaries

2.1 Definitions and Notations

A matrix P is called 0/1-matrix if all the entries of P are 0 or 1. From now on we consider 0/1-matrices only, so we will omit "0/1" if it causes no confusion. Let $M(k, n)$ be the set of $k \times n$-matrices. Clearly, if $k, n \geq 1$, $M(k, n)$ has 2^{kn} elements. For convention we assume that $M(0, 0) = \{\emptyset\}$ and $M(k, 0) = M(0, n) = \emptyset$ for positive integers k and n.

Given a matrix P, a submatrix of P is formed by selecting certain rows and columns from P. For example if $P = \left(\begin{smallmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{smallmatrix}\right)$, then $P(2, 3; 2, 4) = \left(\begin{smallmatrix} f & h \\ j & l \end{smallmatrix}\right)$.

Given two matrices P and Q, we say P contains Q, whenever Q is equal to a submatrix of P. Otherwise say P avoids Q. For example $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ contains $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ but avoids $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For a matrix P and a set α of matrices, we say that P avoids the type set α if P avoids all the matrices in α. If it causes no confusion we will simply say that P avoids α.

Given a set α of matrices, let $\phi(k, n; \alpha)$ be the number of $k \times n$ matrices avoiding α. From the definition of $M(k, n)$, for any set α, we have $\phi(0, 0; \alpha) = 1$ and $\phi(k, 0; \alpha) = \phi(0, n; \alpha) = 0$ for positive integers k and n. Let $\Phi(x, y; \alpha)$ be the exponential generating function for $\phi(k, n; \alpha)$, i.e.,

$$\Phi(x, y; \alpha) := \sum_{n \geq 0} \sum_{k \geq 0} \phi(k, n; \alpha) \frac{x^k}{k!} \frac{y^n}{n!} = 1 + \sum_{n \geq 1} \sum_{k \geq 1} \phi(k, n; \alpha) \frac{x^k}{k!} \frac{y^n}{n!} .$$

Let $\Phi(z;\alpha)$ be the exponential generating function for $\phi(n,n;\alpha)$, i.e.,

$$\Phi(z;\alpha) := \sum_{n\geq 0} \phi(n,n;\alpha)\,\frac{z^n}{n!}\,.$$

2.2 Type A_1 (Lonesum matrices)

This is related to the lonesum matrices. A lonesum matrix is a 0/1-matrix determined uniquely by its column-sum and row-sum vectors. For example, $\left(\begin{smallmatrix}1&0&1\\0&0&1\\1&0&1\end{smallmatrix}\right)$ is a lonesum matrix since it is a unique matrix determined by the column-sum vector $(2,0,3)$ and the row-sum vector $(2,1,2)^{\mathrm{t}}$. However $\left(\begin{smallmatrix}1&0&1\\0&0&1\\1&0&0\end{smallmatrix}\right)$ is not, since $\left(\begin{smallmatrix}1&0&1\\1&0&0\\0&0&1\end{smallmatrix}\right)$ has the same column-sum vector $(2,0,2)$ and row-sum vector $(2,1,1)^{\mathrm{t}}$.

Theorem 2.1 (Brewbaker [2])**.** *A matrix is a lonesum matrix if and only if it avoids A_1.*

Theorem 2.1 implies that $\phi(k,n;A_1)$ is equal to the number of $k\times n$ lonesum matrices.

Definition 2.2. *Bernoulli number B_n is defined as following:*

$$B_0 = 1, \qquad \sum_{i=0}^{m}\binom{m+1}{i} B_i = 0.$$

The exponential generating function for the Bernoulli number is

$$\sum_{n\geq 0} B_n \frac{x^n}{n!} = \frac{x\,e^x}{e^x-1}.$$

Note that

$$B_n = \sum_{m=0}^{n} (-1)^{m+n}\,\frac{m!\,S(n,m)}{m+1},$$

where $S(n,m)$ is the Stirling number of the second kind. The poly-Bernoulli number, first introduced by Kaneko [6], is defined as

$$B_n^{(k)} = \sum_{m=0}^{n} (-1)^{m+n}\,\frac{m!\,S(n,m)}{(m+1)^k},$$

and its exponential generating function is

$$\sum_{n\geq 0} B_n^{(k)} \frac{x^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}},$$

where $\mathrm{Li}_k(x)$ is the polylogarithm

$$\mathrm{Li}_k(x) := \sum_{m\geq 1} \frac{x^m}{m^k}.$$

Bernoulli numbers are nothing but poly-Bernoulli numbers with $k = 1$. Sanchez-Peregrino [10] proved that $B_n^{(-k)}$ has the following simple expression:

$$B_n^{(-k)} = \sum_{m=0}^{\min(k,n)} (m!)^2 \, S(n+1, m+1) \, S(k+1, m+1) \, .$$

Brewbaker [2] and Kim [7] proved that the number of $k \times n$ lonesum matrices is the poly-Bernoulli number $B_n^{(-k)}$. So we have the following result.

Proposition 2.3. *The number of $k \times n$ matrices avoiding A_1 is equal to $B_n^{(-k)}$, i.e.,*

$$\phi(k, n; A_1) = \sum_{m=0}^{\min(k,n)} (m!)^2 \, S(n+1, m+1) \, S(k+1, m+1) \, . \tag{1}$$

In particular, for the square matrices of size n, we have

$$\phi(n, n; A_1) = B_n^{(-n)} = \sum_{m=0}^{n} (m!)^2 \, S(n+1, m+1)^2 \, .$$

The generating function $\Phi(x, y; A_1)$, given by Kaneko [6], is

$$\Phi(x, y; A_1) = e^{x+y} \sum_{m \geq 0} [(e^x - 1)(e^y - 1)]^m = \frac{e^{x+y}}{e^x + e^y - e^{x+y}} . \tag{2}$$

We also easily obtain $\Phi(z; A_1)$ as follows:

$$\begin{aligned} \Phi(z; A_1) &= \sum_{n \geq 0} \phi(n, n; A_1) \, \frac{z^n}{n!} \\ &= \sum_{n \geq 0} \sum_{m \geq 0} (-1)^{m+n} m! S(n, m)(m+1)^n \, \frac{z^n}{n!} \\ &= \sum_{m \geq 0} (-1)^m m! \sum_{n \geq 0} S(n, m) \, \frac{(-(m+1)z)^n}{n!} \\ &= \sum_{m \geq 0} (1 - e^{-(m+1)z})^m \, . \end{aligned} \tag{3}$$

3 Main Results

3.1 Type A_2 (or type A_3)

By row exchange and column exchange we can change the original matrix into a block matrix as in Figure 1. Here [**0**] (resp. [**1**]) stands for a 0-block (resp.1-block) and [**0***] stands for a 0-block or an empty block. Diagonal blocks are [**1**]'s except for the last block [**0***], and the off-diagonal blocks are [**0**]'s.

[1]	[0]	[0]	[0*]
[0]	[1]	[0]	[0*]
[0]	[0]	[1]	[0*]
[0*]	[0*]	[0*]	[0*]

Figure 1: A matrix avoiding A_2 can be changed into a block diagonal matrix.

Theorem 3.1. *The number of $k \times n$ matrices avoiding A_2 is given by*

$$\phi(k, n; A_2) = \sum_{m=0}^{\min(k,n)} m!\, S(n+1, m+1)\, S(k+1, m+1). \tag{4}$$

In particular, for the square matrix of size n, we have

$$\phi(n, n; A_2) = \sum_{m=0}^{n} m!\, S(n+1, m+1)^2.$$

Proof. Let $\mu = \{C_1, C_2, \ldots, C_{m+1}\}$ be a set partition of $[n+1]$ into $m+1$ blocks. Here the block C_l's are ordered by the largest element of each block. Thus $n+1$ is contained in C_{m+1}. Likewise, let $\nu = \{D_1, D_2, \ldots, D_{m+1}\}$ be a set partition of $[k+1]$ into $m+1$ blocks. Choose $\sigma \in S_{m+1}$ with $\sigma(m+1) = m+1$, where S_{m+1} is the set of all permutations of length $m+1$. Given (μ, ν, σ) we define a $k \times n$ matrix $M = (a_{i,j})$ by

$$a_{i,j} := \begin{cases} 1, & (i,j) \in C_l \times D_{\sigma(l)} \text{ for some } l \in [m] \\ 0, & \text{otherwise} \end{cases}.$$

It is obvious that the matrix M avoids the type A_2.

Conversely, let M be a $k \times n$ matrix avoiding type A_2. Set $(k+1) \times (n+1)$ matrix $\widetilde{M}$ by augmenting zeros to the last row and column of M. By row exchange and column exchange we can change $\widetilde{M}$ into a block diagonal matrix B, where each diagonal is 1-block except for the last diagonal. By tracing the position of columns (resp. rows) in $\widetilde{M}$, B gives a set partition of $[n+1]$ (resp. $[k+1]$). Let $\{C_1, C_2, \ldots, C_{m+1}\}$ (resp. $\{D_1, D_2, \ldots, D_{m+1}\}$) be the set partition of $[n+1]$ (resp. $[k+1]$). Note that the block C_i's and D_i's are ordered by the largest element of each block. Let σ be a permutation on $[m]$ defined by $\sigma(i) = j$ if C_i and D_j form a 1-block in B.

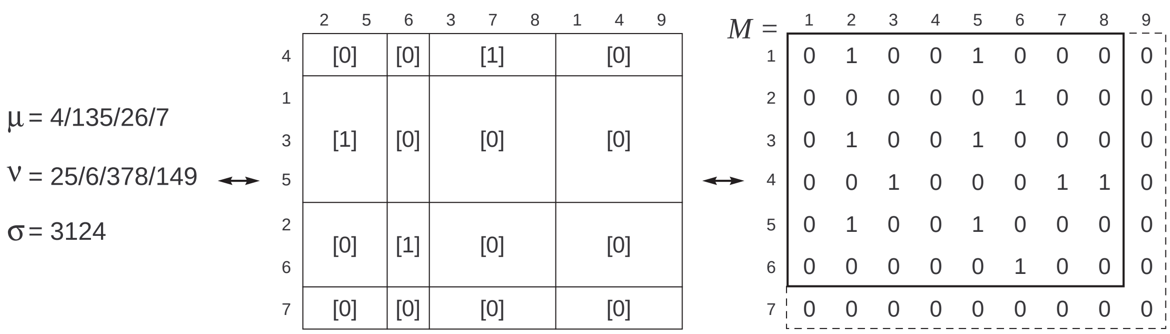

Figure 2: A matrix avoiding A_2 corresponds to two set partitions with a permutation.

The number of set partitions π of $[n+1]$ is $S(n+1, m+1)$, and the number of set partitions π' of $[k+1]$ is $S(n+1, k+1)$. The cardinality of the set of σ's is the cardinality of S_m, i.e., $m!$. Since the number of blocks $m+1$ runs through 1 to $\min(k, n)+1$, the sum of $S(k+1, m+1)\, S(n+1, m+1)\, m!$ gives the required formula. $\square$

Example 1. Let $\mu = 4/135/26/7$ be a set partition of $[7]$ and $\nu = 25/6/378/149$ of $[9]$ into 4 blocks. Let $\sigma = 3124$ be a permutation in S_4 such that $\sigma(4) = 4$. From (μ, ν, σ) we can construct the 6×8 matrix M which avoids type A_2 as in Figure 2.

To find the generating function for $\phi(k, n; A_2)$ the following formula (see [5]) is helpful.

$$\sum_{n \geq 0} S(n+1, m+1)\, \frac{x^n}{n!} = e^x\, \frac{(e^x-1)^m}{m!}. \tag{5}$$

From Theorem 3.1 and (5), we can express $\Phi(x, y; A_2)$ as follows:

$$\begin{aligned}
\Phi(x, y; A_2) &= 1 + \sum_{n,k \geq 1} \phi(k, n; A_2)\, \frac{x^k}{k!} \frac{y^n}{n!} \\
&= \sum_{n,k \geq 0} \sum_{m \geq 0} m!\, S(n+1, m+1)\, S(k+1, m+1)\, \frac{x^k}{k!} \frac{y^n}{n!} \\
&= \sum_{n,m \geq 0} S(n+1, m+1) \frac{y^n}{n!}\; m! \sum_{k \geq 0} S(k+1, m+1)\, \frac{x^k}{k!} \\
&= \sum_{m \geq 0} e^x (e^x-1)^m \sum_{n \geq 0} S(n+1, m+1) \frac{y^n}{n!} \\
&= \sum_{m \geq 0} e^x (e^x-1)^m \frac{1}{m!} e^y (e^y-1)^m \\
&= \exp[(e^x-1)(e^y-1) + x + y]\,. \qquad (6)
\end{aligned}$$

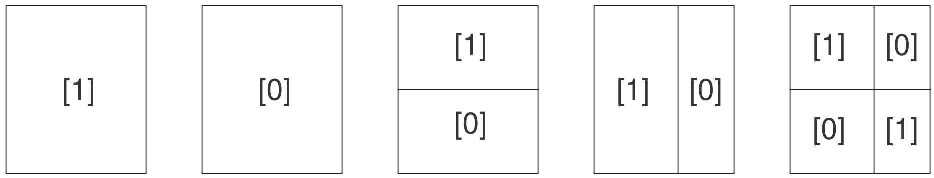

Figure 3: Possible reduced forms of matrices avoiding $A_2 \cup A_3$

Remark 1. It seems to be difficult to find a simple expression of $\Phi(z; A_2)$. The sequence $\phi(n, n; A_2)$ is not listed in the OEIS [9]. The first few terms of $\phi(n, n; A_2)$ $(0 \leq n \leq 9)$ are as follows:

$$1, 2, 12, 128, 2100, 48032, 1444212, 54763088, 2540607060, 140893490432, \ldots$$

3.2 Type $A_2 \cup A_3$

As mentioned in Section 1, if we add a new relation – exchanging 0 and 1 – on matrices, then $A_2 \cup A_3$ becomes a new equivalent class. The reduced form of a matrix M avoiding $A_2 \cup A_3$ is very simple as in Figure 3. In this case if the first row and the first column of M are determined then the rest of the entries of M are determined uniquely. Hence the number $\phi(k, n; A_2 \cup A_3)$ of such matrices is

$$\phi(k, n; A_2 \cup A_3) = 2^{k+n-1} \qquad (k, n \geq 1), \tag{7}$$

and its exponential generating function is

$$\Phi(x, y; A_2 \cup A_3) = 1 + \frac{1}{2}\,(e^{2x} - 1)(e^{2y} - 1)\,. \tag{8}$$

Clearly, $\phi(n, n; A_2 \cup A_3) = 2^{2n-1}$ for $n \geq 1$. Thus its exponential generating function is

$$\Phi(z; A_2 \cup A_3) = \frac{1}{2}\,(e^{4z} + 1)\,. \tag{9}$$

3.3 Type A_4 (or type A_5)

Given a 0/1-matrix, 1-column (resp. 0-column) is a column in which all entries consist of 1's (resp. 0's). We denote a 1-column (resp. 0-column) by $\mathbf{1}$ (resp. $\mathbf{0}$). A mixed column is a column which is neither $\mathbf{0}$ nor $\mathbf{1}$. For $k = 0$, we have $\phi(0, n; A_4) = \delta_{0,n}$. In case $k \geq 1$, i.e., there being at least one row, we can enumerate as follows:

- case 1: there are no mixed columns. Then each column should be $\mathbf{0}$ or $\mathbf{1}$. The number of such $k \times n$ matrices is 2^n.

- case 2: there is one mixed column. In this case each column should be $\mathbf{0}$ or $\mathbf{1}$ except for one mixed column. The number of $k \times n$ matrices of this case is $2^{n-1}\,n\,(2^k - 2)$.

- case 3: there are two mixed columns. As in case 2, each column should be **0** or **1** except for two mixed columns, say, v_1 and v_2. The number of $k \times n$ matrices of this case is the sum of the following three subcases:
  - $v_1 + v_2 = \mathbf{1}:\ \ 2^{n-2}\binom{n}{2}\, 2!\, S(k,2)$
  - $v_1 + v_2$ has an entry $0:\ \ 2^{n-2}\binom{n}{2}\, 3!\, S(k,3)$
  - $v_1 + v_2$ has an entry $2:\ \ 2^{n-2}\binom{n}{2}\, 3!\, S(k,3)$

- case 4: there are m ($m \geq 3$) mixed columns $v_1, \ldots, v_m$. The number of $k \times n$ matrices of this case is the sum of the following four subcases:
  - $v_1 + \cdots + v_m = \mathbf{1}:\ \ 2^{n-m}\binom{n}{m}\, m!\, S(k,m)$
  - $v_1 + \cdots + v_m = (m-1)\mathbf{1}:\ \ 2^{n-m}\binom{n}{m}\, m!\, S(k,m)$
  - $v_1 + \cdots + v_m$ has an entry $0:\ \ 2^{n-m}\binom{n}{m}\, (m+1)!\, S(k,m+1)$
  - $v_1 + \cdots + v_m$ has an entry $m:\ \ 2^{n-m}\binom{n}{m}\, (m+1)!\, S(k,m+1)$

Adding up all numbers in the previous cases yields the following theorem.

Theorem 3.2. *For k, $n \geq 1$ the number of $k \times n$ matrices avoiding A_4 is given by*

$$\phi(k,n;A_4) = 2\sum_{l\geq 1}\binom{n}{l-1} l^k + (n^2-n-4)\,2^{n-2} - n(n+3)\,2^{n+k-3}. \qquad (10)$$

Proof.

$$\begin{aligned}
\phi(k,n;A_4) &= 2^n + 2^{n-1}\binom{n}{1}(2^k-2) + 2^{n-2}\binom{n}{2}\left(\, 2!\, S(k,2) + 3!\, 2\, S(k,3)\,\right) \\
&\quad + \sum_{m=3}^{n} 2^{n-m+1}\binom{n}{m}\left(\, m!\, S(k,m) + (m+1)!\, S(k,m+1)\,\right) \\
&= 2\sum_{m=0}^{n} 2^{n-m}\binom{n}{m} m! S(k+1,m+1) + \ (n^2-n-4)\,2^{n-2} - n(n+3)\,2^{n+k-3} \\
&= 2\sum_{l\geq 1}\binom{n}{l-1} l^k + (n^2-n-4)\,2^{n-2} - n(n+3)\,2^{n+k-3}.
\end{aligned}$$

□

Note that in the proof of Theorem 3.2 we use the identity

$$\sum_{m\geq 0}\binom{n}{m} m!\, S(k,m)\, 2^{n-m} = \sum_{l\geq 0}\binom{n}{l} l^k\,,$$

where both sides count the number of functions f from $[k]$ to $[n]$ such that each element of $[n] \setminus f([k])$ has two colors.

The generating function $\Phi(x,y;A_4)$ is given by

$$\begin{aligned}
\Phi(x,y;A_4) &= 1+\sum_{n\geq 1}\sum_{k\geq 1} 2\sum_{l\geq 1}\binom{n}{l-1} l^k \frac{x^k}{k!}\frac{y^n}{n!} \\
&\quad +\sum_{n\geq 1}\sum_{k\geq 1}(n^2-n-4)\,2^{n-2}\frac{x^k}{k!}\frac{y^n}{n!}-\sum_{n\geq 1}\sum_{k\geq 1} n(n+3)\,2^{n+k-3}\frac{x^k}{k!}\frac{y^n}{n!} \\
&= 1+\big(2e^x(e^{y(e^x+1)}-1)-2e^{2y}+2\big) \\
&\quad +(e^x-1)\big((y^2-1)e^{2y}+1\big)-\frac{1}{2}y(y+2)e^{2y}(e^{2x}-1) \\
&= 2e^{y(e^x+1)+x}-\frac{y^2+2y}{2}e^{2x+2y}+(y^2-1)e^{x+2y}-e^x-\frac{y^2-2y+2}{2}e^{2y}+2\,. \qquad (11)
\end{aligned}$$

For the $n\times n$ matrices we have

$$\phi(n,n;A_4)=2\sum_{l\geq 1}\binom{n}{l-1} l^n+(n^2-n-4)\,2^{n-2}-n(n+3)\,2^{2n-3}.$$

Thus the generating function $\Phi(n,n;A_4)$ is given by

$$\begin{aligned}
\sum_{n\geq 0}\phi(n,n;A_4)\frac{z^n}{n!} &= 2\sum_{n\geq 0}\sum_{l\geq 1}\binom{n+1}{l} l^{n+1}\frac{z^n}{(n+1)!} \\
&\quad +\sum_{n\geq 0}\frac{n^2-n-4}{4}\frac{(2z)^n}{n!}-\frac{n(n+3)}{8}\frac{(4z)^n}{n!} \\
&= \frac{2}{z}\sum_{l\geq 1}\frac{(lz)^l}{l!}\sum_{n\geq l-1}\frac{(lz)^{n-l+1}}{(n-l+1)!}+(z^2-1)e^{2z}-2z(z+1)e^{4z} \\
&= \frac{2}{z}\sum_{l\geq 1}\frac{l^l}{l!}(ze^z)^l+(z^2-1)e^{2z}-2z(z+1)e^{4z} \\
&= \frac{2}{z}\,(ze^z\,W'(-ze^z))+(z^2-1)e^{2z}-2z(z+1)e^{4z} \\
&= \frac{-2\,W(-ze^z)}{z+z\,W(-ze^z)}+(z^2-1)e^{2z}-2z(z+1)e^{4z}, \qquad (12)
\end{aligned}$$

where

$$W(x):=\sum_{n\geq 1}(-n)^{n-1}\frac{x^n}{n!}$$

is the Lambert W-function which is the inverse function of $f(W)=We^W$. See [3] for extensive study about the Lambert W-function.

Remark 2. The sequence $\phi(n,n;A_4)$ is not listed in the OEIS [9]. The first few terms of $\phi(n,n;A_4)$ $(0\leq n\leq 9)$ are as follows:

$$1, 2, 14, 200, 3536, 67472, 1423168, 34048352, 927156224, 28490354432, \ldots$$

3.4 Type $A_4 \cup A_5$

If we add a new relation – transpose – on matrices, then $A_4 \cup A_5$ becomes a new equivalent class. By the symmetry of $A_4 \cup A_5$, we have

$$\phi(k, n; A_4 \cup A_5) = \phi(n, k; A_4 \cup A_5).$$

So it is enough to consider the case $k \geq n$. For $k \leq 2$ or $n \leq 1$, we have

$$\begin{gathered} \phi(0, n; A_4 \cup A_5) = \delta_{0,n}, \quad \phi(1, n; A_4 \cup A_5) = 2^n, \\ \phi(k, 0; A_4 \cup A_5) = \delta_{k,0}, \quad \phi(k, 1; A_4 \cup A_5) = 2^k, \\ \phi(2, 2; A_4 \cup A_5) = 12. \end{gathered}$$

Given a 0/1-vector v with length of at least 3, v is called 1-dominant (resp. 0-dominant) if all entries of v are 1's (resp. 0's) except one entry.

Theorem 3.3. *For $k \geq 3$ and $n \geq 2$, the number of $k \times n$ matrices avoiding $A_4 \cup A_5$ is equal to twice the number of rook positions in the $k \times n$ chessboard. In other words,*

$$\phi(k, n; A_4 \cup A_5) = 2 \sum_{m=0}^{\min(k,n)} \binom{k}{m} \binom{n}{m} m!\,. \tag{13}$$

Proof. Suppose M is a $k \times n$ matrix avoiding $A_4 \cup A_5$. It is easy to show each of the following steps:

(i) If M has a mixed column v, then v should be either 0-dominant or 1-dominant.

(ii) Assume that v is 0-dominant. This implies that other mixed columns(if any) in M should be 0-dominant.

(iii) Any non-mixed column in M should be a 0-column.

(iv) The location of 1's in M corresponds to a rook position in the $k \times n$ chessboard.

If we assume v is 1-dominant in (ii) then the locations of 0's again corresponds to a rook position. The summand of RHS in (13) is the number of rook positions in the $k \times n$ chessboard with m rooks. This completes the proof. □

The generating function $\Phi(x, y; A_4 \cup A_5)$ is given by

$$\begin{aligned} \Phi(x, y; A_4 \cup A_5) \;=\; & 2e^{xy+x+y} - \frac{(xy)^2}{2} \\ & - 2xy + 3 - 2e^x - 2e^y + x(e^y - 2y - 1)(e^y - 1) + y(e^x - 2x - 1)(e^x - 1). \end{aligned} \tag{14}$$

Note that the crucial part of the equation (14) can be obtained as follows:

$$\begin{aligned}
\sum_{k,n\geq 0}\left(\sum_{m\geq 0}\binom{k}{m}\binom{n}{m}m!\right)\frac{x^k}{k!}\frac{y^n}{n!} &= \sum_{m\geq 0} m!\left(\sum_{k\geq 0}\binom{k}{m}\frac{x^k}{k!}\right)\left(\sum_{n\geq 0}\binom{n}{m}\frac{y^n}{n!}\right) \\
&= \sum_{m\geq 0}\left(\frac{x^m}{m!}e^x\right)\left(\frac{y^m}{m!}e^y\right) \\
&= e^{xy+x+y}.
\end{aligned}$$

We remark that the summands in the second line of (14) contribute the coefficient of $x^k y^n$ where k or n are less than 2.

For the $n \times n$ matrices we have

$$\phi(0,0;A_4\cup A_5)=1,\quad \phi(1,1;A_4\cup A_5)=2,\quad \phi(2,2;A_4\cup A_5)=12,\quad \text{and}$$
$$\phi(n,n;A_4\cup A_5)=2\sum_{m=0}^{n}\binom{n}{m}^2 m!\,.\quad (n\geq 3)$$

Thus the generating function $\Phi(z;A_4\cup A_5)$ is given by

$$\Phi(z;A_4\cup A_5)=\frac{2\,e^{\frac{z}{1-z}}}{1-z}-1-2z-z^2. \tag{15}$$

Note that we use the equation

$$\sum_{n\geq 0}\left(\sum_{m=0}^{n}\binom{n}{m}^2 m!\right)\frac{z^n}{n!}=\frac{e^{\frac{z}{1-z}}}{1-z},$$

which appears in [4, pp. 597–598].

3.5 Type $A_6 \cup A_7$

If we add a relation – exchanging 0 and 1 – on 0/1-matrices, then $A_6\cup A_7$ becomes a new equivalent class. Due to the symmetry of $A_6\cup A_7$, it is obvious that

$$\phi(k,n;A_6\cup A_7)=\phi(n,k;A_6\cup A_7).$$

The k-color bipartite Ramsey number $br(G;k)$ of a bipartite graph G is the minimum integer n such that in any k-coloring of the edges of $K_{n,n}$ there is a monochromatic subgraph isomorphic to G. Beineke and Schwenk [1] had shown that $br(K_{2,2};2)=5$. From this we can see that

$$\phi(k,n;A_6\cup A_7)=0\quad (k,n\geq 5)\,.$$

For $k=1$ and 2, we have

$$\begin{aligned}
\phi(1,n;A_6\cup A_7) &= 2^n, \\
\phi(2,n;A_6\cup A_7) &= (n^2+3n+4)2^{n-2}.
\end{aligned}$$

$k \diagdown n$	1	2	3	4	5	6	7	$\cdots$
1	2	4	8	16	32	64	128	$\cdots$
2	4	14	44	128	352	928	2368	$\cdots$
3	8	44	156	408	720	720	0	0
4	16	128	408	840	720	720	0	0
5	32	352	720	720	0	0	0	0
6	64	928	720	720	0	0	0	0
7	128	2368	0	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	0	0	0	0	0	0

Table 2: The sequence $\phi(k, n; A_6 \cup A_7)$

Note that the sequence $(n^2 + 3n + 4)\, 2^{n-2}$ appears in [9, A007466] and its exponential generating function is $(1 + x)^2 e^{2x}$.

For $k \geq 3$, we have

$$\begin{aligned} \phi(3, n; A_6 \cup A_7) = \phi(4, n; A_6 \cup A_7) &= 0 \qquad \text{for } n \geq 7, \\ \phi(5, n; A_6 \cup A_7) = \phi(6, n; A_6 \cup A_7) &= 0 \qquad \text{for } n \geq 5, \\ \phi(k, n; A_6 \cup A_7) &= 0 \qquad \text{for } k \geq 7 \text{ and } n \geq 3. \end{aligned}$$

For exceptional cases, due to the symmetry of $A_6 \cup A_7$, it is enough to consider the followings:

$$\phi(3, 3; A_6 \cup A_7) = 156, \quad \phi(3, 4; A_6 \cup A_7) = 408, \quad \phi(4, 4; A_6 \cup A_7) = 840,$$
$$\phi(3, 5; A_6 \cup A_7) = \phi(3, 6; A_6 \cup A_7) = \phi(4, 5, A_6 \cup A_7) = \phi(4, 6; A_6 \cup A_7) = 720\,.$$

The sequence $\phi(k, n; A_6 \cup A_7)$ is listed in Table 2.

The generating function $\Phi(x, y; A_6 \cup A_7)$ is given by

$$\begin{aligned} \Phi(x, y; A_6 \cup A_7) \;=\; & 1 + x\, e^{2y} + y\, e^{2x} + x^2(1 + y)^2 e^{2y} + y^2(1 + x)^2 e^{2x} \\ & - \left(x + y + \frac{x^2}{2!} + \frac{y^2}{2!} + 2xy + 2x^2y + 2xy^2 + 14\frac{x^2y^2}{2!2!} \right) + 156\frac{x^3y^3}{3!3!} + 840\frac{x^4y^4}{4!4!} \\ & + 720 \left(\frac{x^3y^5}{3!5!} + \frac{x^5y^3}{5!3!} + \frac{x^3y^6}{3!6!} + \frac{x^6y^3}{6!3!} + \frac{x^4y^5}{4!5!} + \frac{x^5y^4}{5!4!} + \frac{x^4y^6}{4!6!} + \frac{x^6y^4}{6!4!} \right) . \end{aligned} \tag{16}$$

In particular, the generating function $\Phi(z; A_6 \cup A_7)$ is given by

$$\Phi(z; A_6 \cup A_7) = 1 + 2z + 7z^2 + 26z^3 + 35z^4\,. \tag{17}$$

4 Concluding remarks

Table 3 summarizes our results. Due to the amount of difficulty, we are not able to enumerate the number $\phi(k, n; A_6)$ or $\phi(k, n; A_7)$. Note that $\phi(k, n; A_6)$ is equal to the following:

α	$\phi(k,n;\alpha)$	$\Phi(x,y;\alpha)$	$\Phi(z;\alpha)$
A_1	(1)	(2)	(3)
A_2 (or A_3)	(4)	(6)	complicated
$A_2 \cup A_3$	(7)	(8)	(9)
A_4 (or A_5)	(10)	(11)	(12)
$A_4 \cup A_5$	(13)	(14)	(15)
A_6 (or A_7)	unknown	unknown	unknown
$A_6 \cup A_7$	Table 2	(16)	(17)

Table 3: Formulas and generating functions according to each avoiding type α.

(a) The number of labeled (k,n)-bipartite graphs with girth of at least 6, i.e., the number of C_4-free labeled (k,n)-bipartite graphs, where C_4 is a cycle of length 4.

(b) The cardinality of the set $\{(B_1, B_2, \ldots, B_k) : B_i \subseteq [n]\ \forall i,\ \ |B_i \cap B_j| \leq 1\ \forall i \neq j\}$.

For further research, we suggest the following problems.

1. Enumeration of sets of $k \times n$ 0/1-matrices avoiding each individual matrix instead of each equivalent class by row/column exchange. For example, $\phi(k,n;\{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\})$.

2. In addition to 0/1-matrices, one can consider $0/1/\cdots/r$-matrices with $r \geq 2$.

3. Consideration of the results of adding the line sum condition to each individual case given in the first column of Table 3.

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