On d-divisible graceful α-labelings of $C_{4k} \times P_m$

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Abstract

In [9] the concept of a d-divisible graceful α-labeling has been introduced as a generalization of classical α-labelings and it has been shown how it is useful to obtain certain cyclic graph decompositions. In the present paper it is proved the existence of d-divisible graceful α-labelings of $C_{4k} \times P_m$ for any integers $k \geq 1$, $m \geq 2$ for several values of d.

Keywords: graceful labeling; α-labeling; graph decomposition.
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1 Introduction

We assume familiarity with the basic concepts about graphs.
As usual, we denote by K_v and $K_{m\times n}$ the *complete graph on v vertices* and the *complete m-partite graph with parts of size n*, respectively. Also, let C_k, $k \geq 3$, be the cycle on k vertices and let P_m, $m \geq 2$, be the path on m vertices. Graphs of the form $C_k \times P_m$ can be viewed as grids on cylinders and they are bipartite if and only if k is even. If $m = 2$, $C_k \times P_2$ is nothing but the prism T_{2k} on $2k$ vertices. For any graph Γ we write $V(\Gamma)$ for the set of its vertices and $E(\Gamma)$ for the set of its edges. If $|E(\Gamma)| = e$, we say that Γ has *size e*.

Given a subgraph Γ of a graph K, a Γ-*decomposition of* K is a set of graphs, called *blocks*, isomorphic to Γ whose edges partition the edge-set of K. Such a decomposition is said to be *cyclic* when it is invariant under a cyclic permutation of all vertices of K. In the case that $K = K_v$ one also speaks of a Γ-*system of order v*. The problem of establishing the set of values of v for which such a system exists is in general quite difficult. For a survey on graph decompositions see [2].

The concept of a *graceful labeling* of Γ, introduced by A. Rosa [10], is quite related to the existence problem of cyclic Γ-systems. A *graceful labeling*

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of a graph Γ of size e is an injective function $f : V(\Gamma) \to \{0,1,2,\dots,e\}$ such that

$$\{|f(x) - f(y)| \mid [x,y] \in E(\Gamma)\} = \{1,2,\dots,e\}.$$

In the case that Γ is bipartite and f has the additional property that its maximum value on one of the two bipartite sets does not reach its minimum on the other one, one says that f is an α-labeling. In [10], Rosa proved that if a graph Γ of size e admits a graceful labeling then there exists a cyclic Γ-system of order $2e+1$ and that if it admits an α-labeling then there exists a cyclic Γ-system of order $2en+1$ for any positive integer n. For a very rich survey on graceful labelings we refer to [5].

Many variations of graceful labelings have been considered. In particular Gnana Jothi [6] defines an *odd graceful labeling* of a graph Γ of size e as an injective function $f : V(\Gamma) \to \{0,1,2,\dots,2e-1\}$ such that

$$\{|f(x) - f(y)| \mid [x,y] \in E(\Gamma)\} = \{1,3,5,\dots,2e-1\}.$$

In a recent paper, see [9], we have introduced the following new definition which is, at the same time, a generalization of the concepts of a graceful labeling (when $d = 1$) and of an odd graceful labeling (when $d = e$).

Definition 1.1. *Let Γ be a graph of size $e = d \cdot m$. A d-*divisible graceful labeling *of Γ is an injective function $f : V(\Gamma) \to \{0,1,2,\dots,d(m+1)-1\}$ such that*

$$\{|f(x) - f(y)| \mid [x,y] \in E(\Gamma)\} = \{1,2,3,\dots,d(m+1)\} - \{m+1, 2(m+1),\dots,d(m+1)\}.$$

Namely the set $\{|f(x) - f(y)| \mid [x,y] \in E(\Gamma)\}$ can be divided into d parts $P^0, P^1, \dots, P^{d-1}$ where $P^i := \{(m+1)i+1, (m+1)i+2, \dots, (m+1)i+m\}$ for any $i = 0,1,\dots,d-1$.

The α-labelings can be generalized in a similar way.

Definition 1.2. *A d-*divisible graceful α-labeling *of a bipartite graph Γ is a d-divisible graceful labeling of Γ having the property that its maximum value on one of the two bipartite sets does not reach its minimum value on the other one.*

We have to point out that in [9] the above labelings have been called "d-graceful (α-)labelings", but the author was unaware that this name is already used in the literature whit a different meaning, see [8] and [11].

It is known that there is a close relationship between graceful labelings and difference families, see [1]. In [9] we established relations between d-divisible graceful (α-)labelings and a generalization of difference families introduced in [3], proving the following theorems.

Theorem 1.3. *If there exists a d-divisible graceful labeling of a graph Γ of size e then there exists a cyclic Γ-decomposition of $K_{(\frac{e}{d}+1)\times 2d}$.*

Theorem 1.4. *If there exists a d-divisible graceful α-labeling of a graph Γ of size e then there exists a cyclic Γ-decomposition of $K_{(\frac{e}{d}+1)\times 2dn}$ for any integer $n \geq 1$.*

In this paper we determine the existence of d-divisible graceful α-labelings of $C_{4k} \times P_m$ for several values of d. In order to obtain these results, first of all we will find d-divisible graceful α-labelings of prisms, which correspond to the case $m = 2$, and then by induction on m we will be able to construct d-divisible graceful α-labelings of $C_{4k} \times P_m$ for any $m \geq 2$. For what said above, these results allow us to obtain new infinite classes of cyclic decompositions of the complete multipartite graph in copies of $C_{4k} \times P_m$.

2 d-divisible graceful α-labelings of prisms

In this section we will investigate the existence of d-divisible graceful α-labelings of prisms. From now on, given two integers a and b, by $[a, b]$ we will denote the set of integers x such that $a \leq x \leq b$.
For convenience, we denote the $2k$ vertices of T_{2k} by $x_1, x_2, \ldots, x_k$; $y_1, y_2, \ldots,$ y_k where the x_i's are the consecutive vertices of one k-cycle and the y_i's are consecutive vertices of the other k-cycle and x_i is connected to y_i. Clearly T_{2k} has size $e = 3k$ and it is bipartite if and only if k is even. In [4] Frucht and Gallian proved that T_{2k} admits an α-labeling if and only if k is even.

Theorem 2.1. *The prism T_{8k} admits a 3-divisible graceful α-labeling for every $k \geq 1$.*

Proof. We set $\mathcal{O}_x = \{x_1, x_3, \ldots, x_{4k-1}\}$, $\mathcal{E}_x = \{x_2, x_4, \ldots, x_{4k}\}$, $\mathcal{O}_y = \{y_1, y_3, \ldots, y_{4k-1}\}$, $\mathcal{E}_y = \{y_2, y_4, \ldots, y_{4k}\}$. Clearly $\mathcal{O}_x \cup \mathcal{E}_y$ and $\mathcal{O}_y \cup \mathcal{E}_x$ are the two bipartite sets of $V(T_{8k})$.
Consider the map $f : V(T_{8k}) \to \{0, 1, \ldots, 12k + 2\}$ defined as follows:

$$f(x_{2i+1}) = \begin{cases} 6k+1 & \text{for } i = 0 \\ 8k+2-i & \text{for } i \in [1,k] \\ 8k+1-i & \text{for } i \in [k+1, 2k-1] \end{cases}$$

$$f(x_{2i}) = 4k + i \qquad \text{for } i \in [1, 2k].$$

$$f(y_{2i+1}) = i \qquad \text{for } i \in [0, 2k-1]$$

$$f(y_{2i}) = \begin{cases} 12k+3-i & \text{for } i \in [1,k] \\ 12k+2-i & \text{for } i \in [k+1, 2k]. \end{cases}$$

We have

$$f(\mathcal{O}_y \cup \mathcal{E}_x) = [0, 2k-1] \cup [4k+1, 6k]$$
$$f(\mathcal{O}_x \cup \mathcal{E}_y) = [6k+1, 7k] \cup [7k+2, 8k+1] \cup [10k+2, 11k+1] \cup \\ \cup [11k+3, 12k+2].$$

Hence f is injective and $\max f(\mathcal{O}_y \cup \mathcal{E}_x) < \min f(\mathcal{O}_x \cup \mathcal{E}_y)$. Now for $i = 1, \ldots, 4k$ set

$$\sigma_i = |f(x_{i+1}) - f(x_i)|, \quad \varepsilon_i = |f(y_{i+1}) - f(y_i)|, \quad \rho_i = |f(x_i) - f(y_i)| \tag{1}$$

where the indices are understood modulo $4k$. By a direct calculation, one can see that

$$\begin{aligned}
\sigma_1 &= 2k, \\
\{\sigma_i \mid i = 2, \ldots, 2k+1\} &= [2k+1, 4k] \\
\{\sigma_i \mid i = 2k+2, \ldots, 4k\} &= [1, 2k-1] \\
\rho_1 &= 6k+1 \\
\{\rho_i \mid i = 2, \ldots, 2k+1\} &= [6k+2, 8k+1], \\
\{\rho_i \mid i = 2k+2, \ldots, 4k\} &= [4k+2, 6k], \\
\{\varepsilon_i \mid i = 1, \ldots, 2k\} &= [10k+3, 12k+2], \\
\{\varepsilon_i \mid i = 2k+1, \ldots, 4k-1\} &= [8k+3, 10k+1], \\
\varepsilon_{4k} &= 10k+2.
\end{aligned}$$

Hence $\{\sigma_i \mid i = 1, \ldots, 4k\} = [1, 4k]$, $\{\rho_i \mid i = 1, \ldots, 4k\} = [4k+2, 8k+1]$ and $\{\varepsilon_i \mid i = 1, \ldots, 4k\} = [8k+3, 12k+2]$. This concludes the proof. □

Theorem 2.2. *The prism T_{8k} admits a 6-divisible graceful α-labeling for every $k \geq 1$.*

Proof. Set $\mathcal{O}_x, \mathcal{E}_x, \mathcal{O}_y, \mathcal{E}_y$ as in the proof of previous theorem. Consider the map $f : V(T_{8k}) \to \{0, 1, \ldots, 12k+5\}$ defined as follows:

$$f(x_{2i+1}) = \begin{cases} 6k+2 & \text{for } i = 0 \\ 8k+4-i & \text{for } i \in [1, k] \\ 8k+2-i & \text{for } i \in [k+1, 2k-1] \end{cases}$$
$$f(x_{2i}) = 4k+1+i \qquad \text{for } i \in [1, 2k].$$
$$f(y_{2i+1}) = i \qquad \text{for } i \in [0, 2k-1]$$
$$f(y_{2i}) = \begin{cases} 12k+6-i & \text{for } i \in [1, k] \\ 12k+4-i & \text{for } i \in [k+1, 2k]. \end{cases}$$

It results

$$\begin{aligned} f(\mathcal{O}_y \cup \mathcal{E}_x) &= [0, 2k-1] \cup [4k+2, 6k+1] \\ f(\mathcal{O}_x \cup \mathcal{E}_y) &= [6k+2, 7k+1] \cup [7k+4, 8k+3] \cup [10k+4, 11k+3] \cup \\ &\cup [11k+6, 12k+5]. \end{aligned}$$

Hence f is injective and $\max f(\mathcal{O}_y \cup \mathcal{E}_x) < \min f(\mathcal{O}_x \cup \mathcal{E}_y)$. Let $\varepsilon_i, \rho_i, \sigma_i$, for $i = 1, \ldots, 4k$, be as in (1). It is not hard to see that

$$\begin{aligned} \{\sigma_i \mid i = 1, \ldots, 4k\} &= [1, 2k] \cup [2k+2, 4k+1] \\ \{\rho_i \mid i = 1, \ldots, 4k\} &= [4k+3, 6k+2] \cup [6k+4, 8k+3] \\ \{\varepsilon_i \mid i = 1, \ldots, 4k\} &= [8k+5, 10k+4] \cup [10k+6, 12k+5]. \end{aligned}$$

Hence f is a 6-divisible graceful α-labeling of T_{8k}. □

Theorem 2.3. *The prism T_{8k} admits a 12-divisible graceful α-labeling for every $k \geq 1$.*

Proof. Also here we set $\mathcal{O}_x, \mathcal{E}_x, \mathcal{O}_y, \mathcal{E}_y$ as in the proof of Theorem 2.1. We are able to prove the existence of a 12-divisible graceful α-labeling of T_{8k} by means of two direct constructions where we distinguish the two cases: k even and k odd.

Case 1: k even.
Consider the map $f : V(T_{8k}) \to \{0, 1, \ldots, 12k+11\}$ defined as follows:

$$f(x_{2i+1}) = \begin{cases} 6k+5 & \text{for } i = 0 \\ 8k+8-i & \text{for } i \in \left[1, \frac{k}{2}\right] \\ 8k+7-i & \text{for } i \in \left[\frac{k}{2}+1, k\right] \\ 8k+5-i & \text{for } i \in [k+1, 2k-1] \end{cases}$$

$$f(x_{2i}) = \begin{cases} 4k+3+i & \text{for } i \in \left[1, \frac{3k}{2}\right] \\ 4k+4+i & \text{for } i \in \left[\frac{3k}{2}+1, 2k\right]. \end{cases}$$

$$f(y_{2i+1}) = \begin{cases} i & \text{for } i \in \left[0, \frac{3k}{2}-1\right] \\ i+1 & \text{for } i \in \left[\frac{3k}{2}, 2k-1\right] \end{cases}$$

$$f(y_{2i}) = \begin{cases} 12k+12-i & \text{for } i \in \left[1, \frac{k}{2}\right] \\ 12k+11-i & \text{for } i \in \left[\frac{k}{2}+1, k\right] \\ 12k+9-i & \text{for } i \in [k+1, 2k] \end{cases}$$

It is easy to see that

$$
\begin{aligned}
f(\mathcal{O}_y) &= \left[0, \frac{3k}{2} - 1\right] \cup \left[\frac{3k}{2} + 1, 2k\right] \\
f(\mathcal{E}_x) &= \left[4k + 4, \frac{11k}{2} + 3\right] \cup \left[\frac{11k}{2} + 5, 6k + 4\right] \\
f(\mathcal{O}_x) &= [6k + 5, 7k + 4] \cup \left[7k + 7, \frac{15k}{2} + 6\right] \cup \left[\frac{15k}{2} + 8, 8k + 7\right] \\
f(\mathcal{E}_y) &= [10k + 9, 11k + 8] \cup \left[11k + 11, \frac{23k}{2} + 10\right] \cup \left[\frac{23k}{2} + 12, 12k + 11\right].
\end{aligned}
$$

Hence f is injective and $\max f(\mathcal{O}_y \cup \mathcal{E}_x) = 6k + 4 < 6k + 5 = \min f(\mathcal{O}_x \cup \mathcal{E}_y)$. Set $\sigma_i, \varepsilon_i, \rho_i$, for $i = 1, \dots, 4k$, as in (1). By a long and tedious calculation, one can see that

$$
\begin{aligned}
\{\sigma_i \mid i = 1, \dots, 4k\} &= [1, 4k + 3] - \{k + 1, 2k + 2, 3k + 3\} \\
\{\rho_i \mid i = 1, \dots, 4k\} &= [4k + 5, 8k + 7] - \{5k + 5, 6k + 6, 7k + 7\} \\
\{\varepsilon_i \mid i = 1, \dots, 4k\} &= [8k + 9, 12k + 11] - \{9k + 9, 10k + 10, 11k + 11\}.
\end{aligned}
$$

This concludes the proof of Case 1.

Case 2: k odd.
Let now $f : V(T_{8k}) \to \{0, 1, \dots, 12k + 11\}$ defined as follows:

$$
f(x_{2i+1}) = \begin{cases} 6k + 5 & \text{for } i = 0 \\ 8k + 8 - i & \text{for } i \in [1, k] \\ 8k + 6 - i & \text{for } i \in \left[k + 1, \frac{3k-1}{2}\right] \\ 8k + 5 - i & \text{for } i \in \left[\frac{3k+1}{2}, 2k - 1\right] \end{cases}
$$

$$
f(x_{2i}) = \begin{cases} 4k + 3 + i & \text{for } i \in \left[1, \frac{k+1}{2}\right] \\ 4k + 4 + i & \text{for } i \in \left[\frac{k+3}{2}, 2k\right]. \end{cases}
$$

$$
f(y_{2i+1}) = \begin{cases} i & \text{for } i \in \left[0, \frac{k-1}{2}\right] \\ i + 1 & \text{for } i \in \left[\frac{k+1}{2}, 2k - 1\right] \end{cases}
$$

$$
f(y_{2i}) = \begin{cases} 12k + 12 - i & \text{for } i \in [1, k] \\ 12k + 10 - i & \text{for } i \in \left[k + 1, \frac{3k-1}{2}\right] \\ 12k + 9 - i & \text{for } i \in \left[\frac{3k+1}{2}, 2k\right] \end{cases}
$$

Arguing exactly as in Case 1, one can check that f is a 12-divisible graceful α-labeling of T_{8k}. □

Example 2.4. *The three graphs in Figure 1 show the 3-divisible graceful α-labeling, the 6-divisible graceful α-labeling and the 12-divisible graceful α-labeling of T_{24} provided by previous theorems.*

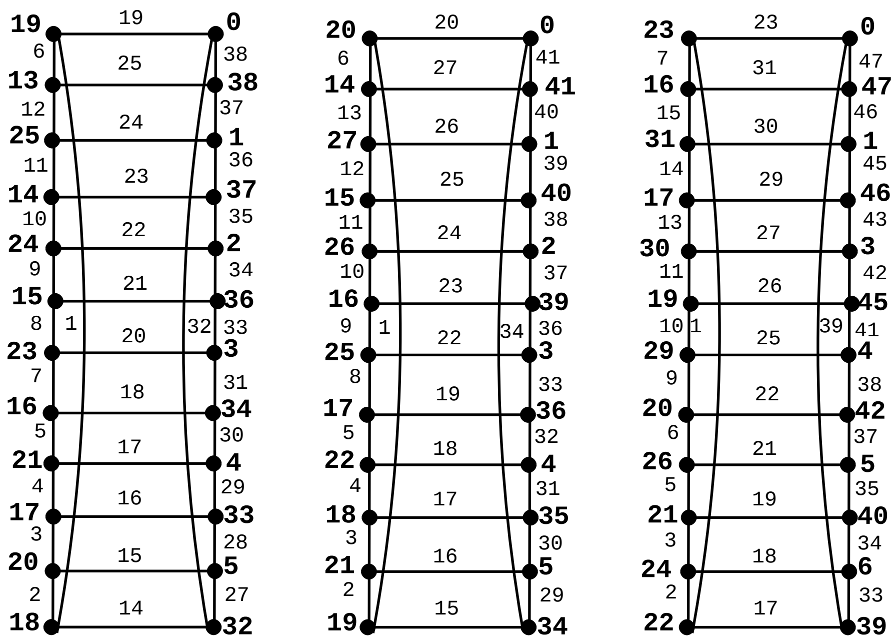

Figure 1: T_{24}

3 d-divisible graceful α-labelings of $C_{4k} \times P_m$

In this section using the results of the previous one we will construct d-divisible graceful α-labelings of $C_{4k} \times P_m$. In particular, since $e = 4k(2m-1)$ we consider $d = 2m-1,\ 2(2m-1),\ 4(2m-1)$. In [7] Jungreis and Reid proved that for any $k, m \geq 2$ not both odd there exists an α-labeling of $C_{2k} \times P_m$. For convenience, we denote the vertices of $C_{4k} \times P_m$ as illustrated in Figure 2 and we set $C^i = ((i,1),(i,2),\dots,(i,4k))$ for any $i = 1,\dots,m$.

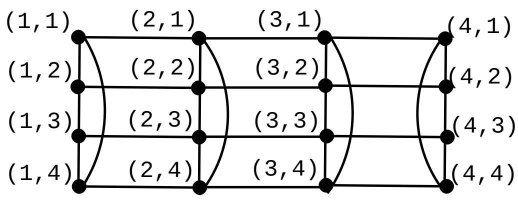

Figure 2: $C_4 \times P_4$

Theorem 3.1. *For any integer $k \geq 1$ and $m \geq 2$, $C_{4k} \times P_m$ admits a $(2m-1)$-divisible graceful α-labeling.*

Proof. We will prove the result by induction on m. If $m = 2$ the thesis follows from Theorem 2.1. Let now $m \geq 2$. Suppose that there exists a

$(2m-1)$-divisible graceful α-labeling f of $C_{4k} \times P_m$ with vertices of C^m so labeled:

$$f(C^m) = (0, (4k+1)(2m-1)-1, 1, (4k+1)(2m-1)-2, 2, \dots,$$
$$(4k+1)(2m-1)-k, k, (4k+1)(2m-1)-(k+2), k+1, \dots,$$
$$2k-1, (4k+1)(2m-1)-(2k+1)).$$

Note that the 3-divisible graceful α-labeling of $C_{4k} \times P_2$ constructed in Theorem 2.1 has this property, in fact $f(C^2) = (0, 12k+2, 1, 12k+1, 2, \dots, k-1, 11k+3, k, 11k+1, k+1, \dots, 2k-1, 10k+2)$. So in order to obtain the thesis it is sufficient to construct a $(2m+1)$-divisible graceful α-labeling g of $C_{4k} \times P_{m+1}$ satisfying the same property, namely such that

$$\begin{aligned} g(C^{m+1}) &= (0, (4k+1)(2m+1)-1, 1, (4k+1)(2m+1)-2, 2, \dots, \\ &\quad (4k+1)(2m+1)-k, k, (4k+1)(2m+1)-(k+2), k+1, \dots, \\ &\quad 2k-1, (4k+1)(2m+1)-(2k+1)). \end{aligned} \tag{2}$$

We set

$$g((i,j)) = f((i,j)) + (4k+1) \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, 4k.$$

By the hypothesis on $f(C^m)$ it results

$$g(C^m) = (4k+1, (4k+1)2m-1, 4k+2, (4k+1)2m-2, 4k+3, \dots,$$
$$(4k+1)2m-k, 5k+1, (4k+1)2m-(k+2), 5k+2, \dots,$$
$$6k, (4k+1)2m-(2k+1)).$$

So there exists $j \in [1, 4n]$ such that $g((m,j)) = (4k+1)2m-1$. We set $g(C^{m+1})$ as in (2) where $g((m+1,j)) = 0$.
Now we will see that $g : V(C_{4k} \times P_{m+1}) \to \{0, \dots, (4k+1)(2m+1)-1\}$ defined as above is indeed a $(2m+1)$-divisible graceful α-labeling of $C_{4k} \times P_{m+1}$. Since $f(V(C_{4k} \times P_m)) \subseteq [0, (4k+1)(2m-1)-1]$, by the definition of g, it follows that

$$g(V(C^1 \cup C^2 \cup \ldots \cup C^m)) \subseteq [4k+1, (4k+1)2m-1].$$

Also we have

$$g(V(C^{m+1})) \subseteq [0, 2k-1] \cup [(4k+1)(2m+1)-(2k+1), (4k+1)(2m+1)-1].$$

hence g is an injective function. Since, by hypothesis f is a $(2m-1)$-divisible graceful α-labeling of $C_{4k} \times P_m$, we have $V(C_{4k} \times P_m) = A \cup B$ with $\max_A f < \min_B f$. Let $V(C_{4k} \times P_{m+1}) = C \cup D$. By the construction, it follows that

$$g(C) = (f(A) + (4k+1)) \cup [0, 2k-1]$$
$$g(D) \subseteq (f(B) + (4k+1)) \cup [(4k+1)(2m+1)-(2k+1), (4k+1)(2m+1)-1]$$

hence $\max_C g < \min_D g$. Now we have to consider the differences between adjacent vertices. Since f is a $(2m-1)$-divisible graceful α-labeling of $C_{4k} \times P_m$, by the construction of g, it results

$$\bigcup_{\substack{i \in [1,m-1] \\ j \in [1,4k]}} |f((i,j)) - f((i+1,j))| \cup \bigcup_{\substack{i \in [1,m] \\ j \in [1,4k]}} |f((i,j)) - f((i,j+1))| =$$

$$[1, (4k+1)(2m-1)] - \{\beta(4k+1) \mid \beta \in [1, 2m-1]\}$$

where the index j is taken modulo $4k$. Finally it is not hard to check that

$$\{|g((m,j)) - g((m+1,j))| \mid j \in [1,4k]\} =$$
$$[(4k+1)2m - 4k, (4k+1)2m - 1]$$

and

$$\{|g((m+1,j)) - g((m+1,j+1))| \mid j \in [1,4k]\} =$$
$$[(4k+1)(2m+1) - 4k, (4k+1)(2m+1) - 1].$$

This concludes the proof. □

Example 3.2. *In Figure 3 we will show the* 5*-divisible graceful* α*-labeling of* $C_4 \times P_3$*, the* 7*-divisible graceful* α*-labeling of* $C_4 \times P_4$ *and the* 9*-divisible graceful* α*-labeling of* $C_4 \times P_5$ *obtained starting from the* 3*-divisible graceful* α*-labeling of* $T_8 = C_4 \times P_2$ *and following the construction illustrated in the proof of Theorem 3.1.*

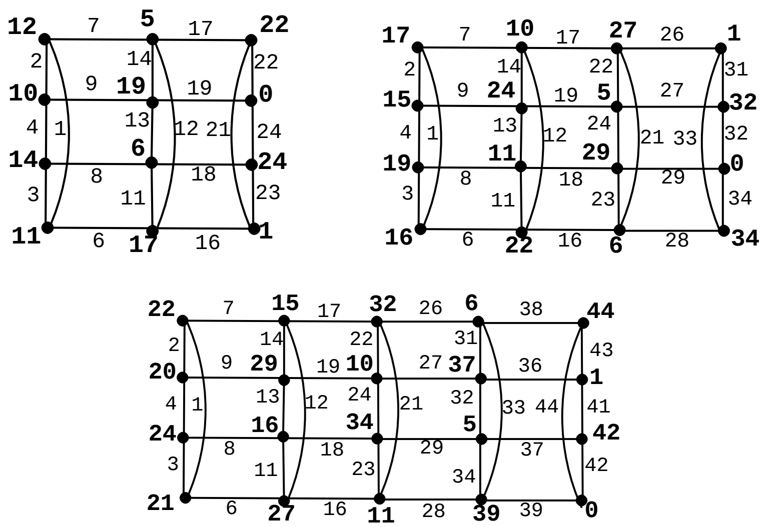

Figure 3:

Theorem 3.3. *For any integer $k \geq 1$ and $m \geq 2$, $C_{4k} \times P_m$ admits a $2(2m-1)$-divisible graceful α-labeling.*

Proof. We will prove the result by induction on m. If $m = 2$ the thesis follows from Theorem 2.2. Let now $m \geq 2$. Suppose that there exists a $2(2m-1)$-divisible graceful α-labeling f of $C_{4k} \times P_m$ with vertices of C^m so labeled:

$$\begin{aligned} f(C^m) =& (0, (4k+2)(2m-1)-1, 1, (4k+2)(2m-1)-2, 2, \dots, \\ & (4k+2)(2m-1)-k, k, (4k+2)(2m-1)-(k+3), k+1, \dots, \\ & 2k-1, (4k+2)(2m-1)-(2k+2)). \end{aligned}$$

We want to show the existence of a $2(2m+1)$-divisible graceful α-labeling g of $C_{4k} \times P_{m+1}$ satisfying the same property, namely such that

$$\begin{aligned} g(C^{m+1}) \quad = \quad & (0, (4k+2)(2m+1)-1, 1, (4k+2)(2m+1)-2, 2, \dots, \\ & (4k+2)(2m+1)-k, k, (4k+2)(2m+1)-(k+3), k+1, \dots, \\ & 2k-1, (4k+2)(2m+1)-(2k+2)). \end{aligned} \tag{3}$$

First of all set

$$g((i,j)) = f((i,j)) + (4k+2) \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, 4k.$$

This implies that there exists $j \in [1, 4n]$ such that $g((m,j)) = (4k+2)2m-1$. We set $g(C^{m+1})$ as in (3) where $g((m+1,j)) = 0$. Arguing exactly as in the previous proof one can prove that g is a $2(2m+1)$-divisible graceful α-labeling g of $C_{4k} \times P_{m+1}$. □

Theorem 3.4. *For any integer $k \geq 1$ and $m \geq 2$, $C_{4k} \times P_m$ admits a $4(2m-1)$-divisible graceful α-labeling.*

Proof. We will prove the result by induction on m. If $m = 2$ the thesis follows from Theorem 2.3. Let now $m \geq 2$. We have to distinguish two cases: k even and k odd.
Let k be even. Suppose that there exists a $4(2m-1)$-divisible graceful α-labeling f of $C_{4k} \times P_m$ with vertices of C^m so labeled:

$$\begin{aligned} f(C^m) =& (0, (4k+4)(2m-1)-1, 1, (4k+4)(2m-1)-2, 2, \dots, \\ & (4k+4)(2m-1)-\frac{k}{2}, \frac{k}{2}, (4k+4)(2m-1)-\left(\frac{k}{2}+2\right), \frac{k}{2}+1, \dots, \\ & (4k+4)(2m-1)-(k+1), k, (4k+4)(2m-1)-(k+4), k+1, \dots, \\ & (4k+4)(2m-1)-\left(\frac{3}{2}k+2\right), \frac{3}{2}k-1, (4k+4)(2m-1)-\left(\frac{3}{2}k+3\right), \\ & \frac{3}{2}k+1, (4k+4)(2m-1)-\left(\frac{3}{2}k+4\right), \dots, \\ & (4k+4)(2m-1)-(2k+2), 2k, (4k+4)(2m-1)-(2k+3)). \end{aligned}$$

We want to show the existence of a $4(2m+1)$-divisible graceful α-labeling g of $C_{4k} \times P_{m+1}$ satisfying the same property, namely such that

$$\begin{aligned} g(C^{m+1}) = {} & (0, (4k+4)(2m+1)-1, 1, (4k+4)(2m+1)-2, 2, \dots, \\ & (4k+4)(2m+1) - \frac{k}{2}, \frac{k}{2}, (4k+4)(2m+1) - \left(\frac{k}{2}+2\right), \frac{k}{2}+1, \dots, \\ & (4k+4)(2m+1) - (k+1), k, (4k+4)(2m+1) - (k+4), k+1, \dots, \\ & (4k+4)(2m+1) - \left(\frac{3}{2}k+2\right), \frac{3}{2}k-1, (4k+4)(2m+1) - \left(\frac{3}{2}k+3\right), \\ & \frac{3}{2}k+1, (4k+4)(2m+1) - \left(\frac{3}{2}k+4\right), \dots, \\ & (4k+4)(2m+1) - (2k+2), 2k, (4k+4)(2m+1) - (2k+3)). \quad (4) \end{aligned}$$

First of all set

$$g((i,j)) = f((i,j)) + (4k+4) \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, 4k.$$

This implies that there exists $j \in [1, 4n]$ such that $g((m,j)) = (4k+4)2m-1$. We set $g(C^{m+1})$ as in (4) where $g((m+1,j)) = 0$.
Let now k be odd. Suppose that there exists a $4(2m-1)$-divisible graceful α-labeling f of $C_{4k} \times P_m$ with vertices of C^m so labeled:

$$\begin{aligned} f(C^m) = {} & (0, (4k+4)(2m-1)-1, 1, (4k+4)(2m-1)-2, 2, \dots, \\ & (4k+4)(2m-1) - \frac{k-1}{2}, \frac{k-1}{2}, (4k+4)(2m-1) - \frac{k+1}{2}, \frac{k+3}{2}, \\ & \dots, (4k+4)(2m-1) - k, k+1, (4k+4)(2m-1) - (k+3), k+2, \dots, \\ & (4k+4)(2m-1) - \frac{3k+3}{2}, \frac{3k+1}{2}, (4k+4)(2m-1) - \frac{3k+7}{2}, \\ & \frac{3k+3}{2}, (4k+4)(2m-1) - \frac{3k+9}{2}, \dots, \\ & (4k+4)(2m-1) - (2k+2), 2k, (4k+4)(2m-1) - (2k+3)). \end{aligned}$$

We want to show the existence of a $4(2m+1)$-divisible graceful α-labeling g of $C_{4k} \times P_{m+1}$ satisfying the same property, namely such that

$$\begin{aligned} g(C^{m+1}) = {} & (0, (4k+4)(2m+1)-1, 1, (4k+4)(2m+1)-2, 2, \dots, \\ & (4k+4)(2m+1) - \frac{k-1}{2}, \frac{k-1}{2}, (4k+4)(2m+1) - \frac{k+1}{2}, \frac{k+3}{2}, \\ & \dots, (4k+4)(2m+1) - k, k+1, (4k+4)(2m+1) - (k+3), k+2, \dots, \\ & (4k+4)(2m+1) - \frac{3k+3}{2}, \frac{3k+1}{2}, (4k+4)(2m+1) - \frac{3k+7}{2}, \\ & \frac{3k+3}{2}, (4k+4)(2m+1) - \frac{3k+9}{2}, \dots, \\ & (4k+4)(2m+1) - (2k+2), 2k, (4k+4)(2m+1) - (2k+3)). \quad (5) \end{aligned}$$

First of all set

$$g((i,j)) = f((i,j)) + (4k+4) \quad \forall i = 1,\dots,m,\ \forall j = 1,\dots,4k.$$

This implies that there exists $j \in [1,4n]$ such that $g((m,j)) = (4k+4)2m-1$. We set $g(C^{m+1})$ as in (5) where $g((m+1,j)) = 0$.
Arguing exactly in the proof of Theorem 3.1 one can prove that, in both cases, g is a $4(2m+1)$-divisible graceful α-labeling g of $C_{4k} \times P_{m+1}$. □

Example 3.5. *In Figure 4 we have the* 10*-divisible graceful* α*-labeling of* $C_{12} \times P_3$ *obtained starting from the* 6*-divisible graceful* α*-labeling of* $T_{24} = C_{12} \times P_2$ *shown in Figure 1 and following the construction explained in the proof of Theorem 3.3 and the* 20*-divisible graceful* α*-labeling of* $C_{12} \times P_3$ *obtained starting from the* 12*-divisible graceful* α*-labeling of* T_{24} *shown in Figure 1 and following the construction illustrated in the proof of Theorem 3.4.*

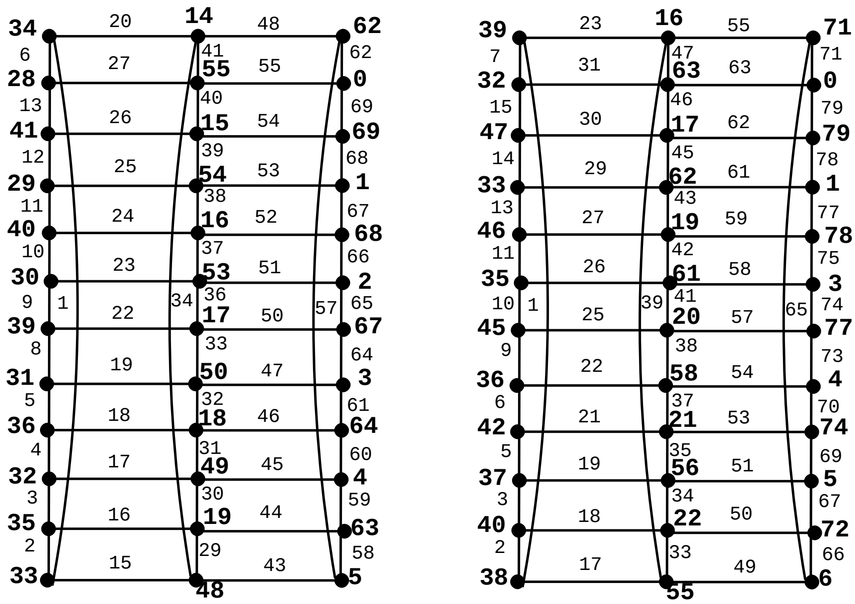

Figure 4: A 10-divisible graceful α-labeling of $C_{12} \times P_3$ and a 20-divisible graceful α-labeling of $C_{12} \times P_3$, respectively.

By virtue of Theorems 1.4, 3.1, 3.3 and 3.4, we have

Proposition 3.6. *There exists a cyclic* $C_{4k} \times P_m$*-decomposition of* $K_{(4k+1)\times 2(2m-1)n}$*, of* $K_{(2k+1)\times 4(2m-1)n}$ *and of* $K_{(k+1)\times 8(2m-1)n}$*, for any integers* $k,n \geq 1$*,* $m > 2$*.*

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