A class of II_1 factors with many non conjugate Cartan subalgebras

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Abstract

We construct a class of II_1 factors that admit a non smooth family of Cartan subalgebras that are non conjugate by an automorphism. We also construct a II_1 factor that admits uncountably many non isomorphic group measure space decompositions, all involving the same group G. So G is a group that admits uncountably many non stably orbit equivalent actions whose crossed product II_1 factors are all isomorphic.

1 Introduction

Free ergodic probability measure preserving (p.m.p.) actions $\Gamma \curvearrowright (X, \mu)$ of countable groups give rise to II_1 factors $L^\infty(X) \rtimes \Gamma$ by Murray and von Neumann's group measure space construction. In [Si55] it was shown that the isomorphism class of $L^\infty(X) \rtimes \Gamma$ only depends on the orbit equivalence relation on (X, μ) given by $x \sim y$ iff $x \in \Gamma \cdot y$. More precisely, two free ergodic p.m.p. actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are orbit equivalent iff there exists an isomorphism of $L^\infty(X) \rtimes \Gamma$ onto $L^\infty(Y) \rtimes \Lambda$ sending $L^\infty(X)$ onto $L^\infty(Y)$.

If $\Gamma \curvearrowright (X, \mu)$ is essentially free and $M := L^\infty(X) \rtimes \Gamma$, then $A := L^\infty(X)$ is a maximal abelian subalgebra of M that is regular: the group of unitaries $u \in \mathcal{U}(M)$ that normalize A in the sense that $uAu^* = A$, spans a dense $*$-subalgebra of M. A regular maximal abelian subalgebra of a II_1 factor is called a *Cartan subalgebra.*

In order to classify classes of group measure space II_1 factors $M := L^\infty(X) \rtimes \Gamma$ in terms of the group action $\Gamma \curvearrowright (X, \mu)$, one must solve two different problems. First one should understand whether the Cartan subalgebra $L^\infty(X) \subset M$ is, in some sense, unique. If this is the case, any other group measure space decomposition $M = L^\infty(Y) \rtimes \Lambda$ must come from an orbit equivalence between $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$. The second problem is then to classify the corresponding class of orbit equivalence relations $\mathcal{R}(\Gamma \curvearrowright X)$ in terms of the group action. Both problems are notoriously hard, but a huge progress has been made over the last 10 years thanks to Sorin Popa's deformation/rigidity theory (see [Po06a, Ga10, Va10a] for surveys). In particular, many uniqueness results for Cartan subalgebras and group measure space Cartan subalgebras have been established recently in [OP07, OP08, Pe09, PV09, FV10, CP10, HPV10, Va10b, Io11, CS11].

In this paper we concentrate on "negative" solutions to the first problem by exhibiting a class of II_1 factors with many Cartan subalgebras. First recall that two Cartan subalgebras $A \subset M$ and $B \subset M$ of a II_1 factor are called *unitarily conjugate* if there exists a unitary $u \in \mathcal{U}(M)$ such that $B = uAu^*$. The Cartan subalgebras are called *conjugate by an automorphism* if there exists $\alpha \in \operatorname{Aut} M$ such that $B = \alpha(A)$. In [CFW81] it is shown that the hyperfinite II_1 factor R has a unique Cartan subalgebra up to conjugacy by an automorphism of R. By [FM75, Theorem 7] (see [Pa83, Section 4] for details) the hyperfinite II_1 factor however admits

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uncountably many Cartan subalgebras that are not unitarily conjugate. In [CJ82] Connes and Jones constructed the first example of a II_1 factor that admits two Cartan subalgebras that are not conjugate by an automorphism. More explicit examples of this phenomenon were found in [OP08, Section 7] and [PV09, Example 5.8].

In [Po06b, Section 6.1] Popa discovered a free ergodic p.m.p. action $\Gamma \curvearrowright (X,\mu)$ such that the orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ has trivial fundamental group, while the II_1 factor $M := \mathrm{L}^\infty(X) \rtimes \Gamma$ has fundamental group $\mathbb{R}_+$. Associated with this is a one parameter family A_t of Cartan subalgebras of M that are not conjugate by an automorphism of M. Indeed, put $A := \mathrm{L}^\infty(X)$ and for every $t \in (0,1]$, choose a projection $p_t \in A$ of trace t and a $*$-isomorphism $\pi_t : p_t M p_t \to M$. Put $A_t := \pi_t(Ap_t)$. Then $A_t \subset M$, $t \in (0,1]$, are Cartan subalgebras that are not conjugate by an automorphism. Note that earlier (see [Po90, Corollary of Theorem 3] and [Po86, Corollary 4.7.2]) Popa found free ergodic p.m.p. actions $\Gamma \curvearrowright (X,\mu)$ such that the equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ has countable fundamental group, while the II_1 factor $\mathrm{L}^\infty(X) \rtimes \Gamma$ has fundamental group $\mathbb{R}_+$. By the same construction, there are uncountably many t such that the Cartan subalgebras A_t are not conjugate by an automorphism. In all these cases, although the Cartan subalgebras $A_t \subset M$ are not conjugate by an automorphism of M, they are by construction conjugate by a stable automorphism of M. More precisely, we say that two Cartan subalgebras $A \subset M$ and $B \subset M$ are *conjugate by a stable automorphism of M* if there exist non zero projections $p \in A$, $q \in B$ and a $*$-isomorphism $\alpha : pMp \to qMq$ satisfying $\alpha(Ap) = Bq$.

In this paper we provide the first class of II_1 factors M that admit uncountably many Cartan subalgebras that are not conjugate by a stable automorphism of M. Actually the equivalence relation "being conjugate by a (stable) automorphism" on the set of Cartan subalgebras of these II_1 factors M is shown to be non smooth. So, the Cartan subalgebras of these II_1 factors M are not concretely classifiable.

The "many" Cartan subalgebras in the II_1 factors M in the previous paragraph are not of group measure space type. In the final section of the paper we give the first example of a II_1 factor that admits uncountably many Cartan subalgebras that are not conjugate by a (stable) automorphism and that all arise from a group measure space construction. They actually all arise as the crossed product with a single group G. This means that we construct a group G that admits uncountably many free ergodic p.m.p. actions $G \curvearrowright (X_i, \mu_i)$ that are non stably orbit equivalent, but such that the II_1 factors $\mathrm{L}^\infty(X_i) \rtimes G$ are all isomorphic. This group G should be opposed to the classes of groups considered in [PV09, FV10, CP10, HPV10, Va10b] for which isomorphism of the group measure space II_1 factors always implies orbit equivalence of the group actions.

Terminology

A *Cartan subalgebra* A of a II_1 factor M is a maximal abelian von Neumann subalgebra whose normalizer

$$\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) \mid uAu^* = A\}$$

generates M as a von Neumann algebra. As mentioned above, we say that two Cartan subalgebras $A \subset M$ and $B \subset M$ are *unitarily conjugate* if there exists a unitary $u \in \mathcal{U}(M)$ such that $uAu^* = B$. We say that they are *conjugate by an automorphism of M* if there exists $\alpha \in \operatorname{Aut} M$ such that $\alpha(A) = B$. Finally A and B are said to by *conjugate by a stable automorphism of M* if there exist non zero projections $p \in A$, $q \in B$ and a $*$-isomorphism $\alpha : pMp \to qMq$ satisfying $\alpha(Ap) = Bq$.

By [FM75] every II_1 equivalence relation $\mathcal{R}$ on (X,μ) gives rise to a II_1 factor $\text{L}\mathcal{R}$ that contains $\text{L}^\infty(X)$ as a Cartan subalgebra. Conversely, if $\text{L}^\infty(X) = A \subset M$ is an arbitrary Cartan subalgebra of a II_1 factor M, we get a natural II_1 equivalence relation $\mathcal{R}$ on (X,μ) and a measurable 2-cocycle on $\mathcal{R}$ with values in $\mathbb{T}$ such that $A \subset M$ is canonically isomorphic with $\text{L}^\infty(X) \subset \text{L}_\Omega\mathcal{R}$.

If $\mathcal{R}_i$ are II_1 equivalence relations on the standard probability spaces (X_i,μ_i), we say that $\mathcal{R}_1$ and $\mathcal{R}_2$ are *orbit equivalent* if there exists a measure space isomorphism $\Delta : X_1 \to X_2$ such that $(x,y) \in \mathcal{R}_1$ iff $(\Delta(x),\Delta(y)) \in \mathcal{R}_2$ almost everywhere. More generally, $\mathcal{R}_1$ and $\mathcal{R}_2$ are called *stably orbit equivalent* if there exist non negligible measurable subsets $\mathcal{U}_i \subset X_i$ such that the restricted equivalence relations $\mathcal{R}_{i|\mathcal{U}_i}$ are orbit equivalent.

Throughout this paper compact groups are supposed to be second countable.

Statement of the main results

Let Γ be a countable group and K a compact abelian group with countable dense subgroup $\Lambda < K$. Consider the II_1 factor

$$M := \text{L}^\infty(K^\Gamma) \rtimes (\Gamma \times \Lambda)$$

where $\Gamma \curvearrowright K^\Gamma$ is the Bernoulli action given by $(g \cdot x)_h = x_{hg}$ and $\Lambda \curvearrowright K^\Gamma$ acts by diagonal translation: $(\lambda \cdot x)_h = \lambda x_h$.

Whenever $K_1 < K$ is a closed subgroup such that $\Lambda_1 := \Lambda \cap K_1$ is dense in K_1, denote

$$\mathcal{C}(K_1) := \text{L}^\infty(K^\Gamma/K_1) \rtimes \Lambda_1 = \text{L}^\infty(K^\Gamma/K_1) \mathbin{\overline{\otimes}} \text{L}\Lambda_1 \; .$$

Note that $\mathcal{C}(K_1)$ is an abelian von Neumann subalgebra of M. In Lemma 6 we prove that $\mathcal{C}(K_1) \subset M$ is always a Cartan subalgebra. As we will see there, the density of $\Lambda \cap K_1$ in K_1 is equivalent with $\mathcal{C}(K_1)$ being maximal abelian in M.

Theorem 1. *Let Γ be any icc property (T) group with $[\Gamma,\Gamma] = \Gamma$, e.g. $\Gamma = \text{SL}(3,\mathbb{Z})$. Denote by $\mathcal{P}$ the set of prime numbers and define $K := \prod_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}$. Consider the dense subgroup $\Lambda := \bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}$. Define as above*

$$M := \text{L}^\infty(K^\Gamma) \rtimes (\Gamma \times \Lambda) \; .$$

Whenever $\mathcal{P}_1 \subset \mathcal{P}$, define the subgroup $K(\mathcal{P}_1) := \prod_{p\in\mathcal{P}_1} \mathbb{Z}/p\mathbb{Z}$. Note that $\Lambda \cap K(\mathcal{P}_1)$ is dense in $K(\mathcal{P}_1)$. Denote by $\mathcal{C}(\mathcal{P}_1)$ the associated Cartan subalgebra of M (denoted by $\mathcal{C}(K(\mathcal{P}_1))$ above). Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P}$. Then the following statements are all equivalent.

- *$\mathcal{C}(\mathcal{P}_1)$ and $\mathcal{C}(\mathcal{P}_2)$ are unitarily conjugate.*
- *$\mathcal{C}(\mathcal{P}_1)$ and $\mathcal{C}(\mathcal{P}_2)$ are conjugate by a (stable) automorphism of M.*
- *The II_1 equivalence relations induced by $\mathcal{C}(\mathcal{P}_1)$ and $\mathcal{C}(\mathcal{P}_2)$ are (stably) orbit equivalent.*
- *The symmetric difference $\mathcal{P}_1 \mathbin{\triangle} \mathcal{P}_2$ is finite.*

In particular, the relation on the set of Cartan subalgebras of M given by conjugacy in any of the above senses, is non smooth. So the Cartan subalgebras of M up to conjugacy in any of the possible senses, are not concretely classifiable.

Note that we do not know whether an arbitrary Cartan subalgebra of M is necessarily conjugate with one of the $\mathcal{C}(\mathcal{P}_1)$, $\mathcal{P}_1 \subset \mathcal{P}$.

Theorem 1 is an immediate consequence of the following more general result that applies to all possible compact abelian groups K with dense countable subgroup Λ.

Theorem 2. *Let Γ be an icc property (T) group with $[\Gamma,\Gamma] = \Gamma$. Let K be a compact abelian group with countable dense subgroup Λ. As above, write $M = \mathrm{L}^\infty(K^\Gamma) \rtimes (\Gamma \times \Lambda)$. Take closed subgroups $K_i < K$, $i = 1, 2$, such that $\Lambda \cap K_i$ is dense in K_i.*

1. *The following statements are equivalent.*
   - *The Cartan subalgebras $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ are unitarily conjugate.*
   - *The subgroups $K_1, K_2 < K$ are commensurate.*
   - $\Lambda K_1 = \Lambda K_2$.
2. *The following statements are equivalent.*
   - *The Cartan subalgebras $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ are conjugate by an automorphism of M.*
   - *The Cartan subalgebras $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ are stably conjugate by an automorphism.*
   - *There exists a continuous automorphism $\delta \in \operatorname{Aut} K$ such that $\delta(\Lambda) = \Lambda$ and $\delta(\Lambda K_1) = \Lambda K_2$.*
3. *The following statements are equivalent.*
   - *The II$_1$ equivalence relations associated with $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ are orbit equivalent.*
   - *The II$_1$ equivalence relations associated with $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ are stably orbit equivalent.*
   - *There exists a continuous automorphism $\delta \in \operatorname{Aut} K$ such that $\delta(\Lambda K_1) = \Lambda K_2$.*

In Theorem 2 the second set of statements is, at least formally, stronger than the third set of statements. Example 3 below shows that this difference really occurs. So in certain cases the Cartan subalgebras $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ give rise to isomorphic equivalence relations, but are nevertheless non conjugate by an automorphism of M. The reason for this is the following : the Cartan inclusions $\mathcal{C}(K_i) \subset M$ come with 2-cocycles Ω_i ; although the associated equivalence relations are isomorphic, this isomorphism does not map Ω_1 onto Ω_2.

We finally observe that if $K_1 < K$ is an infinite subgroup, the Cartan subalgebra $\mathcal{C}(K_1) \subset M$ is never of group measure space type. The induced equivalence relation is generated by a free action of a countable group (see Lemma 6), but the 2-cocycle is non trivial (see Remark 7).

Example 3. Choose elements $\alpha, \beta, \gamma \in \mathbb{T}$ that are rationally independent (i.e. generate a copy of $\mathbb{Z}^3 \subset \mathbb{T}$). Consider $K = \mathbb{T}^3$ and define the countable dense subgroup $\Lambda < K$ generated by $(\alpha, 1, \alpha)$, $(1, \alpha, \beta)$ and $(1, 1, \gamma)$. Put $K_1 := \mathbb{T} \times \{1\} \times \mathbb{T}$ and $K_2 := \{1\} \times \mathbb{T} \times \mathbb{T}$. Then, $\Lambda \cap K_1$ is generated by $(\alpha, 1, \alpha)$, $(1, 1, \gamma)$ and hence dense in K_1. Similarly $\Lambda \cap K_2$ is dense in K_2.

We have $\Lambda K_1 = \mathbb{T} \times \alpha^{\mathbb{Z}} \times \mathbb{T}$ and $\Lambda K_2 = \alpha^{\mathbb{Z}} \times \mathbb{T} \times \mathbb{T}$. Hence the automorphism $\delta(x, y, z) = (y, x, z)$ of $\mathbb{T}^3$ maps ΛK_1 onto ΛK_2. It is however easy to check that there exists no automorphism δ of K such that $\delta(\Lambda K_1) = \Lambda K_2$ and $\delta(\Lambda) = \Lambda$.

So, by Theorem 2, the Cartan subalgebras $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ of M give rise to orbit equivalent relations, but are not conjugate by a (stable) automorphism of M.

In the final section of the paper we give an example of a group G that admits uncountably many non stably orbit equivalent actions that all give rise to the same II_1 factor. The precise statement goes as follows and will be an immediate consequence of the more concrete, but involved, Proposition 11.

Theorem 4. *There exists a group G that admits uncountably many free ergodic p.m.p. actions $G \curvearrowright (X_i, \mu_i)$ that are non stably orbit equivalent but with all the II_1 factors $\mathrm{L}^\infty(X_i) \rtimes G$ being isomorphic with the same II_1 factor M. So this II_1 factor M admits uncountably many group measure space Cartan subalgebras that are non conjugate by a (stable) automorphism of M.*

2 Proof of the main Theorem 2

We first prove the following elementary lemma that establishes the essential freeness of certain actions on compact groups.

Lemma 5. *Let Y be a compact group and $\Lambda < Y$ a countable subgroup. Assume that a countable group Γ acts on Y by continuous group automorphisms $(\alpha_g)_{g\in\Gamma}$ preserving Λ. Assume that for $g \neq e$ the subgroup $\{y \in Y \mid \alpha_g(y) = y\}$ has infinite index in Y. Equip Y with its Haar measure.*

Then the action of the semidirect product $\Lambda \rtimes \Gamma$ on Y given by $(\lambda, g) \cdot y = \lambda\alpha_g(y)$ is essentially free.

Proof. Take $(\lambda, g) \neq e$ and consider the set of y with $(\lambda, g) \cdot y = y$. If this set is non empty, it is a coset of $\{y \in Y \mid \alpha_g(y) = y\}$. If $g \neq e$, the latter is a closed subgroup of Y with infinite index and hence has measure zero. So we may assume that $g = e$ and $\lambda \neq e$, which is the trivial case. □

Take a countable group Γ and a compact abelian group K with dense countable subgroup $\Lambda < K$. Put $X := K^\Gamma$ and consider the action $\Gamma \times \Lambda \curvearrowright X$ given by $((g, s) \cdot x)_h = s x_{hg}$ for all $g, h \in \Gamma$, $s \in \Lambda$, $x \in X$. Write $M = \mathrm{L}^\infty(X) \rtimes (\Gamma \times \Lambda)$.

Whenever $K_1 < K$ is a closed subgroup of K, we put $\Lambda_1 := \Lambda \cap K_1$ and denote

$$\mathcal{C}(K_1) = \mathrm{L}^\infty(X/K_1) \mathbin{\overline{\otimes}} \mathrm{L}\Lambda_1 \subset M \; .$$

Note that we can identify $\mathcal{C}(K_1) = \mathrm{L}^\infty(X/K_1 \times \widehat{\Lambda_1})$.

Lemma 6. *With the above notations, $\mathcal{C}(K_1)' \cap M = \mathcal{C}(\overline{\Lambda_1})$. So $\mathcal{C}(K_1) \subset M$ is maximal abelian iff $\Lambda \cap K_1$ is dense in K_1.*

In that case, $\mathcal{C}(K_1)$ is a Cartan subalgebra of M and the induced equivalence relation on $X/K_1 \times \widehat{\Lambda_1}$ is given by the orbits of the product action of $(\Gamma \times \Lambda K_1/K_1) \times \widehat{K_1}$ on $X/K_1 \times \widehat{\Lambda_1}$.

Observe that if $K_1 < K$ is an arbitrary closed subgroup and if we put $K_2 = \overline{\Lambda \cap K_1}$, then we have that $\Lambda \cap K_2$ is dense in K_2.

Proof. By writing the Fourier decomposition of an element in $M = \mathrm{L}^\infty(X) \rtimes (\Gamma \times \Lambda)$, one easily checks that

$$(\mathrm{L}\Lambda_1)' \cap M = \mathrm{L}^\infty(X/\overline{\Lambda_1}) \rtimes (\Gamma \times \Lambda) \; . \tag{1}$$

View $X = K^\Gamma$ as a compact group and view K_1 as a closed subgroup of X sitting in X diagonally. Put $Y = X/K_1$ and view Λ/Λ_1 as a subgroup of Y sitting in Y diagonally. By Lemma 5 the action $\Gamma \times \Lambda/\Lambda_1 \curvearrowright Y$ is essentially free. From this we conclude that

$$\mathrm{L}^\infty(X/K_1)' \cap \mathrm{L}^\infty(X) \rtimes (\Gamma \times \Lambda) = \mathrm{L}^\infty(X) \rtimes \Lambda_1 \; .$$

In combination with (1) we get that $\mathcal{C}(K_1)' \cap M = \mathcal{C}(\overline{\Lambda_1})$. This proves the first part of the lemma.

Assume now that $\Lambda \cap K_1$ is dense in K_1. We know that $\mathcal{C}(K_1) \subset M$ is maximal abelian. For every $\omega \in \widehat{K}$ we define the unitary $U_\omega \in \mathrm{L}^\infty(X)$ given by $U_\omega(x) = \omega(x_e)$. One checks that $\mathcal{C}(K_1)$ is normalized by all unitaries in $\mathcal{C}(K_1)$, $\{U_\omega \mid \omega \in \widehat{K}\}$, $\{u_g \mid g \in \Gamma\}$ and $\{u_s \mid s \in \Lambda\}$. These unitaries generate M so that $\mathcal{C}(K_1)$ is indeed a Cartan subalgebra of M.

It remains to analyze which automorphisms of $\mathcal{C}(K_1) = \mathrm{L}^\infty(X/K_1 \times \widehat{\Lambda_1})$ are induced by the above normalizing unitaries. The automorphism $\operatorname{Ad} U_\omega$ only acts on $\widehat{\Lambda_1}$ by first restricting ω to a character on Λ_1 and then translating by this character on $\widehat{\Lambda_1}$. The automorphism $\operatorname{Ad} u_g$, $g \in \Gamma$, only acts on X/K_1 by the quotient of the Bernoulli shift. Finally the automorphism $\operatorname{Ad} u_s$, $s \in \Lambda$, also acts only on X/K_1 by first projecting s onto $\Lambda/\Lambda_1 = \Lambda K_1/K_1$ and then diagonally translating with this element. The resulting orbit equivalence relation indeed corresponds to the direct product of $\mathcal{R}(\Gamma \times \Lambda K_1/K_1 \curvearrowright X/K_1)$ and $\mathcal{R}(\widehat{K_1} \curvearrowright \widehat{\Lambda_1})$. □

Remark 7. Lemma 6 says that the equivalence relation induced by $\mathcal{C}(K_1) \subset M$ is the direct product of the orbit relation $\mathcal{R}((\Gamma \times \Lambda K_1/K_1) \curvearrowright X/K_1)$ and the hyperfinite II_1 equivalence relation. Nevertheless, if Γ is an icc property (T) group, the II_1 factor M is not McDuff, i.e. cannot be written as a tensor product of some II_1 factor with the hyperfinite II_1 factor. Indeed, by property (T), all central sequences in M lie asymptotically in the relative commutant $(\mathrm{L}\Gamma)' \cap M = \mathrm{L}\Lambda$. Since $\mathrm{L}\Lambda$ is abelian, it follows that M is not McDuff. So whenever Γ is an icc property (T) group, it follows that the 2-cocycle associated with the Cartan subalgebra $\mathcal{C}(K_1) \subset M$ is non trivial.

To prove Theorem 2 we first have to classify the orbit equivalence relations of the product actions $(\Gamma \times \Lambda K_1/K_1) \times \widehat{K_1}$ on $X/K_1 \times \widehat{\Lambda_1}$, when K_1 runs through closed subgroups of K with $\Lambda \cap K_1$ dense in K_1. These orbit equivalence relations are the direct product of the orbit equivalence relation of $(\Gamma \times \Lambda K_1/K_1) \curvearrowright X/K_1$ and the unique hyperfinite II_1 equivalence relation. In [PV06, Theorem 4.1], the actions $(\Gamma \times \Lambda K_1/K_1) \curvearrowright X/K_1$ were classified up to stable orbit equivalence. In the Lemma 9 we redo the proof in the context of a direct product with a hyperfinite equivalence relation. The main difficulty is to make sure that the orbit equivalence 'does not mix up too much' the first and second factor in the direct product. The orbit equivalence will however not simply split as a direct product of orbit equivalences. The main reason for this is that the II_1 factor $N := \mathrm{L}^\infty(X/K_1) \rtimes (\Gamma \times \Lambda K_1/K_1)$ has property Gamma so that it is unclear whether automorphisms of $N \mathbin{\overline{\otimes}} R$ essentially are the tensor product of two automorphisms.

To state and prove Lemma 9, we use the following point of view on stable orbit equivalences. Assume that $\mathcal{G}_i \curvearrowright Z_i$, $i = 1, 2$, are free ergodic p.m.p. actions of countable groups $\mathcal{G}_i$. By definition, a stable orbit equivalence between $\mathcal{G}_1 \curvearrowright Z_1$ and $\mathcal{G}_2 \curvearrowright Z_2$ is a measure space isomorphism $\Delta : \mathcal{U}_1 \to \mathcal{U}_2$ between non negligible subsets $\mathcal{U}_i \subset Z_i$ satisfying

$$\Delta(\mathcal{G}_1 \cdot x \cap \mathcal{U}_1) = \mathcal{G}_2 \cdot \Delta(x) \cap \mathcal{U}_2$$

for a.e. $x \in \mathcal{U}_1$. By ergodicity of $\mathcal{G}_1 \curvearrowright Z_1$, we can choose a measurable map $\Delta_0 : Z_1 \to \mathcal{U}_1$ satisfying $\Delta_0(x) \in \mathcal{G}_1 \cdot x$ for a.e. $x \in Z_1$. Denote $\Psi := \Delta \circ \Delta_0$. By construction Ψ is a local isomorphism from Z_1 to Z_2, meaning that $\Psi : Z_1 \to Z_2$ is a Borel map such that Z_1 can be partitioned into a sequence of non negligible subsets $\mathcal{W} \subset Z_1$ such that the restriction of Ψ to any of these subsets $\mathcal{W}$ is a measure space isomorphism of $\mathcal{W}$ onto some non negligible subset of Z_2. Also by construction Ψ is orbit preserving, meaning that for a.e. $x, y \in Z_1$ we have that $x \in \mathcal{G}_1 \cdot y$ iff $\Psi(x) \in \mathcal{G}_2 \cdot \Psi(y)$.

Using the inverse $\Delta^{-1} : \mathcal{U}_2 \to \mathcal{U}_1$ we analogously find an orbit preserving local isomorphism $\overline{\Psi} : Z_2 \to Z_1$. By construction $\overline{\Psi}(\Psi(x)) \in \mathcal{G}_1 \cdot x$ for a.e. $x \in Z_1$ and $\Psi(\overline{\Psi}(y)) \in \mathcal{G}_2 \cdot y$ for a.e. $y \in Z_2$. We call $\overline{\Psi}$ a *generalized inverse* of Ψ.

A measurable map $\Psi : Z_1 \to Z_2$ between standard measure spaces is called a *factor map* iff $\Psi^{-1}(\mathcal{V})$ has measure zero whenever $\mathcal{V} \subset Z_2$ has measure zero. If $\mathcal{G}_i \curvearrowright Z_i$, $i = 1, 2$, are free ergodic p.m.p. actions of the countable groups $\mathcal{G}_i$ and if $\Psi : Z_1 \to Z_2$, $\overline{\Psi} : Z_2 \to Z_1$ are factor maps satisfying

$$\Psi(\mathcal{G}_1 \cdot x) \subset \mathcal{G}_2 \cdot \Psi(x) \;\text{ and }\; \overline{\Psi}(\Psi(x)) \in \mathcal{G}_1 \cdot x \;\text{ for a.e. } x \in Z_1 \;,$$
$$\overline{\Psi}(\mathcal{G}_2 \cdot y) \subset \mathcal{G}_1 \cdot \overline{\Psi}(y) \;\text{ and }\; \Psi(\overline{\Psi}(y)) \in \mathcal{G}_2 \cdot y \;\text{ for a.e. } y \in Z_2 \;,$$

then Ψ and $\overline{\Psi}$ are local isomorphisms, stable orbit equivalences and each other's generalized inverse.

If $\Psi, \Psi' : Z_1 \to Z_2$ are factor maps that send a.e. orbits into orbits, we say that Ψ and Ψ' are *similar* if $\Psi'(x) \in \mathcal{G}_2 \cdot \Psi(x)$ for a.e. $x \in Z_1$.

If $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ are p.m.p. actions of second countable locally compact groups and if $\delta : \Gamma \to \Lambda$ is an isomorphism, we call $\Delta : X \to Y$ a *δ-conjugacy* if Δ is a measure space isomorphism satisfying $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ a.e.

We start off by the following well known and elementary lemma. For the convenience of the reader, we give a proof.

Lemma 8. *Let $\mathcal{G}_i \curvearrowright Z_i$ be free ergodic p.m.p. actions of countable groups. Let $\Psi : Z_1 \to Z_2$ be a stable orbit equivalence.*

- *If $\Lambda < \mathcal{G}_1$ is a subgroup and $\Psi(g \cdot x) = \Psi(x)$ for all $g \in \Lambda$ and a.e. $x \in Z_1$, then Λ is a finite group.*
- *If $\delta : \mathcal{G}_1 \to \mathcal{G}_2$ is an isomorphism of $\mathcal{G}_1$ onto $\mathcal{G}_2$ and $\Psi(g \cdot x) = \delta(g) \cdot \Psi(x)$ for all $g \in \mathcal{G}_1$ and a.e. $x \in Z_1$, then Ψ is a measure space isomorphism.*

Proof. To prove the first point take a non negligible subset $\mathcal{U} \subset Z_1$ such that $\Psi_{|\mathcal{U}}$ is a measure space isomorphism of $\mathcal{U}$ onto $\mathcal{V} \subset Z_2$. Since $\mathcal{G}_1 \curvearrowright Z_1$ is essentially free, it follows that $g \cdot \mathcal{U} \cap \mathcal{U}$ is negligible for all $g \in \mathcal{G}_1 - \{e\}$. Hence the sets $(g \cdot \mathcal{U})_{g \in \mathcal{G}_1}$ are essentially disjoint. Since Z_1 has finite measure and $\mathcal{U}$ is non negligible, it follows that Λ is finite.

We now prove the second point. Since Ψ is a local isomorphism, the image $\Psi(Z_1)$ is a measurable, $\delta(\mathcal{G}_1)$-invariant, non negligible subset of Z_2. Since $\delta(\mathcal{G}_1) = \mathcal{G}_2$ and $\mathcal{G}_2 \curvearrowright Z_2$ is ergodic, it follows that Ψ is essentially surjective. Assume that Ψ is not a measure space isomorphism. Since Ψ is a local isomorphism, we find non negligible disjoint subsets $\mathcal{U}_1, \mathcal{U}_2 \subset Z_1$ such that $\Psi_{|\mathcal{U}_i}$, $i = 1, 2$, are measure space isomorphisms onto the same subset $\mathcal{V} \subset Z_2$. Since Ψ is a stable orbit equivalence, $(\Psi_{|\mathcal{U}_2})^{-1}(\Psi(x)) \in \mathcal{G}_1 \cdot x$ for a.e. $x \in \mathcal{U}_1$. Making $\mathcal{U}_1, \mathcal{U}_2$ smaller, we may assume that there exists a $g \in \mathcal{G}_1$ such that $(\Psi_{|\mathcal{U}_2})^{-1}(\Psi(x)) = g \cdot x$ for a.e. $x \in \mathcal{U}_1$. Since $\mathcal{U}_1$ and $\mathcal{U}_2$ are disjoint, it follows that $g \neq e$. But it also follows that $\delta(g) \cdot \Psi(x) = \Psi(x)$ for a.e. $x \in \mathcal{U}_1$. Since $\mathcal{G}_2 \curvearrowright Z_2$ is essentially free, we arrive at the contradiction that $\delta(g) = e$. □

Lemma 9. *Let Γ be an icc property (T) group with $[\Gamma,\Gamma]=\Gamma$. As above take a compact abelian group K with dense countable subgroup $\Lambda < K$. Put $X := K^\Gamma$.*

Let $K_i < K$, $i=1,2$, be closed subgroups such that $\Lambda \cap K_i$ is dense in K_i. Denote $G_i := \Gamma \times \Lambda K_i$ with its natural action on X. Let $H_i \curvearrowright Y_i$ be free ergodic p.m.p. actions of the countable abelian groups H_i. Assume that

$$\Delta : X/K_1 \times Y_1 \to X/K_2 \times Y_2$$

is a stable orbit equivalence between the product actions $G_i/K_i \times H_i \curvearrowright X/K_i \times Y_i$. Then there exists

- *a compact subgroup $K_1' < \Lambda K_1$ that is commensurate with K_1, with corresponding canonical stable orbit equivalence $\Delta_1 : X/K_1' \to X/K_1$ between the actions $G_1/K_1' \curvearrowright X/K_1'$ and $G_1/K_1 \curvearrowright X/K_1$;*
- *an automorphism $\delta_1 \in \operatorname{Aut}\Gamma$ and an isomorphism $\delta_2 : \Lambda K_1/K_1' \to \Lambda K_2/K_2$;*
- *a stable orbit equivalence $\Psi : Y_1 \to Y_2$ between the actions $H_i \curvearrowright Y_i$;*
- *a measurable family $(\Theta_y)_{y\in Y_1}$ of $(\delta_1\times\delta_2)$-conjugacies between the actions $G_1/K_1' \curvearrowright X/K_1'$ and $G_2/K_2 \curvearrowright X/K_2$;*
- *a δ_1-conjugacy Θ_1 of $\Gamma \curvearrowright X/K$;*

such that the stable orbit equivalence $\Delta \circ (\Delta_1 \times \mathrm{id})$ is similar with the stable orbit equivalence

$$\Theta : X/K_1' \times Y_1 \to X/K_2 \times Y_2 : (x,y) \mapsto (\Theta_y(x), \Psi(y))$$

and such that $\Theta_y(xK_1')K = \Theta_1(xK)$ a.e.

Note that we turn G_i into a locally compact group by requiring that $K_i < G_i$ is a compact open subgroup.

Proof. Denote

$$\Theta : X \times Y_1 \to X/K_2 \times Y_2 : \Theta(x,y) = \Delta(xK_1, y)\ .$$

Associated with the orbit map Θ are two measurable families of cocycles:

$$\begin{aligned} \omega^y &: G_1 \times X \to G_2/K_2 \times H_2 \quad \text{determined by} \quad \Theta(g\cdot x, y) = \omega^y(g,x)\cdot\Theta(x,y)\ ,\\ \mu^x &: H_1 \times Y_1 \to G_2/K_2 \times H_2 \quad \text{determined by} \quad \Theta(x, h\cdot y) = \mu^x(h,y)\cdot\Theta(x,y)\ . \end{aligned}$$

By a combination of Popa's cocycle superrigidity theorem for $\Gamma \curvearrowright K^\Gamma$ (see [Po05, Theorem 0.1]) and the elementary [PV08, Lemma 5.5], we can replace Θ by a similar factor map and find measurable families of continuous group homomorphisms

$$\delta_y : G_1 \to G_2/K_2 \quad \text{and} \quad \eta_y : G_1 \to H_2$$

such that

$$\omega^y(g,x) = (\delta_y(g), \eta_y(g)) \quad \text{and} \quad \Theta(g\cdot x, y) = (\delta_y(g),\eta_y(g))\cdot\Theta(x,y) \quad \text{a.e.}$$

Since $[\Gamma,\Gamma]=\Gamma$ and H_2 is abelian, it follows that $\eta_y(g)=e$ for all $g\in\Gamma$ and a.e. $y\in Y_1$. Writing $\Theta(x,y) = (\Theta_1(x,y), \Theta_2(x,y))$, it follows that $\Theta_2(g\cdot x,y) = \Theta_2(x,y)$ for all $g\in\Gamma$ and a.e. $(x,y) \in X \times Y_1$. By ergodicity of $\Gamma \curvearrowright X$, it follows that $\Theta_2(x,y) = \Psi(y)$ for some factor map $\Psi : Y_1 \to Y_2$. But then

$$\eta_y(g) \cdot \Psi(y) = \eta_y(g) \cdot \Theta_2(x,y) = \Theta_2(g \cdot x, y) = \Psi(y)$$

for all $g \in G_1$ and a.e. $(x,y) \in X \times Y_1$. It follows that $\eta_y(g) = e$ for all $g \in G_1$ and a.e. $y \in Y_1$. Denote $\mu^x(h,y) = (\mu_1^x(h,y), \mu_2^x(h,y))$. Because

$$\mu_2^x(h,y) \cdot \Psi(y) = \mu_2^x(h,y) \cdot \Theta_2(x,y) = \Theta_2(x, h \cdot y) = \Psi(h \cdot y)$$

for all $h \in H_1$ and a.e. $(x,y) \in X \times Y_1$, it follows that $\mu_2^x(h,y)$ is essentially independent of the x-variable. So we write $\mu_2(h,y)$ instead of $\mu_2^x(h,y)$.

Next we observe that $\Theta_1(g \cdot x, h \cdot y)$ can be computed in two ways, leading to the equality

$$\delta_{h \cdot y}(g)\, \mu_1^x(h,y) \cdot \Theta_1(x,y) = \mu_1^{g \cdot x}(h,y)\, \delta_y(g) \cdot \Theta_1(x,y) \quad \text{a.e.}$$

So, $\mu_1^{g\cdot x}(h,y) = \delta_{h\cdot y}(g)\, \mu_1^x(h,y)\, \delta_y(g)^{-1}$ for all $g \in G_1$, $h \in H_1$ and a.e. $(x,y) \in X \times Y_1$.

It follows that for all $h \in H_1$ and a.e. $y \in Y_1$, the non negligible set $\{(x,x') \in X \times X \mid \mu_1^x(h,y) = \mu_1^{x'}(h,y)\}$ is invariant under the diagonal action of Γ on $X \times X$. Since $\Gamma \curvearrowright X$ is weakly mixing, we conclude that $\mu_1^x(h,y)$ is essentially independent of the x-variable. We write $\mu_1(h,y)$ instead of $\mu_1^x(h,y)$. The above formula becomes

$$\delta_{h\cdot y}(g) = \mu_1(h,y)\, \delta_y(g)\, \mu_1(h,y)^{-1} \; . \tag{2}$$

It follows that the map $y \mapsto \operatorname{Ker} \delta_y$ is H_1-invariant and hence, by ergodicity of $H_1 \curvearrowright Y_1$, there is a unique closed subgroup $K_1' < G_1$ such that $\operatorname{Ker} \delta_y = K_1'$ for a.e. $y \in Y_1$. Since G_2/K_2 is a countable group, we know that its compact subgroup $\delta_y(K_1)$ must be finite. Hence, $K_1' \cap K_1 < K_1$ has finite index. We claim that also $K_1' \cap K_1 < K_1'$ has finite index.

To prove this claim, first observe that the formula $\Theta(g \cdot x, y) = \delta_y(g) \cdot \Theta(x,y)$ implies that we can view Θ as a map from $X/K_1' \times Y_1$ to $X/K_2 \times Y_2$. In particular Θ can be viewed as a map from $X/(K_1 \cap K_1') \times Y_1$ to $X/K_2 \times Y_2$. Since $X/(K_1 \cap K_1')$ is a finite covering of X/K_1 and since the original Δ was a local isomorphism, also

$$\Theta : X/(K_1 \cap K_1') \times Y_1 \to X/K_2 \times Y_2$$

is a local isomorphism and hence a stable orbit equivalence. Since moreover $\Theta(g\cdot x, y) = \Theta(x,y)$ for all $g \in K_1'/(K_1 \cap K_1')$, it follows from Lemma 8 that $K_1'/(K_1 \cap K_1')$ is a finite group. This proves the claim, meaning that K_1' and K_1 are commensurate subgroups of G_1. Since Γ is icc, Γ has no finite normal subgroups so that $K_1' < \Lambda K_1$.

By construction the map

$$\Theta : X/K_1' \times Y_1 \to X/K_2 \times Y_2$$

is a stable orbit equivalence with the following properties:

- Θ is similar with $\Delta \circ (\Delta_1 \times \mathrm{id})$;
- Θ is of the form $\Theta(x,y) = (\Theta_y(x), \Psi(y))$;
- and Θ satisfies $\Theta_y(g \cdot x) = \delta_y(g) \cdot \Theta_y(x)$ a.e. where $\delta_y : G_1/K_1' \to G_2/K_2$ is a measurable family of injective group homomorphisms.

It remains to prove that the δ_y are group isomorphisms that do not depend on y, that Ψ is a stable orbit equivalence and that a.e. Θ_y is a measure space isomorphism of X/K_1' onto X/K_2.

Applying the same reasoning to the inverse of the stable orbit equivalence $\Delta \circ (\Delta_1 \times \mathrm{id})$, we find a closed subgroup $K_2' < \Lambda K_2$ that is commensurate with K_2, a measurable family of injective group homomorphisms $\overline{\delta}_y : G_2/K_2' \to G_1/K_1'$ and a map

$$\overline{\Theta} : X/K_2' \times Y_2 \to X/K_1' \times Y_1$$

of the form $\overline{\Theta}(x,y) = (\overline{\Theta}_y(x), \overline{\Psi}(y))$ satisfying the following two properties:

$$\overline{\Theta}_y(g \cdot x) = \overline{\delta}_y(g) \cdot \overline{\Theta}_y(x) \quad \text{and} \quad \Theta(\overline{\Theta}(xK_2', y)) \in (G_2/K_2 \times H_2) \cdot (xK_2, y) \quad \text{a.e.}$$

The second of these formulae yields a measurable map $\varphi : X \times Y_2 \to G_2/K_2$ such that

$$\Theta_{\overline{\Psi}(y)}(\overline{\Theta}_y(xK_2')) = \varphi(x,y) \cdot xK_2 \quad \text{a.e.}$$

Computing $\Theta_{\overline{\Psi}(y)}(\overline{\Theta}_y(gK_2' \cdot xK_2'))$ it follows that

$$\delta_{\overline{\Psi}(y)}(\overline{\delta}_y(gK_2'))\, \varphi(x,y) = \varphi(g \cdot x, y)\, gK_2 \quad \text{for all } g \in G_2 \text{ and a.e. } (x,y) \in X \times Y_2.$$

So, for almost every $y \in Y_2$, the non negligible set $\{(x,x') \in X \times X \mid \varphi(x,y) = \varphi(x',y)\}$ is invariant under the diagonal action of Γ on $X \times X$. Since $\Gamma \curvearrowright X$ is weakly mixing, it follows that $\varphi(x,y)$ is essentially independent of the x-variable. We write $\varphi(y)$ instead of $\varphi(x,y)$ and find that

$$(\delta_{\overline{\Psi}(y)} \circ \overline{\delta}_y)(gK_2') = \varphi(y)\, gK_2\, \varphi(y)^{-1} \; .$$

It follows that $K_2' = K_2$ and that $\delta_{\overline{\Psi}(y)} \circ \overline{\delta}_y$ is an inner automorphism of G_2/K_2. It follows in particular that $\delta_{\overline{\Psi}(y)}$ is surjective, so that δ_y is an isomorphism of G_1/K_1' onto G_2/K_2 for all y in a non negligible subset of Y_1. Because of (2) and the ergodicity of $H_1 \curvearrowright Y_1$, it follows that δ_y is an isomorphism for a.e. $y \in Y_1$.

It also follows that $\Psi(\overline{\Psi}(y)) \in H_2 \cdot y$ for a.e. $y \in Y_2$ and similarly with $\overline{\Psi} \circ \Psi$. Hence Ψ and $\overline{\Psi}$ are local isomorphisms and hence a stable orbit equivalence between $H_1 \curvearrowright Y_1$ and $H_2 \curvearrowright Y_2$. Since $\Theta(x,y) = (\Theta_y(x), \Psi(y))$ and since Θ is a local isomorphism, it then follows that a.e. Θ_y is a local isomorphism. But Θ_y is also a δ_y-conjugation w.r.t. the isomorphism δ_y. Lemma 8 implies that Θ_y is a measure space isomorphism for a.e. $y \in Y_1$.

Since $[\Gamma,\Gamma] = \Gamma$ and $\Lambda K_1/K_1'$ is abelian, it follows that δ_y is the direct product of an automorphism δ_y^1 of Γ and an isomorphism δ_y^2 of $\Lambda K_1/K_1'$ onto $\Lambda K_2/K_2$. Formula (2) and the fact that $\Lambda K_2/K_2$ is abelian imply that $\delta^2_{h\cdot y} = \delta^2_y$ a.e. By the ergodicity of $H_1 \curvearrowright Y_1$, it follows that δ^2_y is essentially independent of y. We write δ_2 instead of δ_y^2. Next denote by $\mu_\Gamma(h,y)$ the projection of $\mu^1(h,y) \in \Gamma \times \Lambda K_2/K_2$ into Γ. Formula (2) says that

$$\delta^1_{h\cdot y} = \mathrm{Ad}\, \mu_\Gamma(h,y) \circ \delta^1_y \; .$$

Since Γ has property (T), Γ is finitely generated and $\mathrm{Aut}\,\Gamma$ is a countable group. In particular, $\mathrm{Out}\,\Gamma = \mathrm{Aut}\,\Gamma/\mathrm{Inn}\,\Gamma$ is a countable group. Then the map $y \mapsto (\delta_y^1 \mod \mathrm{Inn}\,\Gamma)$ is an H_1-invariant measurable map from Y_1 to $\mathrm{Out}\,\Gamma$ and hence is essentially constant. So we find a measurable map $\varphi : Y_1 \to \Gamma$ and an automorphism $\delta_1 \in \mathrm{Aut}\,\Gamma$ such that $\delta_y = \mathrm{Ad}\,\varphi(y) \circ \delta_1$ a.e. Replacing $\Theta_y(x)$ by $\varphi(y)^{-1} \cdot \Theta_y(x)$ we may assume that δ_y is a.e. equal to $\delta_1 \times \delta_2$. After this replacement, also $\mu_\Gamma(h,y) = e$ a.e. meaning that $\mu^1(h,y) \in \Lambda K_2/K_2$.

Since Θ_y is a δ_2-conjugacy and $\Lambda K_1, \Lambda K_2$ are dense subgroups of K, the Θ_y induce δ_1-conjugacies ρ_y of the action $\Gamma \curvearrowright X/K$ given by $\rho_y(xK) = \Theta_y(xK_1')K$. Since $\mu^1(h,y) \in \Lambda K_2/K_2$, it follows that $\rho_{h\cdot y}(x) = \rho_y(x)$ a.e. Hence ρ_y is a.e. equal to a single δ_1-conjugacy Θ_1 of the action $\Gamma \curvearrowright X/K$. $\square$

Proof of Theorem 2

Proof. **Proof of the equivalence of the statements in 1.** We first prove the equivalence of the first two statements. Put $\Lambda_i = \Lambda \cap K_i$. If $K_1, K_2 < K$ are commensurate, also $\Lambda_1, \Lambda_2 < \Lambda$ are commensurate. It follows that $\mathrm{L}^\infty(K^\Gamma/(K_1K_2)) \overline{\otimes} \mathrm{L}(\Lambda_1 \cap \Lambda_2)$ is a subalgebra of finite index in both $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$. By [Po01, Theorem A.1] the Cartan subalgebras $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ are unitarily conjugate. Conversely, if $K_1, K_2 < K$ are not commensurate, also $\Lambda_1, \Lambda_2 < \Lambda$ are not commensurate. Assume for instance that $\Lambda_1 \cap \Lambda_2 < \Lambda_1$ has infinite index. We find a sequence $\lambda_n \in \Lambda_1$ such that for every finite subset $\mathcal{F} \subset \Lambda$ the element λ_n lies outside $\mathcal{F}\Lambda_2$ eventually. Denote by $u_n := u_{\lambda_n}$ the unitaries in $\mathcal{C}(K_1)$ that correspond to the group elements λ_n. Denoting by $E_{\mathcal{C}(K_2)}$ the trace preserving conditional expectation, one deduces that

$$\|E_{\mathcal{C}(K_2)}(au_nb)\|_2 \to 0 \quad \text{for all} \quad a, b \in M \; .$$

So, by [Po01, Theorem A.1], the Cartan subalgebras $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ are not unitarily conjugate.

We next prove the equivalence of the last two statements. Equip ΛK_i with the locally compact topology that turns K_i into a compact open subgroup. Assume that $\Lambda K_1 = \Lambda K_2$. Since a locally compact topology is entirely determined by the Borel structure, K_1 is a compact open subgroup of ΛK_2. Hence K_1 and K_2 are commensurate. Conversely assume that K_1, K_2 are commensurate. Then $\Lambda(K_1 \cap K_2)$ is a finite index dense subgroup in both ΛK_1 and ΛK_2. So $\Lambda(K_1 \cap K_2)$ is equal to both ΛK_1 and ΛK_2.

Note for later use that the same methods give the following: if $K_1 < K$ is a compact subgroup such that $\Lambda \cap K_1$ is dense in K_1 and if $K_1' < \Lambda K_1$ is any compact open subgroup, then also $\Lambda \cap K_1'$ is dense in K_1' and $\Lambda K_1' = \Lambda K_1$.

Proof of the equivalence of the statements in 3. Denote $X = K^\Gamma$. Assume first that there exists a continuous automorphism $\delta \in \operatorname{Aut} K$ satisfying $\delta(\Lambda K_1) = \Lambda K_2$. View δ as an isomorphism between the locally compact groups ΛK_1 and ΛK_2. Since δ and δ^{-1} are Borel maps and group isomorphisms, they are homeomorphisms. Hence $K_1' := \delta^{-1}(K_2)$ is a compact open subgroup of ΛK_1. By the previous paragraphs the Cartan subalgebras $\mathcal{C}(K_1')$ and $\mathcal{C}(K_1)$ are unitarily conjugate. In particular, they give rise to isomorphic equivalence relations. So we may replace K_1 by K_1' and assume that $\delta(K_1) = K_2$. By applying 'everywhere' δ, the actions $(\Gamma \times \Lambda K_i/K_i) \curvearrowright X/K_i$ are conjugate, hence orbit equivalent. Also their respective direct products with the hyperfinite equivalence relation are orbit equivalent. By Lemma 6 the equivalence relations induced by $\mathcal{C}(K_i)$, $i = 1, 2$, are orbit equivalent.

Conversely assume that the equivalence relations induced by $\mathcal{C}(K_i)$, $i = 1, 2$, are stably orbit equivalent. Put $G_i := \Gamma \times \Lambda K_i$. Combining Lemmas 6 and 9, and again replacing K_1 by a commensurate group, we find a measure space isomorphism $\Delta : X/K_1 \to X/K_2$, an automorphism $\delta_1 \in \operatorname{Aut}\Gamma$ and an isomorphism $\delta_2 : \Lambda K_1/K_1 \to \Lambda K_2/K_2$ such that Δ is a $(\delta_1 \times \delta_2)$-conjugacy. In particular, Δ is a δ_1-conjugacy of the Γ-action. By [PV06, Lemma 5.2] there exists an isomorphism $\alpha : K_1 \to K_2$ and a $(\delta_1 \times \alpha)$-conjugacy $\overline{\Delta} : X \to X$ satisfying $\overline{\Delta}(x)K_2 = \Delta(xK_1)$ for a.e. $x \in X$. Since Δ is a δ_2-conjugacy, it follows that $\overline{\Delta}(g \cdot x) \in \Lambda K_2 \cdot \Delta(x)$ for all $g \in \Lambda K_1$ and a.e. $x \in X$. We find a measurable map $\omega : \Lambda K_1 \times X \to \Lambda K_2$ such that $\overline{\Delta}(g \cdot x) = \omega(g, x) \cdot \overline{\Delta}(x)$ a.e. Since $\overline{\Delta}$ is a δ_1-conjugacy, it follows that $\omega(g, \gamma \cdot x) = \omega(g, x)$ for all $\gamma \in \Gamma$, $g \in \Lambda K_1$ and a.e. $x \in X$. By ergodicity of $\Gamma \curvearrowright X$, the map $\omega(g, x)$ is essentially independent of the x-variable. So we can extend α to a homomorphism $\alpha : \Lambda K_1 \to \Lambda K_2$. By symmetry (i.e. considering $\overline{\Delta}^{-1}$) it follows that α is an isomorphism of ΛK_1 onto ΛK_2. Viewing K as a closed subgroup of the group $\operatorname{Aut}(X)$ of measure preserving automorphisms of X (up to equality almost everywhere), it follows that $\overline{\Delta}\Lambda K_1 \overline{\Delta}^{-1} = \Lambda K_2$. Taking closures we also have $\overline{\Delta} K \overline{\Delta}^{-1} = K$. This means that α can be extended to a continuous automorphism of K. This concludes the proof of the equivalence of the statements in 3.

Proof of the equivalence of the statements in 2. Denote $X = K^\Gamma$. If $\delta \in \operatorname{Aut} K$ is a continuous automorphism satisfying $\delta(\Lambda) = \Lambda$ and $\delta(\Lambda K_1) = \Lambda K_2$, we may replace K_1 by the commensurate $\delta^{-1}(K_2)$ and assume that $\delta(K_1) = K_2$. Then also $\delta(\Lambda_1) = \Lambda_2$. So, 'applying everywhere' δ yields an automorphism $\alpha \in \operatorname{Aut} M$ satisfying $\alpha(\mathcal{C}(K_1)) = \mathcal{C}(K_2)$.

Conversely assume that $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ are stably conjugate by an automorphism. Interchanging the roles of K_1 and K_2 if necessary, we find a projection $p \in \mathcal{C}(K_2)$ and an isomorphism $\alpha : M \to pMp$ satisfying $\alpha(\mathcal{C}(K_1)) = \mathcal{C}(K_2)p$. This α induces a stable orbit equivalence between the equivalence relations associated with $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ respectively. If one of the K_i is finite, the equivalence of the statements in 3 implies that the other one is finite as well and that $\Lambda = \Lambda K_1 = \Lambda K_2$. So we can exclude this trivial case.

We claim that there exist automorphisms $\delta_1 \in \operatorname{Aut} \Gamma$, $\delta \in \operatorname{Aut} K$ and a $(\delta_1 \times \delta)$-conjugacy $\Delta \in \operatorname{Aut}(X)$ of the action $\Gamma \times K \curvearrowright X$ such that $\delta(K_1) = K_2$ and such that after a unitary conjugacy of α and p, we have

$$p \in \mathrm{L}\Lambda_2 \quad \text{and} \quad \alpha(F) = (F \circ \Delta^{-1})\, p \quad \text{for all } \; F \in \mathrm{L}^\infty(X/K) \; .$$

To prove this claim, denote $Y_i = \widehat{\Lambda_i}$ and $H_i = \widehat{K_i}$. Also denote $G_i = \Gamma \times \Lambda K_i$. Lemma 6 describes the equivalence relations associated with $\mathcal{C}(K_i)$. The isomorphism $\alpha : M \to pMp$ therefore induces a stable orbit equivalence $\Pi : X/K_1 \times Y_1 \to X/K_2 \times Y_2$ between the product actions $G_i/K_i \times H_i \curvearrowright X/K_i \times Y_i$. We apply Lemma 9 to Π. We find in particular a compact subgroup $K_1' < \Lambda K_1$ that is commensurate with K_1. Denote $\Lambda_1' := \Lambda \cap K_1'$, $H_1' := \widehat{K_1'}$ and $Y_1' := \widehat{\Lambda_1'}$. By the equivalence of the statements in 1, we know that the Cartan subalgebras $\mathcal{C}(K_1')$ and $\mathcal{C}(K_1)$ are unitarily conjugate. The corresponding orbit equivalence between the product actions $G_1/K_1' \times H_1' \curvearrowright X/K_1' \times Y_1'$ and $G_1/K_1 \times H_1 \curvearrowright X/K_1 \times Y_1$ is similar with $\Delta_1 \times \Delta_2$ where Δ_1, Δ_2 are the natural stable orbit equivalences (whose compression constants are each other's inverse). So replacing $\mathcal{C}(K_1)$ by $\mathcal{C}(K_1')$ amounts to replacing Π by $\Pi \circ (\Delta_1 \times \Delta_2)$. Lemma 9 says that $\Pi \circ (\Delta_1 \times \mathrm{id})$ is similar to a stable orbit equivalence of a very special form. Then $\Pi \circ (\Delta_1 \times \Delta_2)$ is still of this very special form. So after replacing $\mathcal{C}(K_1)$ by $\mathcal{C}(K_1')$ we find that Π is similar to a stable orbit equivalence $\Theta : X/K_1 \times Y_1 \to X/K_2 \times Y_2$ satisfying the following properties.

- Θ is of the form $\Theta(x, y) = (\Theta_y(x), \Psi(y))$ where $\Psi : Y_1 \to Y_2$ is a stable orbit equivalence between the actions $H_i \curvearrowright Y_i$ and where $(\Theta_y)_{y \in Y_1}$ is a measurable family of measure preserving conjugacies between $G_1/K_1 \curvearrowright X/K_1$ and $G_2/K_2 \curvearrowright X/K_2$.
- There exist automorphisms $\delta_1 \in \operatorname{Aut} \Gamma$, $\delta \in \operatorname{Aut} K$ and a $(\delta_1 \times \delta)$-conjugacy $\Delta \in \operatorname{Aut}(X)$ such that $\delta(K_1) = K_2$ and $\Theta_y(x) K = \Delta(xK)$ for a.e. $(x, y) \in X/K_1 \times Y_1$.

To deduce the precise form of Δ in the last item, we proceed as in the proof above of the equivalence of the statements in 3. The compression constant of Ψ is equal to the compression constant of Π and hence equal to $\tau(p)$. So replacing Ψ by a similar stable orbit equivalence, we may assume that Ψ is a measure space isomorphism (scaling the measure by the factor $\tau(p)$) of Y_1 onto a measurable subset $\mathcal{U} \subset Y_2$ of measure $\tau(p)$. Keeping $\Theta_y(x)$ unchanged, the map Θ then becomes a measure space isomorphism of $X/K_1 \times Y_1$ onto $X/K_2 \times \mathcal{U}$. Define the projection $p \in \mathrm{L}\Lambda_2 = \mathrm{L}^\infty(Y_2)$ by $p = \chi_{\mathcal{U}}$. Since Π and Θ are similar stable orbit equivalences and Θ is moreover a measure space isomorphism, we can unitarily conjugate α so that $\alpha_{|\mathcal{C}(K_1)} = \Theta_*$. This proves the claim above.

It remains to prove that $\delta(\Lambda) = \Lambda$.

Since every automorphism $\delta_1 \in \operatorname{Aut}\Gamma$ defines a natural automorphism of M that globally preserves all the Cartan subalgebras $\mathcal{C}(K_1)$, we may assume that $\delta_1 = \mathrm{id}$. For $g \in \Gamma \times \Lambda$, we denote by $u_g \in M = \mathrm{L}^\infty(X) \rtimes (\Gamma \times \Lambda)$ the canonical unitaries. Since the relative commutant of $\mathrm{L}^\infty(X/K)$ inside M equals $\mathrm{L}^\infty(X) \rtimes \Lambda$, it follows that

$$\alpha(u_g) = \omega_g u_g p \quad \text{for all } g \in \Gamma, \text{ where} \quad \omega_g \in \mathcal{U}(p(\mathrm{L}^\infty(X) \rtimes \Lambda)p) \; .$$

Denote by $(\sigma_g)_{g\in\Gamma}$ the action of Γ on $\mathrm{L}^\infty(X) \rtimes \Lambda$, implemented by $\operatorname{Ad} u_g$ and corresponding to the Bernoulli action on $\mathrm{L}^\infty(X)$ and the trivial action on $\mathrm{L}\Lambda$. Since p is invariant under $(\sigma_g)_{g\in\Gamma}$ we can restrict σ_g to $p(\mathrm{L}^\infty(X) \rtimes \Lambda)p$. Note that

$$\omega_{gh} = \omega_g \, \sigma_g(\omega_h) \quad \text{for all } \; g, h \in \Gamma \; .$$

Denote $N^\infty := \mathrm{B}(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} N$ whenever N is a von Neumann algebra. By Theorem 10 there exists a projection $q \in (\mathrm{L}\Lambda)^\infty$ with $(\operatorname{Tr} \otimes \tau)(q) = \tau(p)$, a partial isometry $v \in \mathrm{B}(\mathbb{C}, \ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} (\mathrm{L}^\infty(X) \rtimes \Lambda)$ and a group homomorphism $\gamma : \Gamma \to \mathcal{U}(q(\mathrm{L}\Lambda)^\infty q)$ satisfying

$$v^* v = p \;\; , \;\; vv^* = q \quad \text{and} \quad \omega_g = v^* \, \gamma_g \, \sigma_g(v) \;\; \text{for all } \; g \in \Gamma \; .$$

We replace α by $\operatorname{Ad} v \circ \alpha$. Then $\alpha : M \to qM^\infty q$. Note that $\mathrm{L}^\infty(X/K)$ commutes with v so that

$$\alpha(F) = (F \circ \Delta^{-1})q \;\; \text{for all } \; F \in \mathrm{L}^\infty(X/K) \quad \text{and} \quad \alpha(u_g) = \gamma_g u_g = u_g \gamma_g \;\; \text{for all } \; g \in \Gamma \; .$$

Denote $\Delta_* \in \operatorname{Aut}(\mathrm{L}^\infty(X))$ given by $\Delta_*(F) = F \circ \Delta^{-1}$ and note that $\Delta_*(\mathrm{L}^\infty(X/K)) = \mathrm{L}^\infty(X/K)$. Also $\alpha(F) = \Delta_*(F)q$ whenever $F \in \mathrm{L}^\infty(X/K)$.

Recall that $X = K^\Gamma$. Whenever $\omega \in \widehat{K}$, denote by $U_\omega \in \mathcal{U}(\mathrm{L}^\infty(X))$ the unitary given by $U_\omega(x) = \omega(x_e)$. Fix $\omega \in \widehat{K}$ and $g \in \Gamma$. Observe that $U_\omega \, u_g U_\omega^* u_g^*$ belongs to $\mathrm{L}^\infty(X/K)$. Hence, since Δ is a conjugacy for $\Gamma \curvearrowright X$, we have

$$\begin{aligned} \alpha(U_\omega \, u_g U_\omega^* u_g^*) &= \Delta_*(U_\omega \, \sigma_g(U_\omega^*))q = \Delta_*(U_\omega) \, \sigma_g(\Delta_*(U_\omega^*)) \, q \\ &= \Delta_*(U_\omega) \, u_g \, \Delta_*(U_\omega)^* \, u_g^* \, q \; . \end{aligned}$$

On the other hand

$$\alpha(U_\omega \, u_g U_\omega^* u_g^*) = \alpha(U_\omega) \, u_g \, \gamma_g \, \alpha(U_\omega)^* \, \gamma_g^* u_g^* \; .$$

Combining both formulae we conclude that

$$u_g^* \, \Delta_*(U_\omega)^* \, \alpha(U_\omega) \, u_g = \Delta_*(U_\omega)^* \, \gamma_g \, \alpha(U_\omega) \, \gamma_g^* \quad \text{for all } \; \omega \in \widehat{K} \; , \; g \in \Gamma \; . \tag{3}$$

We claim that for all $\omega \in \widehat{K}$, $\Delta_*(U_\omega)^* \, \alpha(U_\omega) \in (\mathrm{L}\Lambda)^\infty$. To prove this claim, fix $\omega \in \widehat{K}$ and put $a := \Delta_*(U_\omega)^* \, \alpha(U_\omega)$. Define for every finite subset $\mathcal{F} \subset \Gamma$, the von Neumann subalgebra $N_\mathcal{F} \subset (\mathrm{L}^\infty(X) \rtimes \Lambda)^\infty$ given by

$$N_\mathcal{F} := (\mathrm{L}^\infty(K^\mathcal{F}) \rtimes \Lambda)^\infty \; .$$

Denote by $E_\mathcal{F}$ the trace preserving conditional expectation of $(\mathrm{L}^\infty(X) \rtimes \Lambda)^\infty$ onto $N_\mathcal{F}$. Note that $(\mathrm{L}\Lambda)^\infty \subset N_\mathcal{F}$ for all $\mathcal{F} \subset \Gamma$. Choose $\varepsilon > 0$ and denote by $\| \cdot \|_2$ the 2-norm on $(\mathrm{L}^\infty(X) \rtimes \Lambda)^\infty$

given by the semi-finite trace $\mathrm{Tr} \otimes \tau$. Since $\alpha(U_\omega)$ commutes with $\mathrm{L}^\infty(X/K)q$, it follows that $\alpha(U_\omega) \in q(\mathrm{L}^\infty(X) \rtimes \Lambda)^\infty q$. So we can take a large enough finite subset $\mathcal{F} \subset \Gamma$ such that

$$\|\alpha(U_\omega) - E_\mathcal{F}(\alpha(U_\omega))\|_2 < \varepsilon \quad \text{and} \quad \|q\Delta_*(U_\omega) - E_\mathcal{F}(q\Delta_*(U_\omega))\|_2 < \varepsilon \; .$$

It follows that for all $g \in \Gamma$, the element $\Delta_*(U_\omega)^* \, \gamma_g \, \alpha(U_\omega) \, \gamma_g^*$ lies at distance at most 2ε from $N_\mathcal{F}$. By (3), the element a then lies at distance at most 2ε from $N_{g\mathcal{F}}$. Since a also lies at distance at most 2ε from $N_\mathcal{F}$, we conclude that a lies at distance at most 4ε from $N_{\mathcal{F} \cap g\mathcal{F}}$ for all $g \in \Gamma$. We can choose g such that $\mathcal{F} \cap g\mathcal{F} = \emptyset$ and conclude that a lies at distance at most 4ε from $(\mathrm{L}\Lambda)^\infty$. Since $\varepsilon > 0$ is arbitrary, the claim follows.

Put $V_\omega := \Delta_*(U_\omega)^* \, \alpha(U_\omega)$ and $q_\omega := V_\omega V_\omega^*$. Denote by $z \in \mathrm{L}\Lambda$ the central support of q in $(\mathrm{L}\Lambda)^\infty$. Define

$$q_1 = \bigvee_{\omega \in \widehat{K}} V_\omega V_\omega^* \; .$$

Since $V_\omega \in (\mathrm{L}\Lambda)^\infty$ and $V_\omega^* V_\omega = q$, it follows that $q_1 \in (\mathrm{L}\Lambda)^\infty$ and $q_1 \leq z$. Since

$$V_\omega V_\omega^* = \Delta_*(U_\omega)^* \, q \, \Delta_*(U_\omega)$$

it also follows that q_1 commutes with all the unitaries $\Delta_*(U_\omega)$, $\omega \in \widehat{K}$. Being an element of $(\mathrm{L}\Lambda)^\infty$, the projection q_1 certainly commutes with $\mathrm{L}^\infty(X/K) = \Delta_*(\mathrm{L}^\infty(X/K))$. Hence, q_1 commutes with the whole of $\Delta_*(\mathrm{L}^\infty(X)) = \mathrm{L}^\infty(X)$. So, $q_1 \in \mathrm{B}(\ell^2(\mathbb{N})) \otimes 1$. Since $q_1 \leq z$, it follows that $z = 1$. Because $(\mathrm{Tr} \otimes \tau)(q) = \tau(p) \leq 1$, we must have $p = 1$ and find an element $w \in \mathrm{B}(\ell^2(\mathbb{N}), \mathbb{C}) \overline{\otimes} \mathrm{L}\Lambda$ satisfying $ww^* = 1$ and $w^* w = q$. Replacing α by $\mathrm{Ad}\, w \circ \alpha$, we have found that α is an automorphism of M satisfying

$$\begin{aligned} &\alpha(F) = \Delta_*(F) \;\; \text{for all } F \in \mathrm{L}^\infty(X/K) \;\;, \;\; \alpha(U_\omega) = \Delta_*(U_\omega) \, V_\omega \;\; \text{for all } \omega \in \widehat{K}, \text{ and} \\ &\alpha(u_g) = \gamma_g u_g \;\; \text{for all } g \in \Gamma. \end{aligned}$$

Here $V_\omega \in \mathrm{L}\Lambda$ are unitaries and $g \mapsto \gamma_g$ is a homomorphism from Γ into the abelian group $\mathcal{U}(\mathrm{L}\Lambda)$. Since $[\Gamma, \Gamma] = \Gamma$, we actually have that $\gamma_g = 1$ for all $g \in \Gamma$. In particular $\alpha(\mathrm{L}\Gamma) = \mathrm{L}\Gamma$. Taking the relative commutant, it follows that $\alpha(\mathrm{L}\Lambda) = \mathrm{L}\Lambda$.

Since $(\mathrm{Ad}\, U_\omega)(u_s) = \omega(s) \, u_s$ for all $s \in \Lambda$, $\omega \in \widehat{K}$, we also have that $(\mathrm{Ad}\, \alpha(U_\omega))(\alpha(u_s)) = \omega(s) \, \alpha(u_s)$ for all $s \in \Lambda$, $\omega \in \widehat{K}$. Since $\alpha(U_\omega) = \Delta_*(U_\omega) \, V_\omega$ and V_ω belongs to the abelian algebra $\mathrm{L}\Lambda$, we conclude that $(\mathrm{Ad}\, \Delta_*(U_\omega))(\alpha(u_s)) = \omega(s) \, u_s$ for all $s \in \Lambda$, $\omega \in \widehat{K}$. On the other hand, since Δ is a δ-conjugation for the action $K \curvearrowright X$, we also have that $(\mathrm{Ad}\, \Delta_*(U_\omega))(u_k) = \omega(\delta^{-1}(k)) \, u_k$ for all $k \in \Lambda$ and $\omega \in \widehat{K}$. Writing the Fourier decomposition

$$\alpha(u_s) = \sum_{k \in \Lambda} \lambda_k^s \, u_k \quad \text{with} \;\; \lambda_k^s \in \mathbb{C} \; ,$$

we conclude that

$$\omega(s) \, \lambda_k^s = \omega(\delta^{-1}(k)) \, \lambda_k^s \quad \text{for all } s, k \in \Lambda, \, \omega \in \widehat{K}.$$

So, for every $s \in \Lambda$ there is precisely one $k \in \Lambda$ with $\lambda_k^s \neq 0$ and this element $k \in \Lambda$ moreover satisfies $\omega(s) = \omega(\delta^{-1}(k))$ for all $\omega \in \widehat{K}$. So, $k = \delta(s)$. In particular $\delta(s) \in \Lambda$ for all $s \in \Lambda$. So, $\delta(\Lambda) \subset \Lambda$. Since we can make a similar reasoning on $\alpha^{-1}(u_s)$, $s \in \Lambda$, it follows that $\delta(\Lambda) = \Lambda$.

□

3 A non commutative cocycle superrigidity theorem

We prove the following twisted version of Popa's cocycle superrigidity theorem [Po05, Theorem 0.1] for Bernoulli actions of property (T) groups.

Theorem 10. *Let K be a compact group with countable subgroup $\Lambda < K$. Let Γ be a property (T) group. Put $X = K^\Gamma$ and denote by $\Lambda \curvearrowright X$ the action by diagonal translation. Put $N = \mathrm{L}^\infty(X) \rtimes \Lambda$. Denote by $(\sigma_g)_{g\in\Gamma}$ the action of Γ on N such that σ_g is the Bernoulli shift on $\mathrm{L}^\infty(X)$ and the identity on $\mathrm{L}\Lambda$. Let $p \in \mathrm{L}\Lambda$ be a non zero projection.*

- *Assume that $q \in \mathrm{B}(\ell^2\mathbb{N}) \overline{\otimes} \mathrm{L}\Lambda$ is a projection, that $\gamma : \Gamma \to \mathcal{U}(q(\mathrm{B}(\ell^2\mathbb{N})\overline{\otimes}\mathrm{L}\Lambda)q)$ is a group homomorphism and that $v \in \mathrm{B}(\mathbb{C}, \ell^2(\mathbb{N})) \overline{\otimes} N$ is a partial isometry satisfying $v^*v = p$ and $vv^* = q$. Then the formula*
$$\omega_g := v^* \, \gamma_g \, \sigma_g(v)$$
defines a 1-cocycle for the action $(\sigma_g)_{g\in\Gamma}$ on pNp, i.e. a family of unitaries satisfying $\omega_{gh} = \omega_g \, \sigma_g(\omega_h)$ for all $g, h \in \Gamma$.
- *Conversely, every 1-cocycle for the action $(\sigma_g)_{g\in\Gamma}$ on pNp is of the above form with γ being uniquely determined up to unitary conjugacy in $\mathrm{B}(\ell^2\mathbb{N}) \overline{\otimes} \mathrm{L}\Lambda$.*

Note that if Λ is icc, we can take $q = p$, i.e. every 1-cocycle for the action $(\sigma_g)_{g\in\Gamma}$ on pNp is cohomologous with a homomorphism from Γ to $\mathcal{U}(p\mathrm{L}(\Lambda)p)$. If Λ is not icc, this is no longer true since q and $e_{11} \otimes p$ need not be equivalent projections in $\mathrm{B}(\ell^2\mathbb{N}) \overline{\otimes} \mathrm{L}\Lambda$, although they are equivalent in $\mathrm{B}(\ell^2\mathbb{N}) \overline{\otimes} N$.

Proof. It is clear that the formulae in the theorem define 1-cocycles. Conversely, let $p \in \mathrm{L}\Lambda$ be a projection and assume that the unitaries $\omega_g \in pNp$ define a 1-cocycle for the action $(\sigma_g)_{g\in\Gamma}$ of Γ on pNp. Consider the diagonal translation action $\Lambda \curvearrowright X \times K$ and put $\mathcal{N} := \mathrm{L}^\infty(X \times K) \rtimes \Lambda$. We embed $N \subset \mathcal{N}$ by identifying the element $Fu_\lambda \in N$ with the element $(F \otimes 1)u_\lambda \in \mathcal{N}$ whenever $F \in \mathrm{L}^\infty(X), \lambda \in \Lambda$. Also $(\sigma_g)_{g\in\Gamma}$ extends naturally to a group of automorphisms of $\mathcal{N}$ with $\sigma_g(1 \otimes F) = 1 \otimes F$ for all $F \in \mathrm{L}^\infty(K)$. Define $P = \mathrm{L}^\infty(K) \rtimes \Lambda$ and view P as a subalgebra of $\mathcal{N}$ by identifying the element $Fu_\lambda \in P$ with the element $(1 \otimes F)u_\lambda \in \mathcal{N}$ whenever $F \in \mathrm{L}^\infty(K), \lambda \in \Lambda$. Note that we obtained a commuting square

$$\begin{array}{ccc} N & \subset & \mathcal{N} \\ \cup & & \cup \\ \mathrm{L}\Lambda & \subset & P \end{array} .$$

Define $\Delta : X \times K \to X \times K : \Delta(x, k) = (k \cdot x, k)$ and denote by Δ_* the corresponding automorphism of $\mathrm{L}^\infty(X \times K)$ given by $\Delta_*(F) = F \circ \Delta^{-1}$. One checks easily that the formula

$$\Psi : \mathrm{L}^\infty(X) \overline{\otimes} P \to \mathcal{N} : \Psi(F \otimes Gu_\lambda) = \Delta_*(F \otimes G)u_\lambda \quad \text{for all } \; F \in \mathrm{L}^\infty(X), G \in \mathrm{L}^\infty(K), \lambda \in \Lambda$$

defines a $*$-isomorphism satisfying $\Psi \circ (\sigma_g \otimes \mathrm{id}) = \sigma_g \circ \Psi$ for all $g \in \Gamma$. Put $\mu_g := \Psi^{-1}(\omega_g)$. It follows that $(\mu_g)_{g\in\Gamma}$ is a 1-cocycle for the action $\Gamma \curvearrowright X$ with values in the Polish group $\mathcal{U}(pPp)$. By Popa's cocycle superrigidity theorem [Po05, Theorem 0.1] and directly applying Ψ again, we find a unitary $w \in \mathcal{U}(p\mathcal{N}p)$ and a group homomorphism $\rho : \Gamma \to \mathcal{U}(pPp)$ such that

$$\omega_g = w^* \, \rho_g \, \sigma_g(w) \quad \text{for all } \; g \in \Gamma \; .$$

Consider the basic construction for the inclusion $N \subset \mathcal{N}$ denoted by $\mathcal{N}_1 := \langle \mathcal{N}, e_N \rangle$. Put $T := w e_N w^*$. Since $\sigma_g(w) = \rho_g^* \, w \, \omega_g$, it follows that $\sigma_g(T) = \rho_g^* \, T \, \rho_g$. Also note that $T \in \mathrm{L}^2(\mathcal{N}_1)$, that $T = pT$ and that $\mathrm{Tr}(T) = \tau(p)$.

Since we are dealing with a commuting square, we can identify the basic construction $P_1 := \langle P, e_{\mathrm{L}\Lambda} \rangle$ for the inclusion $\mathrm{L}\Lambda \subset P$ with the von Neumann subalgebra of $\mathcal{N}_1$ generated by P and e_N. In this way, one checks that there is a unique unitary

$$W : \mathrm{L}^2(\mathcal{N}_1) \to \mathrm{L}^2(X) \otimes \mathrm{L}^2(P_1) \quad \text{satisfying} \;\; W(ab) = a \otimes b \quad \text{for all} \;\; a \in \mathrm{L}^\infty(X), b \in \mathrm{L}^2(P_1) \; .$$

The unitary W satisfies $W(\rho_g \, \sigma_g(\xi) \, \rho_g^*) = (\sigma_g \otimes \mathrm{Ad}\, \rho_g)(W(\xi))$ for all $g \in \Gamma$, $\xi \in \mathrm{L}^2(\mathcal{N}_1)$. The projection $T \in \mathrm{L}^2(\mathcal{N}_1)$ satisfies $T = \rho_g \, \sigma_g(T) \, \rho_g^*$. Since the unitary representation $(\sigma_g)_{g \in \Gamma}$ on $\mathrm{L}^2(X) \ominus \mathbb{C}1$ is a multiple of the regular representation, it follows that $W(T) \in 1 \otimes \mathrm{L}^2(P_1)$. This means that $T \in P_1$ and $\rho_g \, T = T \, \rho_g$ for all $g \in \Gamma$.

So we can view T as the orthogonal projection of $\mathrm{L}^2(P)$ onto a right $\mathrm{L}\Lambda$ submodule of dimension $\tau(p)$ that is globally invariant under left multiplication by ρ_g, $g \in \Gamma$. Since $pT = T$, the image of T is contained in $p\mathrm{L}^2(P)$. Take projections $q_n \in \mathrm{L}\Lambda$ with $\sum_n \tau(q_n) = \tau(p)$ and a right $\mathrm{L}\Lambda$-linear isometry

$$\Theta : \bigoplus_{n \in \mathbb{N}} q_n \mathrm{L}^2(\mathrm{L}\Lambda) \to \mathrm{L}^2(P)$$

onto the image of T. Denote by $w_n \in \mathrm{L}^2(P)$ the image under Θ of q_n sitting in position n. Note that

$$w_n = p w_n \;\; , \;\; E_{\mathrm{L}\Lambda}(w_n^* w_m) = \delta_{n,m} q_n \quad \text{and} \quad T = \sum_{n \in \mathbb{N}} w_n e_{\mathrm{L}\Lambda} w_n^*$$

where in the last formula we view T as an element of P_1. Identifying P_1 as above with the von Neumann subalgebra of $\mathcal{N}_1$ generated by P and e_N, it follows that

$$T = \sum_{n \in \mathbb{N}} w_n e_N w_n^* \; .$$

Since $T = w e_N w^*$, we get that $e_N = \sum_n w^* w_n e_N w_n^* w$. We conclude that $w^* w_n \in \mathrm{L}^2(N)$ for all n. So $w_n^* w_m \in \mathrm{L}^1(N)$ for all n, m. Since also $w_n^* w_m \in \mathrm{L}^1(P)$, we have $w_n^* w_m \in \mathrm{L}^1(\mathrm{L}\Lambda)$. But then,

$$\delta_{n,m} q_n = E_{\mathrm{L}\Lambda}(w_n^* w_m) = w_n^* w_m \; .$$

So, the elements w_n are partial isometries in P with mutually orthogonal left supports lying under p and with right supports equal to q_n. Since $\sum_n \tau(q_n) = \tau(p)$, we conclude that the formula

$$W := \sum_n e_{1,n} \otimes w_n$$

defines an element $W \in \mathrm{B}(\ell^2(\mathbb{N}), \mathbb{C}) \overline{\otimes} P$ satisfying $WW^* = p$, $W^*W = q$ where $q \in (\mathrm{L}\Lambda)^\infty = \mathrm{B}(\ell^2\mathbb{N}) \overline{\otimes} \mathrm{L}\Lambda$ is the projection given by $q := \sum_n e_{nn} \otimes q_n$. We also have that $v := W^* w$ belongs to $\mathrm{B}(\mathbb{C}1, \ell^2(\mathbb{N})) \overline{\otimes} N$ and satisfies $v^* v = p$, $vv^* = q$.

Finally, let $\gamma : \Gamma \to \mathcal{U}(q(\mathrm{L}\Lambda)^\infty q)$ be the unique group homomorphism satisfying

$$\Theta(\gamma_g \xi) = \rho_g \Theta(\xi) \quad \text{for all} \;\; g \in \Gamma, \xi \in \bigoplus_n q_n \mathrm{L}^2(\mathrm{L}\Lambda) \; .$$

By construction $W \gamma_g = \rho_g W$ for all $g \in \Gamma$. Since $\omega_g = w^* \, \rho_g \, \sigma_g(w)$ and since W is invariant under $(\sigma_g)_{g \in \Gamma}$, we conclude that $\omega_g = v^* \, \gamma_g \, \sigma_g(v)$.

We finally prove that γ is unique up to unitary conjugacy. So assume that we also have q_1, v_1 and $\varphi : \Gamma \to \mathcal{U}(q_1(\mathrm{L}\Lambda)^\infty q_1)$ satisfying $\omega_g = v_1^* \varphi_g \sigma_g(v_1)$. It follows that the element $v_1 v^* \in q_1 N^\infty q$ satisfies $\sigma_g(v_1 v^*) = \varphi_g^* v_1 v^* \gamma_g$ for all $g \in \Gamma$. As above this forces $v_1 v^*$ to belong to $q_1(\mathrm{L}\Lambda)^\infty q$, providing the required unitary conjugacy between γ and φ. □

4 Factors with many group measure space Cartan subalgebras

In this section we prove Theorem 4. We actually construct a concrete group G and a concrete uncountable family $G \curvearrowright (X_\mathcal{F}, \mu_\mathcal{F})$, indexed by all possible subsets $\mathcal{F} \subset \mathbb{N}$, of free ergodic p.m.p. actions that are non stably orbit equivalent and that have isomorphic II_1 factors.

The basic idea is due to [Po06b, Section 6.1] yielding the following concrete example of *two* group actions $H \curvearrowright (X_i, \mu_i)$, $i = 1, 2$, that are not orbit equivalent (although stably orbit equivalent) and yet have $\mathrm{L}^\infty(X_1) \rtimes H \cong \mathrm{L}^\infty(X_2) \rtimes H$. Let H_0 be a finite non commutative group. Define the infinite direct sum $H_1 := H_0^{(\mathbb{N})}$ with compactification $K := H_0^{\mathbb{N}}$. Consider the action $H_1 \curvearrowright K$ by left translation. Define $X_1 := K^{\mathrm{SL}(3,\mathbb{Z})}$ and consider the action of $H := \mathrm{SL}(3, \mathbb{Z}) \times H_1$ on X_1 where $\mathrm{SL}(3, \mathbb{Z})$ acts by Bernoulli shift and H_1 acts diagonally. Take the first copy of H_0 in H_1 and put $X_2 := X_1/H_0$. We have a natural action of $H \cong H/H_0$ on X_2. By construction $H \curvearrowright X_2$ is stably orbit equivalent (with compression constant $|H_0|^{-1}$) with $H \curvearrowright X_1$. Put $M = \mathrm{L}^\infty(X_1) \rtimes H$. In [Po06b, Section 6.1] it is shown that M has fundamental group $\mathbb{R}_+$, while the orbit equivalence relation of $H \curvearrowright X_1$ has trivial fundamental group. Hence $\mathrm{L}^\infty(X_1) \rtimes H \cong \mathrm{L}^\infty(X_2) \times H$ although $H \curvearrowright X_1$ and $H \curvearrowright X_2$ are not orbit equivalent.

We call two free ergodic p.m.p. actions *W*-equivalent* if their associated group measure space II_1 factors are isomorphic.

We now perform the following general construction. We start with two free ergodic p.m.p. actions $H \curvearrowright X_i$, $i = 1, 2$, that are not orbit equivalent but that are W*-equivalent, and we construct a group G with uncountably many non stably orbit equivalent but W*-equivalent actions.

Assume that Γ is a countable group and that $\Gamma_n < \Gamma$ is a sequence of subgroups of infinite index. Define the disjoint union $I := \bigsqcup_{n \in \mathbb{N}} \Gamma/\Gamma_n$ with the natural action $\Gamma \curvearrowright I$. To avoid trivialities, assume that $\Gamma \curvearrowright I$ is faithful, i.e. the intersection of all $g\Gamma_n g^{-1}$, $g \in \Gamma$, $n \in \mathbb{N}$, reduces to $\{e\}$. Define the generalized wreath product group

$$G := H \wr_I \Gamma = H^{(I)} \rtimes \Gamma .$$

Whenever $\mathcal{F} \subset \mathbb{N}$, define the space

$$X_\mathcal{F} := \prod_{n \in \mathcal{F}} X_1^{\Gamma/\Gamma_n} \times \prod_{n \notin \mathcal{F}} X_2^{\Gamma/\Gamma_n} .$$

Taking an infinite product of copies of the action $H \curvearrowright X_i$, we get an action $H^{(\Gamma/\Gamma_n)} \curvearrowright X_i^{\Gamma/\Gamma_n}$. Taking a further infinite product over these actions over $n \in \mathcal{F}$ (with $i = 1$) and $n \in \mathbb{N} - \mathcal{F}$ (with $i = 2$), we obtain an action of $H^{(I)}$ on $X_\mathcal{F}$. This action is compatible with the Bernoulli shift of Γ on $X_\mathcal{F}$. So we have constructed a free ergodic p.m.p. action

$$G \overset{\sigma_\mathcal{F}}{\curvearrowright} X_\mathcal{F} .$$

Denoting $M := \mathrm{L}^\infty(X_1) \rtimes H \cong \mathrm{L}^\infty(X_2) \rtimes H$, it is easily checked that

$$\mathrm{L}^\infty(X_\mathcal{F}) \rtimes G \cong \big(\bigotimes_{i\in I} M\big) \rtimes \Gamma \quad \text{where } \Gamma \text{ acts on } \bigotimes_{i\in I} M \text{ by shifting the indices.}$$

Hence all the actions $G \curvearrowright X_\mathcal{F}$ are W*-equivalent.

Theorem 4 is an immediate consequence of the following more concrete result, showing that under the right assumptions, the actions $\sigma_\mathcal{F}$ are mutually non stably orbit equivalent. We did not try to formulate the most general conditions under which such a result holds, but provide a concrete example of $\Gamma_n < \Gamma$ for which the proof is rather straightforward.

Proposition 11. *Consider the group actions $H \curvearrowright X_i$, $i = 1, 2$, defined above: $H \curvearrowright X_i$ are W*-equivalent, non orbit equivalent and H is a residually finite group. Define the field*

$$K := \mathbb{Q}(\sqrt{n} \mid n \in \mathbb{N})$$

by adding all square roots of positive integers to $\mathbb{Q}$. For all $n \geq 2$ we put $K_n := \mathbb{Q}(\sqrt{n})$. Define the countable group $\Gamma := \mathrm{PSL}(4, K)$ and the sequence of subgroups

$$\Gamma_n := \{aA \mid a \in K \ , \ A \text{ is an upper triangular matrix with entries in } K_n \ , \ \det(aA) = 1 \ \} \ .$$

As above we put $I = \bigsqcup_{n\geq 2} \Gamma/\Gamma_n$ and $G = H \wr_I \Gamma$. For every subset $\mathcal{F} \subset \mathbb{N} - \{0, 1\}$, we are given the action $\sigma_\mathcal{F}$ of G on $X_\mathcal{F}$.

The actions $\sigma_\mathcal{F}$ and $\sigma_{\mathcal{F}'}$ are stably orbit equivalent iff $\mathcal{F} = \mathcal{F}'$.

We freely make use of the following properties of $\Gamma_n < \Gamma$.

- Γ is a simple group. In particular, Γ has no non trivial amenable quotients.
- The subgroups Γ_n are amenable.
- If $n \geq 2$ and $g \in \Gamma - \Gamma_n$, then $g\Gamma_n g^{-1} \cap \Gamma_n$ has infinite index in Γ_n, i.e. the quasi-normalizer of Γ_n inside Γ equals Γ_n. Equivalently: Γ_n acts with infinite orbits on $\Gamma/\Gamma_n - \{e\Gamma_n\}$.
- If $n \neq m$ and $\delta \in \mathrm{Aut}\,\Gamma$, then $\delta(\Gamma_n) \cap \Gamma_m$ has infinite index in Γ_m. This follows because the automorphism group of Γ is generated by the inner automorphisms, the automorphisms given by automorphisms of the field K and the automorphism transpose-inverse, and because every field automorphism of K globally preserves every K_n.
- The previous two items imply that $(\mathrm{Stab}\, i) \cdot j$ is infinite for all $i \neq j$.

We also use the following lemma. It is a baby version of similar von Neumann algebra results (see [Po03, Theorem 4.1], [Io06, Theorem 0.1] and [IPV10, Theorem 4.2]). Our lemma is also closely related to [CSV09, Theorem 6.7]. We include a short and elementary proof for the convenience of the reader.

Lemma 12. *Let $\Gamma \curvearrowright I$ be any action of a countable group Γ by permutations of a countable set I. Let H be any countable group and put $G := H \wr_I \Gamma$. Assume that $\Lambda < G$ is a subgroup with the relative property (T). Then either a finite index subgroup of Λ is contained in $H \wr_I \mathrm{Stab}\, i$ for some $i \in I$, or there exists a $g \in G$ such that $g\Lambda g^{-1} \subset \Gamma$.*

Proof. Define the unitary representation $\pi : G \to \mathcal{U}(\ell^2(H \times I))$ given by

$$(\pi(h)\xi)(k,i) = \xi(h_i^{-1}k,i) \;\text{ if } h \in H^{(I)}, \text{ and } \quad (\pi(\gamma)\xi)(k,i) = \xi(k,\gamma^{-1}\cdot i) \;\text{ if } \gamma \in \Gamma.$$

Define the function

$$\eta : H \times I \to \mathbb{C} : \eta(k,i) = \begin{cases} 1 & \text{if } k = e, \\ 0 & \text{if } k \neq e. \end{cases}$$

The formal expression $b(g) := \pi(g)\eta - \eta$ provides a well defined 1-cocycle of G with values in $\ell^2(H \times I)$. Since $\Lambda < G$ has the relative property (T), the 1-cocycle b is inner on Λ. So we find $\xi \in \ell^2(H \times I)$ satisfying

$$b(\lambda) = \pi(\lambda)\xi - \xi \quad \text{for all } \; \lambda \in \Lambda \; .$$

Denote by $\delta_h : H \to \mathbb{C}$ the function that equals 1 in h and 0 elsewhere. We get that

$$\xi(k,i) = \xi(h_i^{-1}k,\gamma^{-1}\cdot i) - \delta_e(h_i^{-1}k) + \delta_e(k) \quad \text{for all } \; (h,\gamma) \in \Lambda \; , \; k \in H \; , \; i \in I \; .$$

For every $i \in I$, we write $\xi_i \in \ell^2(H)$ given by $\xi_i(k) = \xi(k,i)$. We also denote by $(\lambda_h)_{h \in H}$ the left regular representation of H on $\ell^2(H)$. The above formula becomes

$$\xi_i = \lambda_{h_i}\xi_{\gamma^{-1}\cdot i} - \delta_{h_i} + \delta_e \quad \text{for all } \; (h,\gamma) \in \Lambda \; , \; i \in I \; . \tag{4}$$

Since $\xi \in \ell^2(H \times I)$, it follows that $\|\xi_i\| \to 0$ as $i \to \infty$. Assume that finite index subgroups of Λ are never contained in $H \wr_I \operatorname{Stab} i$. We then find a sequence $(h_n,\gamma_n) \in \Lambda$ such that for all $i \in I$ we have that $\gamma_n^{-1} \cdot i \to \infty$ as $n \to \infty$. Denote by $V \subset \ell^2(H)$ the subset of vectors $V := \{\delta_e - \delta_h \mid h \in H\}$. Note that $V \subset \ell^2(\mathbb{N})$ is closed. We apply (4) to $(h_n,\gamma_n) \in \Lambda$ and let $n \to \infty$. Since $\|\xi_{\gamma_n^{-1}\cdot i}\| \to 0$, we conclude that $\xi_i \in V$ for every $i \in I$. Denote by $b_i \in H$ the unique element such that $\xi_i = \delta_e - \delta_{b_i}$. Since $\|\xi_i\| \to 0$ as $i \to \infty$, we conclude that $b_i = e$ for all but finitely many $i \in I$. So $b := (b_i)_{i \in I}$ is a well defined element of $H^{(I)}$. Formula (4) becomes

$$b_i^{-1}\, h_i \, b_{\gamma^{-1}\cdot i} = e \quad \text{for all } \; (h,\gamma) \in \Lambda \; , \; i \in I \; ,$$

which precisely means that $b^{-1}\Lambda b \subset \Gamma$. □

We can now prove Proposition 11. We use the terminology and conventions concerning stable orbit equivalences that we introduced after Remark 7.

Proof. Let $\mathcal{F}, \mathcal{F}' \subset \mathbb{N} - \{0,1\}$ be subsets. Assume that $\Delta : X_\mathcal{F} \to X_{\mathcal{F}'}$ is a stable orbit equivalence with corresponding Zimmer 1-cocycle

$$\omega : G \times X_\mathcal{F} \to G \quad , \quad \Delta(g \cdot x) = \omega(g,x) \cdot \Delta(x) \; \text{ for all } g \in G \text{ and a.e. } x \in X_\mathcal{F}.$$

Since both X_1 and X_2 are standard non atomic probability spaces, the restriction of the action $G \overset{\sigma_\mathcal{F}}{\curvearrowright} X_\mathcal{F}$ to Γ is the generalized Bernoulli action $\Gamma \curvearrowright [0,1]^I$. Note that its further restriction to the property (T) group $\Lambda := \mathrm{PSL}(4,\mathbb{Z}) < \Gamma$ is weakly mixing, since $\Lambda \cdot i$ is infinite for every $i \in I$. By Popa's cocycle superrigidity theorem [Po05, Theorem 0.1] we can replace Δ by a similar stable orbit equivalence and assume that $\omega(\lambda,x) = \delta(\lambda)$ for all $\lambda \in \Lambda$ and a.e. $x \in X_\mathcal{F}$, where $\delta : \Lambda \to G$ is a group homomorphism.

For all $i,j \in \{1,2,3,4\}$ denote by e_{ij} the matrix having 1 in position ij and 0 elsewhere. Define, for $i \neq j$ and $k \in K$, the elementary matrix $E_{ij}(k) = 1 + k e_{ij}$. The elementary subgroups $E_{ij}(K)$ generate Γ. Given distinct $i,j \in \{1,2,3,4\}$, denote by i',j' the remaining elements of $\{1,2,3,4\}$ and denote by Λ_{ij} the subgroup of Λ generated by $E_{i'j'}(\mathbb{Z})$ and $E_{j'i'}(\mathbb{Z})$. Note that Λ_{ij} is non amenable and that Λ_{ij} commutes with $E_{ij}(K)$. Since $\Lambda_{ij} \curvearrowright X_{\mathcal{F}}$ is weakly mixing and ω is a homomorphism on Λ_{ij}, [Po05, Proposition 3.6] implies that ω is a homomorphism on $E_{ij}(K)$. Since this holds for all $i \neq j$, we conclude that ω is a homomorphism on the whole of Γ. Denote this homomorphism by $\delta : \Gamma \to G$.

Since Γ is simple, δ is injective. Recall that $G = H \wr_I \Gamma$. We shall prove that there exists a $g \in G$ such that $g\delta(\Gamma)g^{-1} \subset \Gamma$. Denote by Γ_{12} the copy of $\mathrm{PSL}(2, K)$ inside Γ generated by $E_{12}(K)$ and $E_{21}(K)$. We first claim that there is no $i \in I$ such that $\delta(\Gamma_{12}) \subset H \wr_I \mathrm{Stab}\, i$. Assume the contrary. Denote by $\pi : G \to \Gamma$ the natural quotient map. Since Γ_{12} is simple and non amenable, $\pi(\delta(\Gamma_{12})) \subset \mathrm{Stab}\, i$ must be the trivial group. So, $\delta(\Gamma_{12}) \subset H^{(I)}$. Since H is residually finite and δ is injective, this is absurd. This proves the claim. Since Γ_{12} has no non trivial finite index subgroups, we only retain the existence of $\gamma_n \in \Gamma_{12}$ such that for all $i \in I$ we have that $\pi(\delta(\gamma_n)) \cdot i \to \infty$ as $n \to \infty$.

We next claim that there is no finite index subgroup $\Lambda_0 < \Lambda$ and $i \in I$ with $\delta(\Lambda_0) \subset H \wr_I \mathrm{Stab}\, i$. To prove this claim, assume the contrary. Since $\mathrm{Stab}\, i$ is amenable and Λ_0 has property (T), it follows that $\pi(\delta(\Lambda_0))$ is finite. So making Λ_0 smaller but still of finite index, we get that $\delta(\Lambda_0) \subset H^{(I)}$. Since Λ_0 is finitely generated, we can take $I_1 \subset I$ finite such that $\delta(\Lambda_0) \subset H^{(I_1)}$. Recall the subgroup $\Lambda_{12} < \Lambda$ that commutes with the subgroup $\Gamma_{12} < \Gamma$ that we considered in the previous paragraph. Also recall that we have found elements $\gamma_n \in \Gamma_{12}$ such that for all $i \in I$ we have that $\pi(\delta(\gamma_n)) \cdot i \to \infty$ as $n \to \infty$. Since $\delta(\Lambda_0 \cap \Lambda_{12}) \subset H^{(I_1)}$, we have

$$\delta(\gamma_n)\, \delta(\Lambda_0 \cap \Lambda_{12})\, \delta(\gamma_n)^{-1} \in H^{(\pi(\delta(\gamma_n))\cdot I_1)} \; .$$

Since γ_n and Λ_{12} commute, the left hand side equals $\delta(\Lambda_0 \cap \Lambda_{12})$. Letting $n \to \infty$, the sets $\pi(\delta(\gamma_n)) \cdot I_1$ and I_1 are eventually disjoint. We conclude that $\delta(\Lambda_0 \cap \Lambda_{12}) = \{e\}$. Since $\Lambda_0 < \Lambda$ has finite index, the group $\Lambda_0 \cap \Lambda_{12}$ is infinite, contradicting the injectivity of δ.

Combining the claim that we proved in the previous paragraph with Lemma 12, we can conjugate δ, replace Δ by a similar stable orbit equivalence and assume that $\delta(\Lambda) \subset \Gamma$. For all i, j we know that $\delta(\Lambda_{ij})$ is a non amenable subgroup of Γ. Therefore $\delta(\Lambda_{ij}) \cdot k$ is infinite for all $k \in I$ and the centralizer of $\delta(\Lambda_{ij})$ inside G is a subgroup of Γ. Therefore $\delta(E_{ij}(K)) \subset \Gamma$ for all i, j. Hence, $\delta(\Gamma) \subset \Gamma$.

We can perform a similar reasoning on the generalized inverse stable orbit equivalence $\overline{\Delta} : X_{\mathcal{F}'} \to X_{\mathcal{F}}$ and may assume that $\overline{\Delta}(g \cdot x) = \overline{\delta}(g) \cdot \overline{\Delta}(x)$ for all $g \in \Gamma$ and a.e. $x \in X_{\mathcal{F}'}$, where $\overline{\delta} : \Gamma \to \Gamma$ is an injective group homomorphism. By definition $\overline{\Delta}(\Delta(x)) \in G \cdot x$ for a.e. $x \in X_{\mathcal{F}}$. So we find a measurable function $\varphi : X_{\mathcal{F}} \to G$ such that $\overline{\Delta}(\Delta(x)) = \varphi(x) \cdot x$ for a.e. $x \in X_{\mathcal{F}}$. It follows that

$$\varphi(g \cdot x) = \overline{\delta}(\delta(g))\, \varphi(x)\, g^{-1} \quad \text{for all } \; g \in \Gamma \; \text{ and a.e. } \; x \in X_{\mathcal{F}} \; .$$

So the non negligible subset $\{(x, x') \in X_{\mathcal{F}} \times X_{\mathcal{F}} \mid \varphi(x) = \varphi(x')\}$ is invariant under the diagonal Γ-action. Since $\Gamma \curvearrowright X_{\mathcal{F}}$ is weakly mixing, we conclude that φ is essentially constant. This constant value necessarily belongs to Γ and we conclude that $\overline{\delta} \circ \delta$ is an inner automorphism of Γ. A similar reasoning holds for $\delta \circ \overline{\delta}$. In particular $\Delta(\Gamma \cdot x) = \Gamma \cdot \Delta(x)$ for a.e. $x \in X_{\mathcal{F}}$. Lemma 8 implies that Δ is a measure space isomorphism of $X_{\mathcal{F}}$ onto $X_{\mathcal{F}'}$ and that Δ is a δ-conjugacy of the respective Γ-actions. In particular, the compression constant of Δ equals 1.

Denote

$$X^i_{\mathcal{F}} := \begin{cases} X_1 & \text{if } \; i \in \Gamma/\Gamma_n \; \text{ and } \; n \in \mathcal{F} \; , \\ X_2 & \text{if } \; i \in \Gamma/\Gamma_n \; \text{ and } \; n \not\in \mathcal{F} \; . \end{cases}$$

By construction, $X_{\mathcal{F}} = \prod_{i \in I} X^i_{\mathcal{F}}$ and similarly for $X_{\mathcal{F}'}$.

By [PV09, Lemma 6.15] the δ-conjugacy Δ necessarily decomposes as a product of measure space isomorphisms. More precisely, we find a permutation η of the set I and measure space isomorphisms $\Delta_i : X^i_{\mathcal{F}} \to X^{\eta(i)}_{\mathcal{F}'}$ such that

$$\eta(g \cdot i) = \delta(g) \cdot \eta(i) \quad \text{and} \quad \Delta(x)_{\eta(i)} = \Delta_i(x_i) \quad \text{for all } g \in \Gamma \text{ and a.e. } x \in X_{\mathcal{F}}.$$

Since η preserves the orbits of $\Gamma \curvearrowright I$, it defines a permutation $\overline{\eta}$ of $\mathbb{N}$ such that $\eta(i) \in \Gamma/\Gamma_{\overline{\eta}(n)}$ iff $i \in \Gamma/\Gamma_n$. Also Δ_i only depends on the orbit of i, yielding $\Delta_n : X^i_{\mathcal{F}} \to X^{\eta(i)}_{\mathcal{F}'}$ whenever $i \in \Gamma/\Gamma_n$.

Note that $\delta(\operatorname{Stab} i) = \operatorname{Stab} \eta(i)$. For all $n \neq m$ and all $\delta \in \operatorname{Aut} \Gamma$ we have that $\delta(\Gamma_n) \neq \Gamma_m$. Hence $\overline{\eta}$ is the identity. Denote by H^i the copy of H in $H^{(I)}$ sitting in position i. Fix $n \in \mathbb{N}$ and $i \in \Gamma/\Gamma_n$. Put $j := \eta(i)$ and note that $j \in \Gamma/\Gamma_n$. Since the quasi-normalizer of Γ_n inside Γ equals Γ_n and since Γ_n acts with infinite orbits on Γ/Γ_m for all $m \neq n$, it follows that the action

$$\operatorname{Stab} i \curvearrowright \prod_{j, j \neq i} X^j_{\mathcal{F}}$$

is weakly mixing. Since the 1-cocycle ω is a homomorphism on $\operatorname{Stab} i$ and H^i commutes with $\operatorname{Stab} i$, it follows from [Po05, Proposition 3.6] that for all $h \in H^i$, the map $x \mapsto \omega(h, x)$ only depends on the coordinate x_i and that $\omega(h, x)$ commutes with $\delta(\operatorname{Stab} i) = \operatorname{Stab} j$. Since $(\operatorname{Stab} j) \cdot k$ is infinite for all $k \neq j$ and since $\operatorname{Stab} j < \Gamma$ has a trivial centralizer in Γ, it follows that $\omega(h, x) \in H^j$. A similar reasoning holds for the inverse 1-cocycle and we conclude that $\Delta_n : X^i_{\mathcal{F}} \to X^j_{\mathcal{F}'}$ is an orbit equivalence between $H \curvearrowright X^i_{\mathcal{F}}$ and $H \curvearrowright X^j_{\mathcal{F}'}$. Since both $i, j \in \Gamma/\Gamma_n$ and since the actions $H \curvearrowright X_1$ and $H \curvearrowright X_2$ are not orbit equivalent, it follows that either n belongs to both $\mathcal{F}$ and $\mathcal{F}'$ or that n belongs to both $\mathbb{N} - \mathcal{F}$ and $\mathbb{N} - \mathcal{F}'$. This holds for all n and we conclude that $\mathcal{F} = \mathcal{F}'$. □

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