On mean field solutions of kinetic exchange opinion models

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A solution for the discretised version of some recently proposed kinetic exchange opinion dynamics models with two-agent interactions (Lallouache et. al., Phys. Rev E **82**, 056112 (2010)) is provided. A generalisation to include three-agent interactions is proposed. A phase boundary, separating the ordered and disordered phase is obtained. It is shown that if the probability of three-agent interaction stays below a threshold, the phase transition is continuous in nature, while above the threshold it becomes discontinuous. The threshold is a tri-critical point on the phase boundary, having different exponent values from those of the continuous transition.

I. INTRODUCTION

The dynamics of opinion formation in a society and emergence of consensus are being extensively studied recently [1–8]. It is a standard practice to model such complex phenomena using the tools of statistical mechanics. Although many intricacies of real societies are lost in the process, such minimal modelling often yields intriguing features in terms of their social as well as physical aspects.

The key feature in modelling opinion formation is to quantify opinions in terms of real numbers. Depending on the need and variety of the model, opinion is often quantified as discrete or continuous variables between two or more choices. Also the process of interaction between the agents is a vital ingredient. While several choices to model such interactions exist, one way is to consider an interaction as a 'scattering process' where the agents are stochastically influenced by each others opinions (see e.g., [9–12]).

Recently an opinion formation model [13] based on such 'kinetic exchange process' between two individuals was proposed (LCCC model hereafter). Resembling the model for wealth exchange in a society [14], this model has a single parameter that determines the 'conviction' of an individual. It was shown that beyond a certain value of this conviction parameter, the society reaches a 'consensus', where one of the two choices (positive or negative) provided to the individuals prevails, thereby spontaneously breaking a discrete symmetry. The values of the opinions, however, are continuous in [-1,1].

A generalisation was proposed subsequently [15], in which the 'self-conviction' and the ability to influence others were taken as independent variables. This two-parameter model has a simple phase boundary along which apparent non-universality was observed (for detailed discussion on the critical behavior of a class of model of this kind see [16]). Subsequent extensions in terms of including "noise" [17] and also to study a generalised map-version [18] for this class of models were also done.

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In the present paper, a discretised version of the LCCC model and its generalised version are analysed in the mean-field limit, which is exact here. Also a generalisation in terms of three-agent interactions is proposed. From the expression of the order parameter it is seen that for pure three-agent interactions the transition is discontinuous (giving hysteresis like behavior as well) but for mixture of two-agent and three-agent interactions, the nature of transition depends on the relative probabilities of the two types (two-agents and three-agents) of interactions.

The paper is organised as follows: In the next section a description of the model and its generalised version are given with mention to the modifications made here. In the next section the mean-field calculations to find the expression for order parameter is presented. Then in sec. IV the three-agent generalisation is introduced with the analysis of the order of transition. In sec. V, the phase boundary obtained for the model with both two-agent and three-agent interactions are present. Finally the results are discussed.

II. MODEL

A very simple pair-wise interaction model for opinion formation in a well-connected group of individual is presented in Ref.[13]. The opinion of an individual in the society is represented by a real number which can continuously vary within the limit $-1 \leq o_i \leq +1$. At any time t an agent with opinion $o_i(t)$ interacts with another randomly chosen agent with opinion $o_j(t)$. After the interaction the i-th agent retains a fraction of his/her own opinion (which depends on the agent's 'conviction') and is stochastically influenced by the j-the agent. The amount of the influence, of course, depends upon the j-th agent's 'conviction'. The dynamics of the LCCC model evolves following the equation

$$o_i(t+1) = \lambda_i o_i(t) + \lambda_j \epsilon o_j(t). \quad (1)$$

The parameter λ_m represents the conviction of m-th agent and ϵ is a stochastic variable uniformly distributed between [0,1]. If the opinion of an agent reached the higher (lower) extreme $+1$ (-1), then of course its opinion value was prevented from further increase (decrease).

N such exchanges (where N is the total number of agents) constitute a single Monte Carlo time step. For simplicity, the society was assumed to be homogeneous in the sense that all λ_m's were same (say, λ).

The steady state characterisation of this model were done using two measures. One is the average opinion of the agents $O = \frac{1}{N}|\sum_{i=1}^{N} o_i|$ representing the measure of global consensus and the other is the fraction of agents having extreme opinions

$$C = \frac{1}{N}\sum_{i=1}^{N}\left[\delta(1-o_i)+\delta(1+o_i)\right], \tag{2}$$

where $\delta(x)$ is the Dirac-delta function.

Extensive Monte Carlo (MC) study [13] yields that in LCCC model, above $\lambda_c \approx \frac{2}{3}$, $O \neq 0$ and below λ_c, $O = 0$. Similar behavior was also obtained for C. As in usual critical phenomena, the relaxation time shows divergence from both sides of the critical point following a power-law, having same exponent value on either side of the criticality. Although nothing could be predicted about the critical behavior, a mean-field like analysis gave $\lambda_c = 2/3$ for LCCC model (for detailed discussions see Ref.[16]).

In its generalisation [15], it was argued that the 'self-conviction' λ of an agent need not, in general, be equal to his/her ability to 'influence' others. In its generalised form, therefore, the dynamical exchange process reads

$$o_i(t+1) = \lambda o_i(t) + \mu\epsilon o_j(t), \tag{3}$$

where μ represents the j-th agents ability to influence others. In the limit $\lambda = \mu$ one recovers LCCC. For this generalised model, there is a phase boundary in $\lambda - \mu$ plane, having the equation $\lambda_c = 1 - \frac{\mu_c}{2}$. The values of the critical exponents along this phase boundary was reported to have strongly non-universal behavior for O and weakly non-universal behavior for C.

The above mentioned models defy simple treatments to find the order parameter as long as o_i's are continuous. But it is often the case in a society that the opinions can take only discrete values (voting 'yes' or 'no' for a referendum, or that in a two-party political scenario etc.). While retaining the social interpretation, it significantly simplifies analytical treatment. To that end following modifications are made. For the LCCC model, the dynamical exchange equation (Eq.(1)) remains the same. But we make λ stochastic in the sense that we put $\lambda = 1$ with probability p and 0 with probability $1-p$. Also, the parameter ϵ is either 1 or 0 with equal probability.

Under these modifications, on one hand we lift the 'homogeneous society' (all agents having same 'conviction') assumption and on the other hand keep the opinion values discretised. However, the inhomogeneity is the simplest of its kind: only 'high' and 'low' convictions are present. The agents can change between these two states randomly in time (λ is annealed variable). The case of quenched λ, in this case, is a trivial limit where order parameter becomes simply proportional to p.

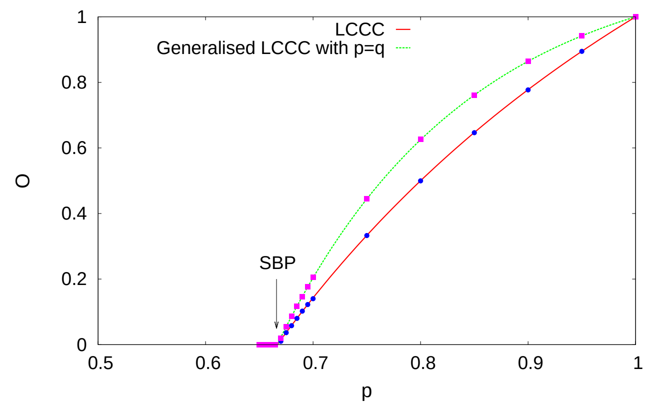

FIG. 1. Variation of the order parameter O for discrete LCCC model and its generalised version (in the limit $p = q$). Analytical expressions given by Eq. (7) and (9) are in good agreement with the simulation points. The critical or symmetry breaking point (SBP) $p_c = 2/3$ is indicated. For the simulation points, $N = 10^5$ system sizes were used.

In the case of the generalised version (Eq.(3)) the additional change is that like λ, we put $\mu = 1$ with probability q and 0 with probability $1-q$.

However, regarding its variation in time: it is explicitly checked numerically throughout the paper that the results do not change whether μ depends on time or not (at least in the MF limit). Therefore, to facilitate analytical treatment, it is assumed to be randomly varying with time. Then of course one could combine the two stochastic variables μ and ϵ in Eq.(3). That would only change a few pre-factors in the following calculations. However, to keep the formal similarity we do not combine them here.

Now, if the initial distribution of the opinion values are between $\pm 1, 0$, then the present modifications ensure that it will remain discretised within that limit. Here of course, the relevant parameters of the problem will be p and/or q, which essentially specify the average values of λ and μ respectively.

In the subsequent sections, a mean-field analysis of this modified-LCCC model and its generalisations to include three-agent terms are presented.

III. MEAN FIELD SOLUTION OF DISCRETE LCCC MODEL

It is our intention to find an expression for the order parameter O in terms of p and to find out the order parameter exponent β defined as $O \sim (p-p_c)^\beta$. Subsequently similar expression in terms of both p and q for the generalised model (see Eq.(3)) is also presented.

Let f_0, f_1 and f_{-1} be the fractions of agents having opinions $0, +1$ and -1 respectively. Now, since the interactions are only pair-wise and both λ and μ can take only two values, one can enumerate all possible interactions between all possible pairs, which contribute to increase

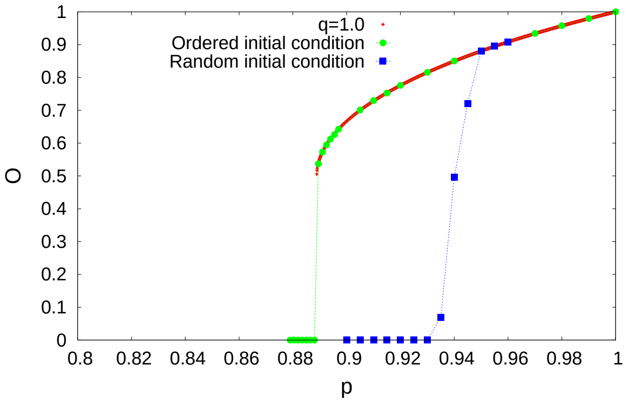

FIG. 2. Variation of the order parameter O for pure three agent interactions. When the initial condition is an ordered state, the discontinuous jump occurs at $p_{c1} = 8/9$ (see Eq.(19)), while starting from the random initial condition, the jump occurs at $p_{c2} \approx 0.930 \pm 0.005$. This clearly shows a hysteresis behavior as is expected for a discontinuous transition. System size is $N = 10^5$ for the simulation points.

and decrease of the order parameter. For example, the probability that both the agents in an exchange process have opinion +1 is f_1^2. Now the probability with which one agent shifts his/her opinion to 0 is $(1-p)$. Therefore the process $(1,1) \to (0,1)$ has probability $f_1^2(1-p)$. This process, of course, contributes in decreasing of the order parameter. One can enumerate all the eight processes that contribute in changing the order parameter. In the steady state one would expect that the terms contributing to increase and decrease should balance. Canceling some of the terms one finds the equation

$$\begin{aligned} & f_1^2(1-p) + f_0 f_{-1}\frac{p}{2} + f_0 f_1(1-p) \\ & = f_{-1}^2(1-p) + f_0 f_1 \frac{p}{2} + f_0 f_{-1}(1-p) \end{aligned} \tag{4}$$

This gives either $f_1 = f_{-1}$, (which implies disorder) or

$$f_0 = \frac{2(1-p)}{p}. \tag{5}$$

It is possible in this this case to show explicitly that in the ordered state agents with opinion +1 and −1 cannot coexist (making O and C identical in this and for all subsequent discussions also). Therefore, the order parameter should be

$$O = \pm(1 - f_0) \tag{6}$$

where the sign will depend on whether f_1 or f_{-1} is nonzero in the ordered (symmetry-broken) phase. On simplification, the above expression yields

$$O = \pm\frac{3(p-\frac{2}{3})}{p}. \tag{7}$$

Now the fact that opinion of only one sign exists in the ordered phase and that fraction goes continuously to zero in the disordered phase suggests that in the disordered phase $f_1 = f_{-1} = 0$ and therefore $f_0 = 1$. Note that the last condition along with Eq.(5) yields $p_c = \frac{2}{3}$. Therefore, Eq.(7) gives $\beta = 1$. In Fig.1, Eq.7 is compared with Monte Carlo simulations to find good agreement.

This can of course be generalised for Eq.(3). Slightly more involved algebra would yield

$$f_0 = \frac{(p-1)(q-2)}{pq}. \tag{8}$$

As before, in the disordered phase $f_0 = 1$, which yields the equation for the phase boundary in the p-q plane as $p_c = 1 - \frac{q_c}{2}$. This gives the expression for order parameter as

$$O = \pm\frac{2(p-p_c) + (q-q_c)}{pq}. \tag{9}$$

Therefore, no matter through which path and which point the phase boundary is crossed, the order parameter exponent is $\beta = 1$. The discretised version of LCCC model presented here belongs to the Directed Percolation (DP) universality class [19]. Accordingly $\beta = 1$ is obtained. Other exponents (not shown) also agree with this fact.

Of course we do not expect to get Eq.(7) from Eq.(9) by putting $p = q$, as this would only mean $\langle\lambda\rangle = \langle\mu\rangle$ and not $\lambda = \mu$.

IV. BEYOND PAIR-WISE INTERACTIONS: THREE-AGENT INTERACTION AND FIRST ORDER TRANSITION

In all previous studies regarding the kinetic exchange processes mentioned here, interactions were always taken to be occurring between two agents. This is partly because two-body exchange is the simplest and also because in the energy exchange of ideal gas too only two body interactions are important. But in opinion formation, exchange between more than two agents is perfectly possible. So we intend to investigate the effect of such interactions in opinion formation.

The simplest possible generalisation towards many-body interaction is to consider three-body exchange. In doing so, the following strategy is followed. Three agents are chosen randomly. Then one agent modifies his/her opinion according to that of the other two only when the other two agrees among themselves. If they do not agree the first agent considers the group to be 'neutral'. Mathematically this can be represented as

$$o_i(t+1) = \lambda o_i(t) + \lambda\epsilon\theta_{jk}(t), \tag{10}$$

where,

$$\begin{aligned} \theta_{jk}(t) &= o_j(t) \quad \text{if} \quad o_j(t) = o_k(t) \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{11}$$

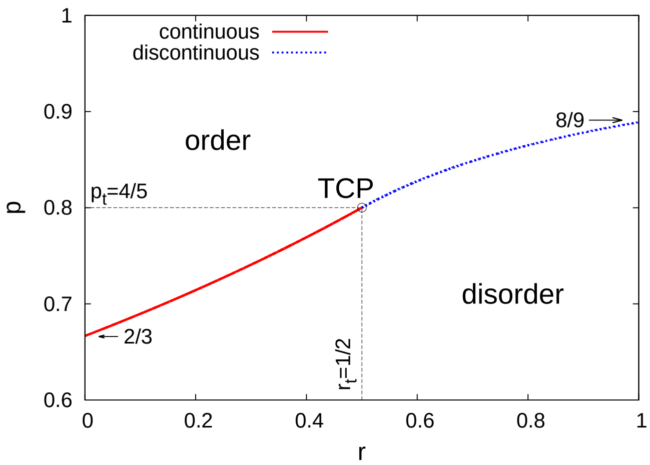

FIG. 3. The phase diagram of the model with mixture of three agent and two agent interactions in the q-r plane, where r denotes the fraction of three agent interactions. The continuous transition line follows Eq.(20), while the discontinuous transition line follows Eq.(21). Clearly, in the limits $r = 0$ and 1 the transition points are 2/3 and 8/9 respectively, as is expected from the discussions in the text.

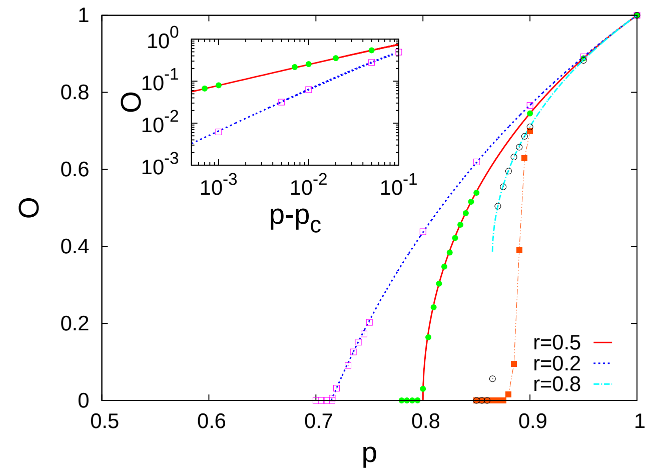

FIG. 4. Variations of the order parameter for $r = 0.2(< r_t)$, $r = 0.5(= r_t)$ and $r = 0.8(> r_t)$. The continuous lines are analytical results (see Eq.(19)) and the points are simulation results with $N = 10^5$. For the first two curves, continuous transitions are seen, as expected. For the last one ($r = 0.8$) a discontinuous transition with signature of hysteresis is seen. Inset shows the log-log plots of O versus $p - p_c$ near the transition points for the continuous transitions. From the slopes of the curves the order parameter exponent β is found to be 1 and 1/2 for $r = 0.2$ and $r = 0.5$ respectively.

Before proceeding further, it is to be noted that $\theta_{jk}(t)$ takes the value $+1, -1$ and 0 with probabilities f_1^2, f_{-1}^2 and $1 - (f_1^2 + f_{-1}^2)$ respectively. Then just as Eq.(1) was treated, one can enumerate all exchange processes that contribute to increase and decrease in the order parameter. Again, in the steady state increase and decrease should balance. As before, in the ordered state opinion of only one sign exists (numerically verified). This makes any term like $f_1^x f_{-1}^y$ (with $x, y \neq 0$) vanish in any state. With this simplification we get

$$\begin{aligned} & f_1^3(1-p) + f_1 f_{-1}^2\left(1-\frac{p}{2}\right) + f_0 f_{-1}^2\frac{p}{2} \\ & + f_1\left[1-(f_1^2+f_{-1}^2)\right](1-p) \\ & = f_{-1}^3(1-p) + f_{-1} f_1^2\left(1-\frac{p}{2}\right) + f_0 f_1^2\frac{p}{2} + \\ & f_{-1}\left[1-(f_1^2+f_{-1}^2)\right](1-p). \end{aligned} \quad (12)$$

This gives either $f_1 = f_{-1}$, (which implies disorder) or

$$f_0 = \frac{1}{2} - \frac{3\sqrt{p-8/9}}{2\sqrt{p}}, \quad (13)$$

(implying order). We have neglected one solution of f_0 in which it increases in the ordered phase. Using this, the order parameter takes the form

$$O = \pm\left(\frac{1}{2} + \frac{3\sqrt{p-8/9}}{2\sqrt{p}}\right). \quad (14)$$

Clearly, the above equation gives real values for O only when $p > 8/9$. Therefore, for $p < 8/9$ the only real solution can be $f_1 = f_{-1}$ i.e., $O = 0$. But from the form of Eq.(14) it is clear that in the ordered phase, the minimum value of O can be 1/2. Therefore, the order-disorder transition is necessarily discontinuous. To verify this further MC simulations were performed. Depending on the initial condition, the discontinuous jump from order to disorder happen at two different points, thus showing hysteresis behavior (see Fig.2). When the initial condition is ordered, Eq.(14) is followed upto $p_{c1} = 8/9$. After that the order parameter jumps to zero. On the other hand, when the initial condition is random (having almost equal number of agents having opinions of opposite signs) then $O = 0$ upto $p_{c2} \approx 0.930 \pm 0.005$ and then suddenly jumps to the ordered (symmetry broken) phase. Of course p_{c2} is the symmetry breaking point. Note that the estimation of p_{c2} is entirely numerical here. To be absolutely sure about occurrence of hysteresis, one particular point ($p = 0.91$) is checked for large enough size ($N = 10^6$) for two different initial conditions. It is clearly seen then the long time saturation values are very different, as is expected.

Hysteresis in opinion formation has been studied before in different contexts. For example, in Ref.[20], hysteresis was observed while modelling the influence of a strong leader in the society. In general, hysteresis signifies the resistance offered by the society to changes in the global opinion, despite the fact that the very reason for its formation has lost its relevance. In the present case too, the hysteresis loop area is somewhat a measure of this ‘social tolerance’.

Similar exercise can be made for the generalised case described by Eq.(3). However, instead of doing so, one could also look at another limit of Eq.(3), where $q = 1$ (studied in [16] as model C). One can show that even in this limit, a discontinuous transition can be obtained with $p_{c1} = (2+\sqrt{2})/4$. So one would in general expect a discontinuous transition for all ranges of Eq.(3).

V. MIXTURE OF TWO-AGENT AND THREE-AGENT INTERACTIONS: PHASE DIAGRAM AND TRI-CRITICAL POINT

Let us now discuss how robust is this discontinuous transition. In the above analysis it was assumed that only three-agent interactions are present as opposed to the previous cases, where only two-agent interactions were considered. Here we consider a situation where both two-agent and three-agent interactions are allowed. In principle interactions of all sizes should be allowed, but this is the simplest generalisation one could make.

With probability r an exchange process is three-agent and otherwise it is two-agent. The exchange equation is same as Eq.(10) but now clearly

$$\begin{aligned} \theta_{jk}(t) &= 1 \quad \text{with probability} \quad rf_1^2 + (1-r)f_1 \\ &= -1 \quad \text{with probability} \quad rf_{-1}^2 + (1-r)f_{-1} \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{15}$$

With this, one may enumerate all possibilities of increase and decrease of the order parameter and in the steady state it must balance:

$$\begin{aligned} & f_1\left[rf_1^2+(1-r)f_1\right](1-p)+\frac{f_0 p}{2}\left[rf_{-1}^2+(1-r)f_{-1}\right] \\ & +f_1\left[1-(f_1^2+f_{-1}^2)r-(1-r)(f_1+f_{-1})\right](1-p) \\ & = f_{-1}\left[rf_{-1}^2+(1-r)f_{-1}\right](1-p)+\frac{f_0 p}{2}\left[rf_1^2+(1-r)f_1\right] \\ & +f_{-1}\left[1-(f_1^2+f_{-1}^2)r-(1-r)(f_1+f_{-1})\right](1-p). \end{aligned} \tag{16}$$

On simplification, this yields either $f_1 = f_{-1}$ which implies disorder, or

$$\frac{pr}{2}f_0^2 - \frac{pf_0}{2} + 1 - p = 0, \tag{17}$$

which gives (the only relevant solution)

$$f_0 = \frac{1}{2r} - \frac{\sqrt{p^2/4 - 2pr(1-p)}}{pr}. \tag{18}$$

Again as before

$$O = \pm\left(\frac{2r-1}{2r} + \frac{\sqrt{p^2/4-2pr(1-p)}}{pr}\right). \tag{19}$$

The first term in the right hand side is negative as long as $r < 1/2$. But the term inside the bracket has to be positive, as it is the magnitude of the order parameter (the sign will depend on which opinion prevails in the ordered state). So, for $r < 1/2$ the transition occurs only when by increasing p the term within the bracket has a positive value. Before that, the other solution i.e., the disorder-state solution will be stable. One can show that the condition of validity of the ordered-state solution gives the critical line

$$p_c = \frac{2}{3-r_c} \qquad \text{for} \quad r < \frac{1}{2}. \tag{20}$$

Across this line a continuous transition takes place. In Eq.(19) one can put $p = 2/(3-r) + \Delta$ (where $\Delta \to 0$) and show that the leading order term comes out to be linear in Δ, implying $\beta = 1$ along this line. This critical line, of course, terminates at $(p_t = \frac{1}{2}, r_t = \frac{1}{2})$. When $r > 1/2$, the ordered-state solution Eq. (19) can be valid whenever it gives real values for O. The last condition gives the phase boundary:

$$p_c = \frac{8r_c}{1+8r_c} \qquad \text{for} \quad r > \frac{1}{2}. \tag{21}$$

When $r > 1/2$, the minimum value possible for O from the ordered state solution (Eq.(19)) is greater then zero. Therefore transition across this line is necessarily discontinuous. This discontinuous nature is verified numerically. A ‘hysteresis’ like behavior, as discussed in the previous section, is also seen. The amount of discontinuity, of course, is given by $1-\frac{1}{2r}$, which is maximum (1/2) for pure three agent interactions ($r = 1$). Note that the phase boundary equations correctly give $p_c = 2/3$ and $p_c = 8/9$ limits respectively for $r = 0$ (from Eq.(20)) and $r = 1$ (from Eq.(21)).

The point $(p_t = \frac{1}{2}, r_t = \frac{1}{2})$ is special where the critical line terminates. It is a Tricritical Point (TCP). As is seen generally, at TCP the exponent values are different. Clearly,

$$O \sim \sqrt{p - 4/5} \tag{22}$$

giving $\beta_{TCP} = 1/2$, which is different from $\beta = 1$ found along the critical line.

To find the other exponent values that characterize this TCP, one can use the following scaling relation for the order parameter

$$O(t) \approx t^{-\delta}\mathcal{F}\left(t^{1/\nu_\parallel}\Delta, t^{d/z}/N\right), \tag{23}$$

where $\Delta = p - p_c$, $\nu_\parallel$ is the time-correlation exponent, z is the dynamical exponent and d is the space dimension. At the critical point, the order parameter follows a power-law relaxation $O(t) \sim t^{-\delta}$ (see inset of Fig.6) with $\delta = 0.50 \pm 0.01$.

One could then plot $O(t)t^{\delta}$ against $t(p-p_c)^{\nu_\parallel}$. By knowing δ, $\nu_\parallel$ can be tuned to find data collapse. From Fig.5 the estimate of $\nu_\parallel$ is 1.00 ± 0.01. Similarly, one can plot $O(t)t^{\delta}$ against $t/N^{z/d}$. Again by tuning z, data collapse is found (see Fig.6). The estimate of z/d is 0.666 ± 0.001. To find z one should put $d = 4$, which is the upper critical dimension. This gives $z \approx 8/3$. Similar analysis for $r < \frac{1}{2}$ gives $z/d \approx 1/2$, here also by putting $d = 4$ one gets $z \approx 2$ which is expected for DP.

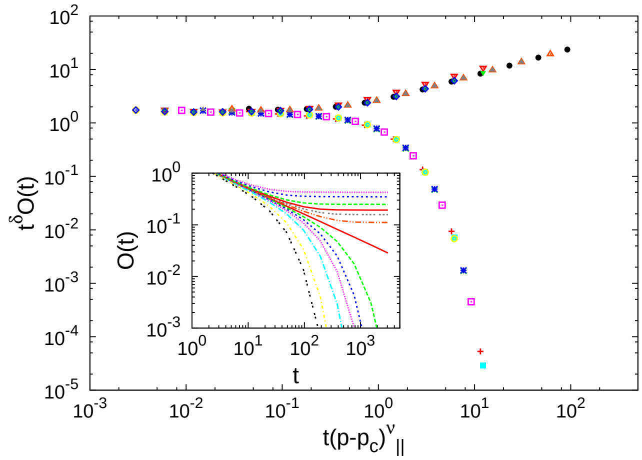

FIG. 5. Data collapse for finding $\nu_{||}$ (see Eq.(23)) for different p values for $r = 0.5$ ($p_c = 0.5$). The estimate is $\nu_{||} = 1.00 \pm 0.01$. Inset shows the uncollapsed data. System size is $N = 10^5$.

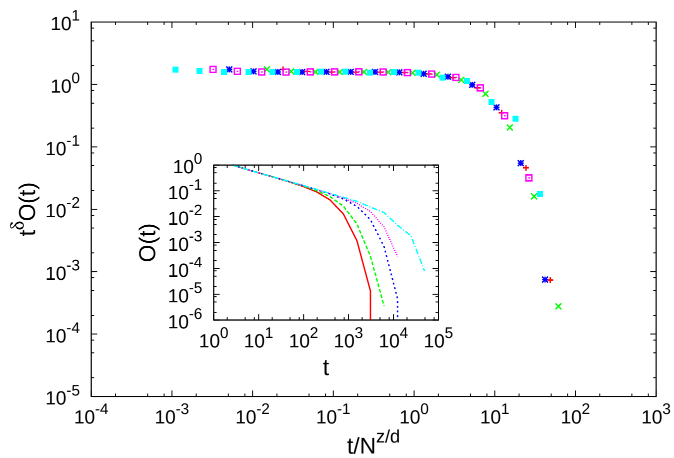

FIG. 6. Data collapse for finding z for different system sizes ($N = 500, 1000, 5000, 10000, 50000$) at $p = 0.5$ and $r = 0.5$ (TCP). The estimate is $z/d = 0.666 \pm 0.001$. Inset shows the uncollapsed data.

The scaling relation $\delta = \beta/\nu_{||}$ is clearly satisfied here.

Therefore we see that the nature of the transition is actually determined by the relative probabilities of the two-agent and three-agent interactions. Also the exponent values at the tri-critical point are different from those along the critical line.

VI. DISCUSSION

In the first part of this paper a mean-field solution of the discretised version of a recently proposed model [13] for opinion dynamics is given. The interactions, as in its simplest form, are between two agents (see Eq.(1)). The exchange process is such that an agent has a 'conviction' with which he/she retains his/her opinion and also gets influenced (stochastically, because it is otherwise impossible to incorporate all social complexities involved in such processes) by the opinion of one randomly chosen agent. It was shown [13, 15] from extensive MC study that beyond a certain value of the 'conviction' parameter the society undergoes a phase transition from disordered to ordered state (where consensus is formed). In the present study that behavior is shown analytically (see Eq.(7)) for a discretised version of the model in mean-field limit (which is exact here). The order parameter exponent has been found to be $\beta = 1$. Even for the generalised version [15] (see Eq.(3)) this exponent remains same along the phase boundary (belonging to DP universality class).

Thereafter a generalisation of this model for three-agent interaction is reported. There is, of course, no single choice for this kind of generalisation. But here we have taken a plausible strategy in which an agent can be influenced by the opinions of two other randomly chosen agents only when those two agents agree among themselves (have same opinion) otherwise the first agent takes the group as 'neutral' (see Eq.(10)). This generalisation has led to an interesting behavior it terms of the order of the transition. It is seen if all interactions are three-agent, a discontinuous transition is obtained (see Eq.(14)) and a hysteresis loop was also observed (Fig.2). It is to be noted that hysteresis in opinion models have been reported before in other contexts (see e.g, [20, 21]). In general, hysteresis in opinion dynamics models can be taken as a signature of the tolerance of the society, or in other words, its resistance to changes in global opinion (as is also indicated in [20]). Although a direct correspondence to a measurable quantity cannot be made from these simplified models, qualitatively this hysteresis loop-area is somewhat a 'measure' of this social 'tolerance' mentioned above.

It is important to find out how far this discontinuous nature is generic or it is an artifact of the restriction of only three-agent interaction, as invoked by Eq.(10). Of course it is not possible (at least very difficult) to allow interactions of all sizes as it should be in a real society. But to the very least one can allow both two-agent and three-agent interactions with some probabilities. In doing so it is found that upto the point when the probability of three-agent interactions is below 1/2, the transition is continuous (phase boundary given by Eq.(20)) and beyond that the transition is discontinuous, phase boundary is given by Eq.(21). The point where the two-agent and three-agent interactions are equally probable, is a special point, because it is a tri-critical point. The transition here is continuous, but the values of the exponents are different from those along the critical line. Along the critical line, the exponents are of course mean-field DP ($\beta = 1$, $z = 2$, $\nu_{||} = 1$, $\delta = 1$) and at TCP they are $\beta = 1/2$, $z/d \approx .666 \pm 0.001 \approx 2/3$ ($z \approx 8/3$), $\delta = 0.50 \pm 0.01 \approx 1/2$, $\nu_{||} \approx 1.00 \pm 0.01$.

At this point it is appropriate to mention that the 'disordered' phase in all these versions is quite special in the sense that all the agents are neutral in this phase. One can avoid such situation (see [17]) and make the 'disordered phase' have coexistence of opinions of different signs. But here such generalisations were not discussed. Even without such complexities, which one can add anyway with this, one finds intriguing features in this model.

One may note that while attempting to interpolate between a continuous and discontinuous transition, a tricritical point was obtained also in Ref.[22]. There too, the tricritical point was situated at the point where the phase boundary changed its curvature.

Finally, one must also note that in all the above cases, the *existence* of a phase transition has not been proved. What are analytically obtained are the two solutions representing ordered and disordered states. The existence of the phase transition, for all practical purposes, has been assumed here.

To conclude, a mean field solution for a kinetic exchange model of opinion formation and its phase transition in terms of forming global consensus is presented here. Its three-agent generalisation is proposed. Surprisingly, the nature of the transition depends on the relative probabilities of the two-agent and three-agent interactions.

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[1] *Econophysics and Sociophysics*, edited by B. K. Chakrabarti, A. Chakraborti, A. Chatterjee (Wiley-VCH, Berlin, 2006).
[2] D. Stauffer, in *Encyclopedia of Complexity and Systems Science*, edited by R. A. Meyers (Springer, New York, 2009); Comp. Sc. Engg. **5**, 71 (2003).
[3] C. Castellano, S. Fortunato, V. Loreto, Rev. Mod. Phys. **81**, 591 (2009).
[4] S. Galam, Y. Gefen, Y. Shapir, J. Math. Sociology **9**, 1 (1982).
[5] T. M. Liggett, *Interacting Particle Systems: Contact, Voter and Exclusion Processes* (Springer-Verlag, Berlin, 1999).
[6] K. Sznajd-Weron, J. Sznajd, Int. J. Mod. Phys. C **11**, 1157 (2000).
[7] S. Galam, Int. J. Mod. Phys. C **19**, 409 (2008).
[8] S. Biswas, P. Sen, Phys. Rev. E **80**, 027101 (2009).
[9] R. Hegselman, U. Krause, J. Art. Soc. Soc. Simul. **5**, 2 (2002).
[10] G. Deffuant, N. Neau, F. Amblard, G. Weisbuch, Adv. Complex Sys. **3**, 87 (2000).
[11] S. Fortunato, Int. J. Mod. Phys. C **16**, 17 (2005).
[12] G. Toscani, Comm. Math. Sc. **4**, 481 (2006).
[13] M. Lallouache, A. S. Chakrabarti, A. Chakraborti, B. K. Chakrabarti, Phys. Rev. E **82**, 056112 (2010).
[14] A. Chakraborti, B. K. Chakrabarti, Eur. Phys. J. B **17**, 167 (2000); A. Chatterjee, B. K. Chakrabarti, S. S. Manna, Physica A **335**, 155 (2004); A. Chatterjee, B. K. Chakrabarti, Eur. Phys. J. B **60**, 135 (2007); A. Chatterjee, Eur. Phys. J. B **67**, 593 (2009).
[15] P. Sen, Phys. Rev. E **83**, 016108 (2011).
[16] S. Biswas, A. K. Chandra, A. Chatterjee, B. K. Chakrabarti, J. Phys.: Conf. Ser. **297**, 012004 (2011).
[17] S. Biswas, A. Chatterjee, P. Sen, arXiv:1102.0902v2 (2011).
[18] A. S. Chakrabarti, arXiv:1101.3617 (2011).
[19] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[20] K. Kacpersky, J. A. Hołyst, J. Stat. Phys. **84**, 169 (1996); J. A. Hołyst, K. Kacpersky, F. Schweitzer, Physica A: Statistical Mechanics and its Applications **285**, 199 (2000).
[21] V. Schwämmle, M. C. González, A. A. Moreira, J. S. Andrade, Jr., H. J. Herrmann, Phys. Rev. E **75**, 066108 (2007).
[22] Nuno A. M. Araújo, José S. Andrade, Jr., Robert M. Ziff, Hans J. Herrmann., Phys. Rev. Lett. **106**, 095703 (2011).