GLUING AND HILBERT FUNCTIONS OF MONOMIAL CURVES

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ABSTRACT. In this article, by using the technique of gluing semigroups, we give infinitely many families of 1-dimensional local rings with non-decreasing Hilbert functions. More significantly, these are local rings whose associated graded rings are not necessarily Cohen-Macaulay. In this sense, we give an effective technique to construct large families of 1-dimensional Gorenstein local rings associated to monomial curves, which support Rossi's conjecture saying that every Gorenstein local ring has non-decreasing Hilbert function.

1. INTRODUCTION

In this article, we study the Hilbert functions of local rings associated to affine monomial curves obtained by using the technique of gluing numerical semigroups. The concept of gluing was introduced by J.C. Rosales in [11] and used by several authors to produce new examples of set-theoretic and ideal-theoretic complete intersection affine or projective varieties (for example [10, 12, 13]). We give large families of local rings with non-decreasing Hilbert functions and generalize the results in [1] and [2] given for nice extensions, which are in fact special types of gluings. In doing this, we also give the definition of a nice gluing which is a generalization of a nice extension. Moreover, by using the technique of nice gluing, we obtain infinitely many families of 1-dimensional local rings with non-Cohen-Macaulay associated graded rings and still having non-decreasing Hilbert functions. We demonstrate that nice gluing is an effective technique to construct large families of 1-dimensional Gorenstein local rings associated to monomial curves, which support the conjecture due to Rossi saying that every Gorenstein local ring has non-decreasing Hilbert function [2].

Our main interest in this article is the following question about gluing:

> *Question.* If the Hilbert functions of the local rings associated to two monomial curves are non-decreasing, is the Hilbert function of the local ring associated to the monomial curve obtained by gluing these two monomial curves also non-decreasing?

Every monomial curve in affine 2-space is obtained by gluing, and it is well-known that every local ring associated to a monomial curve in affine 2-space has a non-decreasing Hilbert function. In affine 3-space, every monomial curve is not obtained by gluing, but every local ring associated to a monomial curve in affine 3-space has also a non-decreasing Hilbert function. This follows from the important result of Elias saying that every one-dimensional Cohen-Macaulay local ring with embedding dimension three has a non-decreasing Hilbert function [6]. Thus, the

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above question is trivial for the monomial curves in affine 2-space and 3-space, which are obtained by gluing, while the question is open even for the monomial curves in 4-space, which are obtained by gluing. What makes this question important is that, if the answer is affirmative even in the case of gluing complete intersection monomial curves, it will follow that the Hilbert function of every local ring associated to a complete intersection monomial curve is non-decreasing. This will be due to a result of Delorme [4], which is restated by Rosales in terms of gluing and says that every complete intersection numerical semigroup minimally generated by at least two elements is a gluing of two complete intersection numerical semigroups [11, Theorem 2.3]. Considering that it is still not known whether the Hilbert function of local rings with embedding dimension four associated to complete intersection monomial curves in affine 4-space is non-decreasing, this will be an important step in proving Rossi's conjecture.

We recall that an *affine monomial curve* $C(n_1,\dots,n_k)$ is a curve with generic zero $(t^{n_1},\dots,t^{n_k})$ in the affine n-space $\mathbb{A}^n$ over an algebraically closed field K, where $n_1<\cdots<n_k$ are positive integers with $\gcd(n_1,n_2,\dots,n_k)=1$ and $\{n_1,n_2,\dots,n_k\}$ is a minimal set of generators for the numerical semigroup $\langle n_1,n_2,\dots,n_k\rangle = \{n \mid n=\sum_{i=1}^k a_in_i,\ a_i\text{'s are non-negative integers}\}$. The local ring associated to the monomial curve $C=C(n_1,\dots,n_k)$ is $K[[t^{n_1},\dots,t^{n_k}]]$, and the Hilbert function of this local ring is the Hilbert function of its associated graded ring $gr_m(K[[t^{n_1},\dots,t^{n_k}]])$, which is isomorphic to the ring $K[x_1,\dots,x_k]/I(C)_*$, where $I(C)$ is the defining ideal of C and $I(C)_*$ is the ideal generated by the polynomials f_* for f in $I(C)$ and f_* is the homogeneous summand of f of least degree. In other words, $I(C)_*$ is the defining ideal of the tangent cone of C at 0.

2. Technique of Gluing Semigroups and Monomial curves

In this section, we first give the definition of gluing for numerical semigroups.

Definition 2.1. [11, Lemma 2.2] Let S_1 and S_2 be two numerical semigroups minimally generated by $m_1<\cdots<m_l$ and $n_1<\cdots<n_k$ respectively. Let $p=b_1m_1+\cdots+b_lm_l\in S_1$ and $q=a_1n_1+\cdots+a_kn_k\in S_2$ be two positive integers satisfying $gcd(p,q)=1$ with $p\notin\{m_1,\dots,m_l\}$, $q\notin\{n_1,\dots,n_k\}$ and $\{qm_1,\dots,qm_l\}\cap\{pn_1,\dots,pn_k\}=\emptyset$. The numerical semigroup $S=\langle qm_1,\dots,qm_l,pn_1,\dots,pn_k\rangle$ is called a gluing of the semigroups S_1 and S_2.

Thus, the monomial curve $C=C(qm_1,\dots,qm_l,pn_1,\dots,pn_k)$ can be interpreted as the gluing of the monomial curves $C_1=C(m_1,\dots,m_l)$ and $C_2=C(n_1,\dots,n_k)$, if p and q satisfy the conditions in Definition 2.1. Moreover, from [11, Theorem 1.4], if the defining ideals $I(C_1)\subset K[x_1,\dots,x_l]$ of C_1 and $I(C_2)\subset K[y_1,\dots,y_k]$ of C_2 are generated by the sets $G_1=\{f_1,\dots,f_s\}$ and $G_2=\{g_1,\dots,g_t\}$ respectively, then the defining ideal of $I(C)\subset K[x_1,\dots,x_l,y_1,\dots,y_k]$ is generated by the set $G=\{f_1,\dots,f_s,g_1,\dots,g_t,x_1^{b_1}\dots x_l^{b_l}-y_1^{a_1}\dots y_k^{a_k}\}$.

Now, consider the local rings $R_1=K[[t^{m_1},\dots,t^{m_l}]]$, $R_2=K[[t^{n_1},\dots,t^{n_k}]]$ and $R=K[[t^{qm_1},\dots,t^{qm_l},t^{pn_1},\dots,t^{pn_k}]]$ associated respectively to the monomial curves C_1, C_2 and C obtained by gluing C_1 and C_2. Our main interest is whether the Hilbert function of R is non-decreasing, given that the Hilbert functions of the local rings R_1 and R_2 are non-decreasing.

We first answer the following question: If C_1 and C_2 have Cohen-Macaulay tangent cones, is the tangent cone of the monomial curve C obtained by gluing these two monomial curves necessarily Cohen-Macaulay? The following example shows that the answer is no.

Example 2.2. Let C_1 and C_2 be the monomial curves $C_1 = C(5, 12)$ and $C_2 = C(7, 8)$. Obviously, they have Cohen-Macaulay tangent cones. By a gluing of C_1 and C_2, we obtain the monomial curve $C = C(21 \times 5, 21 \times 12, 17 \times 7, 17 \times 8)$. The ideal $I(C)$ is generated by the following set $G = \{x_1^{12} - x_2^5, y_1^8 - y_2^7, x_1x_2 - y_1^3\}$. The ideal $I(C)_*$ of the tangent cone of C at the origin is generated by the set $G_* = \{x_1x_2, x_2^5, y_1^{15}, y_2^7, x_2^4y_1^3, x_2^3y_1^6, x_2^2y_1^9, x_2y_1^{12}\}$ which is a Gröbner basis with respect to the negative degree reverse lexicographical ordering with $x_2 > y_2 > y_1 > x_1$. From [1, Theorem 2.1], since x_1 divides $x_1x_2 \in G_*$, the monomial curve C obtained by a gluing of C_1 and C_2 does not have a Cohen-Macaulay tangent cone. It should also be noted that Hilbert function of the local ring corresponding to C is non-decreasing, although the tangent cone of C is not Cohen-Macaulay.

This example leads us to ask the following question:

> *Question.* If two monomial curves have Cohen-Macaulay tangent cones, under which conditions does the monomial curve obtained by gluing these two monomial curves also have a Cohen-Macaulay tangent cone?

To answer this question partly, we first give the definition of a nice gluing, which generalizes the definition of a nice extension given in [2].

Definition 2.3. Let $S_1 = \langle m_1, \dots, m_l \rangle$ and $S_2 = \langle n_1, \dots, n_k \rangle$ be two numerical semigroups minimally generated by $m_1 < \cdots < m_l$ and $n_1 < \cdots < n_k$ respectively. The numerical semigroup $S = \langle qm_1, \dots, qm_l, pn_1, \dots, pn_k \rangle$ obtained by gluing S_1 and S_2 is called a nice gluing, if $p = b_1m_1 + \cdots + b_lm_l \in S_1$ and $q = a_1n_1 \in S_2$ with $a_1 \leq b_1 + \cdots + b_l$.

Remark 2.4. Notice that a nice extension defined in [2] is exactly a nice gluing with $S_2 = \langle 1 \rangle$.

Remark 2.5. It is important to determine the smallest integer among the generators of the numerical semigroup $S = \langle qm_1, \dots, qm_l, pn_1, \dots, pn_k \rangle$ obtained by gluing, since this is essential in checking the Cohen-Macaulayness of the tangent cone of the associated monomial curve. The condition $a_1 \leq b_1 + \cdots + b_l$ with $m_1 < \cdots < m_l$, $n_1 < \cdots < n_k$, $\gcd(p, q) = 1$ and $\{qm_1, \dots, qm_l\} \cap \{pn_1, \dots, pn_k\} = \emptyset$ implies that

$$qm_1 = a_1n_1m_1 \leq (b_1 + \cdots + b_l)n_1m_1 < pn_1 = (b_1m_1 + \cdots + b_lm_l)n_1$$

and qm_1 is the smallest integer among the generators of S.

We can now state the following theorem:

Theorem 2.6. *Let $S_1 = \langle m_1, \dots, m_l \rangle$ and $S_2 = \langle n_1, \dots, n_k \rangle$ be two numerical semigroups minimally generated by $m_1 < \cdots < m_l$ and $n_1 < \cdots < n_k$, and let $S = \langle qm_1, \dots, qm_l, pn_1, \dots, pn_k \rangle$ be a nice gluing of S_1 and S_2. If the associated monomial curves $C_1 = C(m_1, \dots, m_l)$ and $C_2 = C(n_1, \dots, n_k)$ have Cohen-Macaulay tangent cones at the origin, then $C = C(qm_1, \dots, qm_l, pn_1, \dots, pn_k)$ has also Cohen-Macaulay tangent cone at the origin, and thus, the Hilbert function of the local ring $K[[t^{qm_1}, \dots, t^{qm_l}, t^{pn_1}, \dots, t^{pn_k}]]$ is non-decreasing.*

To prove this theorem, we first give a refinement of the criterion for checking the Cohen-Macaulayness of the tangent cone of a monomial curve given in [1, Theorem

2.1], which was used in Example 2.2. The advantage of this modification in the criterion is that instead of first finding the generators of the tangent cone and then computing another Gröbner basis, it only needs the computation of the standard basis of the generators of the defining ideal of the monomial curve with respect to a special local order. Recall that a local order is a monomial ordering with 1 greater than any other monomial. For the examples and properties of local orderings, see [8]. We denote the leading monomial of a polynomial f by $\mathrm{LM}(f)$.

Lemma 2.7. *Let $\langle n_1, \ldots, n_k\rangle$ be a numerical semigroup minimally generated by $n_1 < \cdots < n_k$, $C = C(n_1, \ldots, n_k)$ be the associated monomial curve and $G = \{f_1, \ldots, f_s\}$ be a minimal standard basis of the ideal $I(C) \subset K[x_1, \ldots, x_k]$ with respect to the negative degree reverse lexicographical ordering that makes x_1 the lowest variable. C has Cohen-Macaulay tangent cone at the origin if and only if x_1 does not divide $\mathrm{LM}(f_i)$ for $1 \leq i \leq s$.*

This lemma combines a result of Bayer-Stillman [5, Theorem 15.13] with the well-known fact that a monomial curve $C = C(n_1, \ldots, n_k)$, where n_1 is smallest among the integers $n_1, \ldots, n_k$, has Cohen-Macaulay tangent cone if and only if x_1 is not a zero divisor in the ring $K[x_1, \ldots, x_k]/I(C)_*$ [7, Theorem 7].

Proof. Recalling that f_* is the homogeneous summand of the polynomial f of least degree, if x_1 divides $\mathrm{LM}(f_i)$ for some i, then either $f_{i_*} = x_1m$ or $f_{i_*} = x_1m + \sum c_im_i$, where m_i's are monomials having the same degree with x_1m and c_i's are in K. In the latter case, x_1 must divide each m_i, because we work with the negative degree reverse lexicographical ordering that makes x_1 the lowest variable. This implies that in both cases $f_{i_*} = x_1g$ where g is a homogeneous polynomial. Moreover, $g \notin I(C)_*$. If $g \in I(C)_*$, then there exists $f \in I(C)$ such that $f_* = g$ so $\mathrm{LM}(f) = \mathrm{LM}(g)$. Since $\langle \mathrm{LM}(f_1), \ldots, \mathrm{LM}(f_s)\rangle = \langle \mathrm{LM}(I(C))\rangle$, there exists an $f_j \in G$ such that $\mathrm{LM}(f_j)$ divides $\mathrm{LM}(f) = \mathrm{LM}(g)$ and this contradicts with the minimality of G. Thus, $x_1g \in I(C)_*$, while $g \notin I(C)_*$, which makes x_1 a zero-divisor in $K[x_1, \ldots, x_k]/I(C)_*$. Hence, the tangent cone of the monomial curve C is not Cohen-Macaulay. Conversely, if $K[x_1, \ldots, x_k]/I(C)_*$ is not Cohen-Macaulay, then x_1 is a zero-divisor in $K[x_1, \ldots, x_k]/I(C)_*$. Thus, $x_1m \in I(C)_*$, where m is a monomial and $m \notin I(C)_*$. The ideal generated by the leading monomials of the elements in $I(C)$ obviously contains x_1m. Since G is a standard basis, there exists $f_i \in G$ such that $\mathrm{LM}(f_i) = x_1m'$, where m' divides m and $m' \notin I(C)_*$, because $m \notin I(C)_*$. This completes the proof.

□

We can now prove Theorem 2.6.

Proof of Theorem 2.6. By using the notation in [8], we denote the s-polynomial of the polynomials f and g by spoly(f, g) and the Mora's polynomial weak normal form of f with respect to G by $NF(f|G)$. Let $G_1 = \{f_1, \ldots, f_s\}$ be a minimal standard basis of the ideal $I(C_1) \subset K[x_1, \ldots, x_l]$ with respect to the negative degree reverse lexicographical ordering with $x_2 > \cdots > x_l > x_1$ and $G_2 = \{g_1, \ldots, g_t\}$ be a minimal standard basis of the ideal $I(C_2) \subset K[y_1, \ldots, y_k]$ with respect to the negative degree reverse lexicographical ordering with $y_2 > \cdots > y_k > y_1$. Since C_1 and C_2 have Cohen-Macaulay tangent cones at the origin, we conclude from Lemma 2.7 that x_1 does not divide the leading monomial of any element in G_1 and y_1 does not divide the leading monomial of any element in G_2 for the given orderings. The defining ideal of the monomial curve C obtained by gluing is generated by the set $G = \{f_1, \ldots, f_s, g_1, \ldots, g_t, x_1^{b_1} \ldots x_l^{b_l} - y_1^{a_1}\}$. Moreover, this set is a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $y_2 > \cdots > y_k > y_1 > x_2 > \cdots > x_l > x_1$, because $NF(\text{spoly}(f_i, g_j)|G) = 0$, $NF(\text{spoly}(f_i, x_1^{b_1} \ldots x_l^{b_l} - y_1^{a_1})|G) = 0$ and $NF(\text{spoly}(g_j, x_1^{b_1} \ldots x_l^{b_l} - y_1^{a_1})|G) = 0$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. This is due to the fact that $NF(\text{spoly}(f, g)|G) = 0$, if $\text{lcm}(\text{LM(f)}, \text{LM(g)}) = \text{LM(f)} \cdot \text{LM(g)}$. From Remark 2.5, qm_1 is the smallest integer among the generators of G. Thus, C has Cohen-Macaulay tangent cone at the origin if and only if x_1, which corresponds to qm_1, is not a zero-divisor in $K[x_1, \ldots, x_l, y_1, \ldots, y_k]/I(C)_*$. Since x_1 does not divide the leading monomial of any element in G_1 and G_2, and $\text{LM}(x_1^{b_1} \ldots x_l^{b_l} - y_1^{a_1}) = y_1^{a_1}$, x_1 does not divide the leading monomial of any element in G, which is a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $y_2 > \cdots > y_k > y_1 > x_2 > \cdots > x_l > x_1$. Thus, from Lemma 2.7, C has Cohen-Macaulay tangent cone at the origin.

□

Remark 2.8. From Remark 2.4, every nice extension is a nice gluing. Thus, if the monomial curve $C = C(m_1, \ldots, m_l)$ has a Cohen-Macaulay tangent cone at the origin, then every monomial curve $C' = C(qm_1, \ldots, qm_l, b_1m_1 + \cdots + b_lm_l)$ obtained by a nice gluing has also Cohen-Macaulay tangent cone at the origin. Thus, Theorem 2.6 generalizes the results in [1, Proposition 4.1] and [2, Theorem 3.6].

Example 2.9. Let C_1 and C_2 be the monomial curves $C_1 = C(m_1, m_2)$ with $m_1 < m_2$ and $C_2 = C(n_1, n_2)$ with $n_1 < n_2$. Obviously, they have Cohen-Macaulay tangent cones. From Theorem 2.6, every monomial curve $C = C(qm_1, qm_2, pn_1, pn_2)$ obtained by a nice gluing with $q = a_1n_1$, $p = b_1m_1 + b_2m_2$, $\gcd(p, q) = 1$ and $a_1 \leq b_1 + b_2$ has Cohen-Macaulay tangent cone at the origin, so the local ring $R = K[[t^{qm_1}, t^{qm_2}, t^{pn_1}, t^{pn_2}]]$ associated to the monomial curve C has a non-decreasing Hilbert function. Thus, by starting with fixed m_1, m_2, n_1 and n_2, we can construct infinitely many families of 1-dimensional local rings with non-decreasing Hilbert functions. For example, consider the monomial curves $C_1 = C(2, 3)$ and $C_2 = C(4, 5)$. By choosing $q = 2n_1 = 8$ and $p = (2r)m_1 + m_2 = 4r + 3$, for any $r \geq 1$, we obtain the monomial curve $C(16, 24, 16r + 12, 20r + 15)$, which is a nice gluing of C_1 and C_2. Since C is also a complete intersection monomial curve having a Cohen-Macaulay tangent cone, the associated local rings are Gorenstein with non-decrasing Hilbert functions. Obviously, they support Rossi's conjecture.

This example shows that gluing is an effective method to obtain new families of monomial curves with Cohen-Macaulay tangent cones. Especially in affine 4-space, nice gluing is a very efficent method to obtain large families of complete intersection monomial curves with Cohen-Macaulay tangent cones, since every monomial curve in affine 2-space has a Cohen-Macaulay tangent cone.

3. Monomial Curves with Non-Cohen-Macaulay Tangent Cones

In this section we show that nice gluing is not only an efficient tool to obtain new families of monomial curves with Cohen-Macaulay tangent cones, but more significantly, it is very useful for obtaining families of monomial curves with non-Cohen-Macaulay tangent cones having nondecreasing Hilbert functions. In other words, it is an effective method to obtain families of local rings with non-decreasing Hilbert functions. In this sense, it can be used to obtain families of local rings in proving the conjecture due to Rossi saying that a one-dimensional Gorenstein local ring has a non-decreasing Hilbert function.

Theorem 3.1. *Let $S_1 = \langle m_1, \dots, m_l \rangle$ and $S_2 = \langle n_1, \dots, n_k \rangle$ be two numerical semigroups minimally generated by $m_1 < \cdots < m_l$ and $n_1 < \cdots < n_k$, and let $S = \langle qm_1, \dots, qm_l, pn_1, \dots, pn_k \rangle$ be a nice gluing of S_1 and S_2. (Recall that $p = b_1m_1 + \cdots + b_lm_l \in S_1$ and $q = a_1n_1 \in S_2$ with $a_1 \leq b_1 + \cdots + b_l$.) Let the local ring $K[[t^{m_1}, \dots, t^{m_l}]]$ associated to the monomial curve $C_1 = C(m_1, \dots, m_l)$ have a non-decreasing Hilbert function and let $C_2 = C(n_1, \dots, n_k)$ have Cohen-Macaulay tangent cone at the origin, then the Hilbert function of the local ring $K[[t^{qm_1}, \dots, t^{qm_l}, t^{pn_1}, \dots, t^{pn_k}]]$ associated to the monomial curve $C = C(qm_1, \dots, qm_l, pn_1, \dots, pn_k)$ obtained by gluing is also non-decreasing.*

Proof. Let $G_1 = \{f_1, \dots, f_s\}$ be a minimal standard basis of the ideal $I(C_1) \subset K[x_1, \dots, x_l]$ with respect to the negative degree reverse lexicographical ordering with $x_2 > \cdots > x_l > x_1$ and $G_2 = \{g_1, \dots, g_t\}$ be a minimal standard basis of the ideal $I(C_2) \subset K[y_1, \dots, y_k]$ with respect to the negative degree reverse lexicographical ordering with $y_2 > \cdots > y_k > y_1$. Since C_2 has Cohen-Macaulay tangent cone, y_1 does not divide $\mathrm{LM}(g_i)$ for $1 \leq i \leq t$ from Lemma 2.7. From the proof of Theorem 2.6, $G = \{f_1, \dots, f_s, g_1, \dots, g_t, x_1^{b_1} \dots x_l^{b_l} - y_1^{a_1}\}$ is a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $y_2 > \cdots > y_k > y_1 > x_2 > \cdots > x_l > x_1$, and again from Lemma 2.7, we have $\langle \mathrm{LM}(I(C)_*) \rangle = \langle \mathrm{LM}(f_1), \dots, \mathrm{LM}(f_s), \mathrm{LM}(g_1), \dots, \mathrm{LM}(g_t), y_1^{a_1} \rangle$. Hence, recalling the well-known result going back to Macaulay [9], the Hilbert function of the local ring $K[[t^{qm_1}, \dots, t^{qm_l}, t^{pn_1}, \dots, t^{pn_k}]]$ is equal to the Hilbert function of the graded ring

$$R = K[x_1, \dots, x_l, y_1, \dots, y_k]/\langle \mathrm{LM}(f_1), \dots, \mathrm{LM}(f_s), \mathrm{LM}(g_1), \dots, \mathrm{LM}(g_t), y_1^{a_1} \rangle.$$

By using [3, Proposition 2.4] and recalling that $y_1 \nmid \mathrm{LM}(g_i)$ for $1 \leq i \leq t$, R is isomorphic to $R_1 \otimes_K R_2 \otimes_K R_3$, where $R_1 = K[x_1, \dots, x_l]/\langle \mathrm{LM}(f_1), \dots, \mathrm{LM}(f_s) \rangle$, $R_2 = K[y_2, \dots, y_l]/\langle \mathrm{LM}(g_1), \dots, \mathrm{LM}(g_t) \rangle$ and $R_3 = K[y_1]/\langle y_1^{a_1} \rangle$. Moreover, Hilbert series of R is the product of Hilbert series of R_1, R_2 and R_3. Hilbert series of R_1 can be given as $h_1(t)/(1-t)$, where the polynomial $h_1(t)$ has non-negative coefficients, since from the assumption the local ring associated to the monomial curve C_1 has non-decreasing Hilbert function. The Hilbert series of R_2 can be given as $h_2(t)$, where the polynomial $h_2(t)$ has non-negative coefficients, because R_2 is the Artinian reduction of the Cohen-Macaulay tangent cone of the monomial curve C_2. Observing that the Hilbert series of R_3 is $h_3(t) = 1 + t + \cdots + t^{a_1 - 1}$, we obtain that the Hilbert series of R is $h_1(t)h_2(t)h_3(t)/(1-t)$, where the polynomial $h_1(t)h_2(t)h_3(t)$ has non-negative coefficients. This proves that Hilbert function of R is non-decreasing. □

We can now use this theorem to obtain large families of Gorenstein monomial curves with non-Cohen-Macaulay tangent cones having nondecreasing Hilbert functions to support Rossi's conjecture.

Example 3.2. Let C_1 and C_2 be the monomial curves $C_1 = C(6, 7, 15)$ and $C_2 = C(1)$. C_1 has non-Cohen-Macaulay tangent cone, having a non-decreasing Hilbert function. Obviously, they satisfy the conditions of the Theorem 3.1, which implies that every local ring associated to the monomial curve $C = C(6q, 7q, 15q, 6q + 7)$ obtained by a nice gluing (which is also a nice extension) with $q \not\equiv 0 \pmod 7$ has a non-decreasing Hilbert function. C_1 is a complete intersection monomial curve, with $I(C_1) = \langle x_1^5 - x_3^2, x_1x_3 - x_2^3\rangle$ having a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $x_2 > x_3 > x_1$ given by $\{x_1^5 - x_3^2, x_1x_3 - x_2^3, x_2^3x_3 - x_1^6, x_2^6 - x_1^7\}$. Hence, C is a complete intersection monomial curve, with $I(C) = \langle x_1^5 - x_3^2, x_1x_3 - x_2^3, y_1^q - x_1^q x_2\rangle$ having a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $y_1 > x_2 > x_3 > x_1$ given by $\{x_1^5 - x_3^2, x_1x_3 - x_2^3, x_2^3x_3 - x_1^6, x_2^6 - x_1^7, y_1^q - x_1^q x_2\}$, which shows that C has non-Cohen-Macaulay tangent cone. Thus, we have obtained Gorenstein local rings $K[[t^{6q}, t^{7q}, t^{15q}, t^{6q+7}]]$ with $q \not\equiv 0 \pmod 7$ having non-Cohen-Macaulay associated graded rings and non-decreasing Hilbert functions. In this way, starting with a complete intersection monomial curve C_1 in affine 3-space having non-Cohen-Macaulay tangent cone, we can construct infinitely many families of 1-dimensional Gorenstein local rings with non-Cohen-Macaulay associated graded rings and non-decreasing Hilbert functions. In this way, we can construct infinitely many families of Gorenstein local rings supporting Rossi's conjecture.

Corollary 3.3. Every local ring with embedding dimension 4 associated to a monomial curve obtained by a nice gluing of

a) $C_1 = C(m_1, m_2)$ with $m_1 < m_2$ and $C_2 = C(n_1, n_2)$ with $n_1 < n_2$

b) $C_1 = C(m_1, m_2, m_3)$ with $m_1 < m_2 < m_3$ and $C_2 = C(1)$

c) $C_1 = C(1)$ and $C_2 = C(n_1, n_2, n_3)$ with $n_1 < n_2 < n_3$, whose tangent cone is Cohen-Macaulay

has a non-decreasing Hilbert function.

Proof. In part a), the result follows both from Theorem 2.6 and Theorem 3.1. In part b), the result follows from Theorem 3.1, since every local ring associated to the monomial curve $C_1 = C(m_1, m_2, m_3)$ has non-decreasing Hilbert function due to a result of Elias [6]. In the same way, in part c), the result is a direct consequence of Theorem 3.1. □

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