TRIVIAL ZEROS OF p-ADIC L-FUNCTIONS AT NEAR CENTRAL POINTS

Denis Benois

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Using the $\mathscr{L}$-invariant constructed in our previous paper [Ben2] we proof a Mazur-Tate-Teitelbaum style formula for derivatives of p-adic L-functions of modular forms at near central points

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Introduction

0.1. Trivial zeros of modular forms. In this paper we prove a Mazur-Tate-Teitelbaum style formula for the values of derivatives of p-adic L-functions of modular forms at near central points. Together with the results of Kato-Kurihara-Tsuji and Greenberg-Stevens on the Mazur-Tate-Teitelbaum conjecture this gives a complete proof of the trivial zero conjecture formulated in [Ben2] for elliptic

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modular forms. Namely, let $f=\sum\limits_{n=1}^{\infty}a_nq^n$ be a normalized newform on $\Gamma_0(N)$ of weight $k\geqslant 2$ and character ε and let $L(f,s)=\sum\limits_{n=1}^{\infty}a_nn^{-s}$ be the complex L-function associated to f. It is well known that $L(f,s)$ converges for $\mathrm{Re}(s)>\frac{k+1}{2}$ and decomposes into an Euler product

$$L(f,s)=\prod_l E_l(f,l^{-s})^{-1}$$

where l runs over all primes and $E_l(f,X)=1-a_lX+\varepsilon(l)l^{k-1}X^2$. Moreover $L(f,s)$ has an analytic continuation on the whole complex plane and satisfies the functional equation

$$(2\pi)^{-s}\,\Gamma(s)\,L(f,s)=i^kcN^{k/2-s}(2\pi)^{s-k}\,\Gamma(k-s)\,L(f^*,k-s)$$

where $f^*=\sum\limits_{n=1}^{\infty}\bar{a}_nq^n$ is the dual cusp form and c is some constant (see for example [Mi], Theorems 4.3.12 and 4.6.15). More generally, to any Dirichlet character η we can associate the L-function

$$L(f,\eta,s)=\sum_{n=1}^{\infty}\frac{\eta(n)\,a_n}{n^s}.$$

The theory of modular symbols implies that there exist non-zero complex numbers Ω_f^+ and Ω_f^- such that for any Dirichlet character η one has

$$\widetilde{L}(f,\eta,j)=\frac{\Gamma(j)}{(2\pi i)^{j-1}\,\Omega_f^{\pm}}\;L(f,\eta,j)\in\overline{\mathbb{Q}},\qquad 1\leqslant j\leqslant k-1 \tag{1}$$

where $\pm=(-1)^{j-1}\eta(-1)$. Fix a prime number $p>2$ such that the Euler factor $E_p(f,X)$ is not equal to 1. Let α be a root of the polynomial $X^2-a_pX+\varepsilon(p)p^{k-1}$ in $\overline{\mathbb{Q}}_p$. Assume that α is not critical i.e. that $v_p(\alpha)<k-1$. Let $\omega\,:\,(\mathbb{Z}/p\mathbb{Z})^*\to\mathbb{Q}_p^*$ denote the Teichmüller character. Manin [Mn], Vishik [Vi] and independently Amice-Velu [AV] constructed analytic p-adic L-functions $L_{p,\alpha}(f,\omega^m,s)$ which interpolate algebraic parts of special values of $L(f,s)$[1]. Namely, the interpolation property writes

$$L_{p,\alpha}(f,\omega^m,j)=\mathcal{E}_\alpha(f,\omega^m,j)\,\widetilde{L}(f,\omega^{j-m},j),\qquad 1\leqslant j\leqslant k-1$$

where $\mathcal{E}_\alpha(f,\omega^m,j)$ is an explicit Euler like factor. One says that $L_{p,\alpha}(f,\omega^m,s)$ has a trivial zero at $s=j$ if $\mathcal{E}_\alpha(f,\omega^m,j)=0$. This phenomenon was first studied by Mazur, Tate and Teitelbaum in [MTT] where the following cases were distinguished:

• *The semistable case: $p\parallel N$, k is even and $\alpha=a_p=p^{k/2-1}$.* The p-adic L-function $L_{p,\alpha}(f,\omega^{k/2},s)$ has a trivial zero at the central point $s=k/2$.

• *The crystalline case: $p\nmid N$, k is odd and either $\alpha=p^{\frac{k-1}{2}}$ or $\alpha=\varepsilon(p)\,p^{\frac{k-1}{2}}$.* The p-adic L-function $L_{p,\alpha}(f,\omega^{\frac{k+1}{2}},s)$ (respectively $L_{p,\alpha}(f,\omega^{\frac{k-1}{2}},s)$) has a trivial zero at the near central point $s=\frac{k+1}{2}$ (respectively $s=\frac{k-1}{2}$).

• *The potentially crystalline case: $p\mid N$, k is odd and $\alpha=a_p=p^{\frac{k-1}{2}}$.* The p-adic L-function $L_{p,\alpha}(f,\omega^{\frac{k-1}{2}},s)$

[1] This construction was recently generalized to the critical case by Pollack-Stevens [PS] and Bellaiche [Bel]

has a trivial zero at the near central point $s=\frac{k-1}{2}$.

0.2. The semistable case. Let

$$\rho_f\,:\,\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to \mathrm{GL}(W_f).$$

be the p-adic Galois representation associated to f by Deligne [D1]. Assume that k is even, $p\parallel N$ and $a_p=p^{k/2-1}$. Then the restriction of ρ_f on the decomposition group at p is semistable and non-crystalline in the sense of Fontaine [Fo3]. The associated filtered (φ,N)-module $\mathbf{D}_{\mathrm{st}}(W_f)$ has a basis e_α, e_β such that $e_\alpha=Ne_\beta$, $\varphi(e_\alpha)=a_pe_\alpha$ and $\varphi(e_\beta)=p\,a_pe_\beta$. The jumps of the canonical decreasing filtration of $\mathbf{D}_{\mathrm{st}}(W_f)$ are 0 and $k-1$ and the $\mathscr{L}$-invariant of Fontaine-Mazur is defined to be the unique element $\mathscr{L}(f)\in\overline{\mathbb{Q}}_p$ such that $\mathrm{Fil}^{k-1}\mathbf{D}_{\mathrm{st}}(V_f)$ is generated by $e_\beta-\mathscr{L}(f)e_\alpha$. In [MTT] Mazur, Tate and Teitelbaum conjectured that

$$L_{p,\alpha}(f,\omega^{k/2},k/2)=\mathscr{L}(f)\,\widetilde{L}(f,k/2). \tag{2}$$

We remark that $L(f,k/2)$ can vanish. This conjecture was proved in [GS] in the weight two case and in [St] in general using Hida theory. Another proof, based on the theory of Euler systems was found by Kato, Kurihara and Tsuji (unpublished but see [Ka2], [PR5], [Cz3]). Note that in [St] Stevens uses another definition of the $\mathscr{L}$-invariant proposed by Coleman [Co]. We refer to [CI] and to the survey article [Cz4] for further information and references.

0.3. The crystalline case. Our aim in this paper is to prove an analogue of the formula (2) in the crystalline case. Let f be a newform of an odd weight k. Fix a prime $p\nmid N$ and assume that $\alpha=p^{\frac{k-1}{2}}$ is a root of $X^2-a_pX+\varepsilon(p)p^{k-1}$. Then the p-adic L-function $L_{p,\alpha}(f,\omega^{\frac{k+1}{2}},s)$ vanishes in $s=\frac{k+1}{2}$. The p-adic representation W_f is crystalline at p and we denote by $\mathbf{D}_{\mathrm{cris}}(W_f)$ the filtered Dieudonné module associated to W_f. We assume that the semisimplicity conjecture holds i.e. that the Frobenius operator φ acts semisimply on $\mathbf{D}_{\mathrm{cris}}(W_f)$. The assumption $\alpha=p^{\frac{k-1}{2}}$ together with the semisimplicity of φ implies that $\mathbf{D}_{\mathrm{cris}}\big(W_f(\frac{k+1}{2})\big)^{\varphi=p^{-1}}$ is a one-dimensional vector space which we denote by D_α. The main construction of [Ben2] associates to D_α an element $\mathscr{L}\left(W_f(\frac{k+1}{2}),D_\alpha\right)\in\overline{\mathbb{Q}}_p$ which can be viewed as a direct generalization of Greenberg's $\mathscr{L}$-invariant [Gre] to the non-ordinary case. To simplify notation we set $\mathscr{L}_\alpha(f)=\mathscr{L}\left(W_f(\frac{k+1}{2}),D_\alpha\right)$. The main result of this paper states as follows.

Theorem. *Assume that φ acts semisimply on $\mathbf{D}_{\mathrm{cris}}(W_f)$ and that $\alpha=p^{\frac{k-1}{2}}$ is a root of $X^2-a_pX+\varepsilon(p)\,p^{k-1}$. Then $L_{p,\alpha}\left(f,\omega^{\frac{k+1}{2}},s\right)$ has a trivial zero at $s=\frac{k+1}{2}$ and*

$$L'_{p,\alpha}\left(f,\omega^{\frac{k+1}{2}},\frac{k+1}{2}\right)=-\mathscr{L}_\alpha(f)\,\left(1-\frac{\varepsilon(p)}{p}\right)\widetilde{L}\left(f,\frac{k+1}{2}\right).$$

Remarks. 1) $L(f,\frac{k+1}{2})\neq 0$ by the theorem of Jacquet-Shalika [JS].

2) Let η be a Dirichlet character of conductor M with $(p,M)=1$. The study of trivial zeros of $L_{p,\alpha}\left(f,\eta\omega^{\frac{k+1}{2}},s\right)$ reduces to our theorem by considering the newform $f\otimes\eta$ associated to $f_\eta=\sum\limits_{n=1}^{\infty}\eta(n)\,a_nq^n$ (see section 4.2.2).

3) If $\alpha=p^{(k-1)/2}$ then $\alpha^*=\varepsilon^{-1}(p)\,\alpha$ is a root of the quadratic polynomial associated to the dual form $f^*=\sum\limits_{n=1}^{\infty}\bar a_nq^n$ and $L_{p,\alpha^*}(f^*,\omega^{\frac{k-1}{2}},s)$ has a trivial zero at $s=\frac{k-1}{2}$. Repeating the proof of the main

theorem with obvious modifications or just simply using the compatibility of the trivial zero conjecture with the functional equation ([Ben2], section 2.3.5) we obtain that

$$L'_{p,\alpha^*}\left(f^*,\omega^{\frac{k-1}{2}},\frac{k-1}{2}\right)=-\mathscr{L}_{\alpha^*}(f^*)\,\left(1-\frac{\varepsilon(p)}{p}\right)\widetilde{L}\left(f^*,\frac{k-1}{2}\right).$$

4) The $\mathscr{L}$-invariant of Fontaine-Mazur which appears in the central point case (2) is local i.e. it depends only on the restriction of the p-adic representation ρ_f on the decomposition group at p. However, in the near central point case the $\mathscr{L}$-invariant $\mathscr{L}_\alpha(f)$ is global and contains information about the localisation map $H^1(\mathbb{Q},W_f(\frac{k+1}{2}))\to H^1(\mathbb{Q}_p,W_f(\frac{k+1}{2}))$.

5) Our theorem follows purely formally from the following results:

i) Computation of the image of Kato's Euler systems $\mathbf{z}_{\mathrm{Kato}}$ under the dual exponential map in terms of special values of $L(f,s)$.

ii) Construction of $L_{p,\alpha}(f,s)$ using Euler systems and Perrin-Riou logarithmic map [PR2].

iii) Computation of the derivative of Perrin-Riou's logarithmic map.

We remark that i) and ii) above are deep theorems of Kato ([Ka2], Theorems 12.5 and 16.2). The computation of the derivative of the logarithmic map in terms of the $\mathscr{L}$-invariant is the main technical result of this paper (see Propositions 2.2.2 and 2.2.4).

6) In the potentially crystalline case the restriction of W_f on the decomposition group at p is potentially crystalline and $\mathbf{D}_{\mathrm{cris}}(W_f)=\mathbf{D}_{\mathrm{pcris}}(W_f)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is of rank one. This situation is not covered by our trivial zero conjecture.

0.4. Trivial zeros of Dirichlet L-functions. Let η be a primitive Dirichet character modulo N and let $p\nmid N$ be a fixed prime. The p-adic L-function of Kubota-Leopoldt $L_p(\eta\omega,s)$ satisfies the interpolation property

$$L_p(\eta\omega,1-j)\;=\;(1-(\eta\omega^{1-j})(p)\,p^{j-1})\,L(\eta\omega^{1-j},1-j),\qquad j\geqslant 1.$$

Assume that η is odd and $\eta(p)=1$. Then $L(\eta,0)\neq 0$ but the Euler like factor $1-(\eta\omega^{1-j})\,(p)p^{j-1}$ vanishes at $j=1$ and $L_p(\eta\omega,s)$ has a trivial zero at $s=0$. Fix a finite extension $L/\mathbb{Q}_p$ containing the values of η. Let χ denote the cyclotomic character and let $\mathrm{ord}_p\,:\,\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)\to L$ be the character defined by $\mathrm{ord}_p(\mathrm{Fr}_p)=-1$ where Fr_p is the geometric Frobenius. Then $H^1(\mathbb{Q}_p,L)=\mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p),L)$ is the two-dimensional L-vector space generated by $\log\chi$ and ord_p. Since $p\nmid N$ and $\eta(p)=1$ the restriction of $L(\eta)$ on the decomposition group at p is a trivial representation. The localization map

$$\kappa_\eta\,:\,H^1(\mathbb{Q},L(\eta))\to H^1(\mathbb{Q}_p,L)$$

is injective and identifies $H^1(\mathbb{Q},L(\eta))$ with a one-dimensional subspace of $H^1(\mathbb{Q}_p,L)$. It can be shown that $\mathrm{Im}(\kappa_\eta)$ is generated by an element of the form

$$\log\chi+\mathscr{L}(\eta)\,\mathrm{ord}_p \tag{3}$$

there $\mathscr{L}(\eta)\in L$ is necessarily unique. Applying Proposition 2.2.4 to the Euler system of cyclotomic units we obtain a new proof of the trivial zero conjecture for Dirichlet L-functions

$$L'_p(\eta\omega,0)=-\mathscr{L}(\eta)\,L(\eta,0). \tag{4}$$

This formula was first proved in [Gro] as the combination of the result of Ferrero-Greenberg [FG] giving an explicit formula for $L'_p(\eta\omega,0)$ in terms of the p-adic Γ-function and the Gross-Koblitz formula [GK]. We also remark that Dasgupta, Darmon and Pollack [DDP] recently generalized (4) to totally real number fields F assuming Leopoldt's conjecture and some additional condition on the vanishing of p-adic L-functions.

0.5. The plan of the paper. The main contents of this article is as follows. In §1 we review the necessary preliminaries. In particular, sections 1.1-1.2 are devoted to the theory of (φ,Γ)-modules which plays a key role in our definition of the $\mathscr{L}$-invariant. In section 1.3 we review the construction and main properties of Perrin-Riou's large exponential map.

In §2 we review the construction of the $\mathscr{L}$-invariant $\mathscr{L}(V,D)$ from [Ben2] and prove an explicit formula for the derivative of the large logarithmic map in terms of $\mathscr{L}(V,D)$ and the dual exponential map.

In §3 we apply this formula to Dirichlet L-functions and give a new proof of (4).

Trival zeros of modular forms are studied in §4. In section 4.1 we review basic results about the representations W_f and specialize the general definition of the $\mathscr{L}$-invariant to the case of modular forms. The construction of p-adic L-functions is recalled in section 4.2. Finally in section 4.3 we deduce our main theorem from the fondamental results of Kato [Ka2].

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§1. Preliminaries

1.1. (φ,Γ)-modules.

1.1.1. Definition of (φ,Γ)-modules (see [Fo1], [CC1], [Cz5]). Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and $G_{\mathbb{Q}_p}=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We denote by C the p-adic completion of $\overline{\mathbb{Q}}_p$ and $v_p\,:\,C\to\mathbb{R}\cup\{\infty\}$ the p-adic valuation normalized so that $v_p(p)=1$ and set $|x|_p=\left(\frac{1}{p}\right)^{v_p(x)}$. Write $B(r,1)$ for the p-adic annulus $B(r,1)=\{x\in C\mid r\leqslant |x|_p<1\}$. Fix a system of primitive roots of unity $\varepsilon=(\zeta_{p^n})_{n\geqslant 0}$, such that $\zeta_{p^n}^p=\zeta_{p^{n-1}}$ for all n. Let $K_n=\mathbb{Q}_p(\zeta_{p^n})$ be the cyclotomic extention of $\mathbb{Q}_p$ obtained by adjoining ζ_{p^n} and let $K_\infty=\bigcup_{n=0}^\infty K_n$. Put $\Gamma=\mathrm{Gal}(K_\infty/\mathbb{Q}_p)$, $\Gamma_n=\mathrm{Gal}(K_\infty/K_n)$ and denote by $\chi\,:\,\Gamma\to\mathbb{Z}_p^*$ the cyclotomic character. We fix a topological generator γ of Γ and define a compatible system of generators γ_n of Γ_n setting $\gamma_1=\gamma^{p-1}$ and $\gamma_{n+1}=\gamma_n^p$ for $n\geqslant 1$. Fix a finite extension $L/\mathbb{Q}_p$. For any $0\leqslant r<1$ define

$$\mathscr{R}_L^{(r)}=\left\{f(X)=\sum_{k\in\mathbb{Z}}a_kX^k\mid a_k\in L\text{ and }f\text{ is holomorphic on }B(r,1)\right\},$$

$$\mathscr{E}_L^{(r)}=\left\{f(X)=\sum_{k\in\mathbb{Z}}a_kX^k\mid a_k\in L\text{ and }f\text{ is holomorphic and bounded on }B(r,1)\right\}.$$

Set $\mathscr{E}_L^\dagger=\underset{0\leqslant r<0}{\cup}\mathscr{E}_L^{(r)}$ and $\mathscr{R}_L=\underset{0\leqslant r<0}{\cup}\mathscr{R}_L^{(r)}$. Then $\mathscr{E}_L^\dagger$ is a field endowed with the valuation

$$w\left(\sum_{k\in\mathbb{Z}}a_kX^k\right)=\min\{v_p(a_k)\mid k\in\mathbb{Z}\}$$

and we denote by $\mathcal{O}_{\mathscr{E}_L}^\dagger$ its ring of integers. The rings $\mathcal{O}_{\mathscr{E}_L}^\dagger$, $\mathscr{E}_L^\dagger$ and $\mathscr{R}_L$ are equipped with an L-linear action of Γ and a Frobenius operator φ given by

$$\tau f(X)=f(\tau(X)),\qquad\text{where }\tau(X)=(1+X)^{\chi(\tau)}-1,\quad\tau\in\Gamma,$$
$$\varphi f(X)=f(\varphi(X)),\qquad\text{where }\varphi(X)=(1+X)^p-1.$$

The actions of Γ and φ commute to each other. As usual we set $t=\log(1+X)=\sum_{n=1}^{\infty}(-1)^{n-1}X^n/n$. Note that $\varphi(t)=p\,t$ and $\tau(t)=\chi(\tau)t$. The operator ψ defined by

$$\psi(f(X))=\frac{1}{p}\,\varphi^{-1}\sum_{\zeta^p=1}f(\zeta(1+X)-1)$$

is a left inverse to φ i.e. $\psi\circ\varphi=\mathrm{id}$. We remark that the rings $\mathcal{O}^{\dagger}_{\mathscr{E}_L}$, $\mathscr{E}^{\dagger}_L$ and $\mathscr{R}_L$ are stable under the action of ψ.

Definition. *i) A (φ,Γ_n)-module over $\mathscr{E}^{\dagger}_L$ (resp. $\mathscr{R}_L$) is a free $\mathscr{E}^{\dagger}_L$-module (resp. $\mathscr{R}_L$-module) $\mathbf{D}$ of finite rank d equipped with semilinear actions of Γ_n and φ which commute to each other and such that the induced linear map $\mathscr{E}^{\dagger}_L\otimes_{\varphi}\mathbf{D}\to\mathbf{D}$ (resp. $\mathscr{R}_L\otimes_{\varphi}\mathbf{D}\to\mathbf{D}$) is an isomorphism.*

ii) A (φ,Γ_n)-module $\mathbf{D}$ over $\mathscr{E}^{\dagger}_L$ is said to be etale if there exists a basis of $\mathbf{D}$ such that the matrix of φ in this basis is in $\mathrm{GL}_d(\mathcal{O}^{\dagger}_{\mathscr{E}_L})$.

If $\mathbf{D}$ is a (φ,Γ_n)-module over $A=\mathscr{E}^{\dagger}_L$ or $\mathscr{R}_L$ we write $\mathbf{D}^*$ for the dual module $\mathrm{Hom}_A(\mathbf{D},A)$ and $\mathbf{D}(\chi)$ for the module obtained from $\mathbf{D}$ by twisting the action of Γ_n by the cyclotomic character.

Let $\mathbf{Rep}_L(G_{K_n})$ be the category of p-adic representations of G_{K_n} with coefficients in L i.e. the category of finite dimensional L-vector spaces equipped with a continuous linear action of G_{K_n}.

Theorem 1.1.2 ([Fo1], [CC1]). There exists a natural functor $V\to\mathbf{D}^{\dagger}(V)$ which induces an equivalence between $\mathbf{Rep}_L(G_{K_n})$ and the category of etale (φ,Γ_n)-modules over $\mathscr{E}^{\dagger}_L$.

From Kedlaya's theory it follows (see [Cz5], Proposition 1.4 and Corollary 1.5) that the functor $\mathbf{D}\to\mathscr{R}_L\otimes_{\mathscr{E}^{\dagger}_L}\mathbf{D}$ establishes an equivalence between the category of étale (φ,Γ_n)-modules over $\mathscr{E}^{\dagger}_L$ and the category of (φ,Γ_n)-modules over $\mathscr{R}_L$ of slope 0 in the sense of [Ke]. Together with Theorem 1.1.2 this implies that the functor $V\to\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ defined by $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)=\mathscr{R}_L\otimes_{\mathscr{E}^{\dagger}_L}\mathbf{D}^{\dagger}(V)$ induces an equivalence between the category of p-adic representations and the category of (φ,Γ_n)-modules over $\mathscr{R}_L$ of slope 0.

1.1.3. Crystalline (φ,Γ)-modules (see [Fo3], [Ber3], [Ber4]). Recall that a filtered Dieudonné module over K_n with coefficients in L is a finite dimensional L-vector space M equipped with the following structures:

- an L-linear isomorphism $\varphi\,:\,M\to M$;
- an exhaustive decreasing filtration $(\mathrm{Fil}^iM_{K_n})_{i\in\mathbb{Z}}$ on $M_{K_n}=K_n\otimes_{\mathbb{Q}_p}M$ by $(K_n\otimes_{\mathbb{Q}_p}L)$-submodules.

It is well known (see for example [Fo3]) that filtered Dieudonné modules form a tensor category $\mathbf{MF}^{\varphi}_{K_n}$ which is additive, has kernels and cokernels but is not abelian. The unit object $\mathbf{1}$ of $\mathbf{MF}^{\varphi}_{K_n}$ is the one dimensional vector space L with the trivial action of φ and the filtration given by

$$\mathrm{Fil}^i\mathbf{1}=\begin{cases}L, & \text{if } i\leqslant 0,\\ 0, & \text{if } i>0.\end{cases}$$

If M is a one-dimensional Dieudonné module and m is a basis vector of M, then $\varphi(m)=\alpha\,m$ for some $\alpha\in L$. Set $t_N(M)=v_p(\alpha)$ and denote by $t_H(M)$ the unique filtration jump of M. If M has a dimension $d\geqslant 1$, set $t_N(M)=t_N(\overset{d}{\wedge}M)$ and $t_H(M)=t_H(\overset{d}{\wedge}M)$. A Dieudonné module M is said to be weakly admissible if $t_H(M)=t_N(M)$ and if $t_H(M')\leqslant t_N(M')$ for any φ-submodule M' of M equipped with the induced filtration. Weakly admissible modules form a subcategory of $\mathbf{MF}^{\varphi}_{K_n}$ which we denote by $\mathbf{MF}^{\varphi,f}_{K_n}$.

If $\mathbf{D}$ is a (φ,Γ_n)-module over $\mathscr{R}_L$ we set

$$\mathscr{D}_{\mathrm{cris}}(\mathbf{D})=(\mathbf{D}\otimes_{\mathscr{R}_L}\mathscr{R}_L[1/t])^{\Gamma_n}\,.$$

Then $\mathscr{D}_{\mathrm{cris}}(\mathbf{D})$ is a finite-dimensional L-vector space equipped with a natural action of φ. The embeddings $\varphi^{-m}\,:\,\mathscr{R}_L^{(r)}\to(L\otimes K_\infty)[[t]]$ which sends X onto $\zeta_{p^m}e^{t/p^m}-1$ allow to define an exhaustive decreasing filtration on $\mathscr{D}_{\mathrm{cris}}(D)_{K_n}$ (see [Ber1],). Moreover $\dim_L\mathscr{D}_{\mathrm{cris}}(\mathbf{D})\leqslant \mathrm{rg}(\mathbf{D})$ and we say that $\mathbf{D}$ is crystalline if the equality holds here.

Proposition 1.1.4. *i) The functor $\mathbf{D}\mapsto\mathscr{D}_{\mathrm{cris}}(\mathbf{D})$ induces an equivalence between the category of crystalline (φ,Γ_n)-modules and $\mathbf{MF}^{\varphi}_{K_n}$.*

ii) If V is a p-adic representation of G_{K_n} then $\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$ is canonically and fonctorially isomorphic to Fontaine's crystalline module $\mathbf{D}_{\mathrm{cris}}(V)$. In particular, $\mathscr{D}_{\mathrm{cris}}$ induces an equivalence between the category of crystalline (φ,Γ_n)-modules of slope 0 and $\mathbf{MF}^{\varphi,f}_{K_n}$.

Proof. The first statement is the main result of [Ber4]. The second statement follows from [Ber1], Theorem 0.2.

The description of the filtered module $\mathbf{D}_{\mathrm{cris}}(V)$ is particulary simple if V crystalline over $\mathbb{Q}_p$ (or more generally over an unramified ground field). Set $\mathscr{E}_L^+=\mathscr{E}_L^{\dagger}\cap L[[X]]$ and $\mathscr{R}_L^+=\mathscr{R}_L\cap L[[X]]$. Thus $\mathscr{E}_L^+=O_L[[X]]\left[\frac{1}{p}\right]$ and $\mathscr{R}_L^+$ coincides with the ring of power series which are holomorphic on the open unit disc. If V is a p-adic representation of $G_{\mathbb{Q}_p}$ we let $\mathbf{D}^+(V)$ denote the union of (φ,Γ)-stable $\mathscr{E}_L^+$-submodules of $\mathbf{D}^{\dagger}(V)$. It is easy to see that $\mathbf{D}^+(V)$ is the maximal $\mathscr{E}_L^+$-submodule of $\mathbf{D}^{\dagger}(V)$ stable under φ and Γ. In [Cz2], Theorem 1 Colmez proved that V is crystalline if and only if $\mathrm{rg}_{\mathscr{E}_L^+}\mathbf{D}^+(V)=\dim_L V$. Together with the results of Wach [Wa] this implies that

$$\mathbf{D}_{\mathrm{cris}}(V)\;=\;\left(\mathbf{D}^+(V)\otimes_{\mathscr{E}_L^+}\mathscr{R}_L^+\right)^{\Gamma}$$

(see [Ber3], Proposition 3.4).

1.2. Cohomology of (φ,Γ)-modules.

1.2.1. Fontaine-Herr complexes (see [H1], [H2], [Liu]). Let A be either $\mathscr{E}_L^{\dagger}$ or $\mathscr{R}_L$. Recall that we fixed a generator $\gamma_n\in\Gamma_n$. If $\mathbf{D}$ is a (φ,Γ_n)-module over A we shall write $H^*(\mathbf{D})$ for the cohomology of the complex

$$C_{\varphi,\gamma}(\mathbf{D})\;:\;0\to\mathbf{D}\xrightarrow{f}\mathbf{D}\oplus\mathbf{D}\xrightarrow{g}\mathbf{D}\to 0$$

where $f(x)=((\varphi-1)\,x,(\gamma_n-1)\,x)$ and $g(y,z)=(\gamma_n-1)\,y-(\varphi-1)\,z$. A short exact sequence of (φ,Γ_n)-modules

$$0\to\mathbf{D}'\to\mathbf{D}\to\mathbf{D}''\to 0$$

gives rise to an exact cohomology sequence:

$$0\to H^0(\mathbf{D}')\to H^0(\mathbf{D})\to H^0(\mathbf{D}'')\to H^1(\mathbf{D}')\to\cdots\to H^2(\mathbf{D}'')\to 0.$$

The cohomology of (φ,Γ_n)-modules over $\mathscr{R}_L$ satisfies the following fondamental properties (see [Liu], Theorem 0.2):

- *Euler chracteristic formula.* $H^*(\mathbf{D})$ are finite dimensional L-vector spaces and the usual formula for the Euler characteristic holds

$$\sum_{i=0}^{\infty}(-1)^i\dim_L H^i(\mathbf{D})=-[K_n:\mathbb{Q}_p]\,\mathrm{rg}_{\mathscr{R}_L}(\mathbf{D}).$$

- *Poincaré duality.* For each $i=0,1,2$ there exist functorial pairings

$$H^i(\mathbf{D})\times H^{2-i}(\mathbf{D}^*(\chi))\xrightarrow{\cup}H^2(\mathscr{R}_L(\chi))\simeq L$$

which are compatible with the connecting homomorphisms in the usual sense.

Proposition 1.2.2. *Let V be a p-adic representation of G_{K_n}. Then*

i) The continuous Galois cohomology $H^*(K_n,V)$ is canonically (up to the choice of γ_n) and functorially isomorphic to $H^*(\mathbf{D}^\dagger(V))$.

ii) The natural map $\mathbf{D}^\dagger(V)\to \mathbf{D}^\dagger_{\mathrm{rig}}(V)$ induces a quasi-isomorphism of complexes $C_{\varphi,\gamma}(\mathbf{D}^\dagger(V))\to C_{\varphi,\gamma}(\mathbf{D}^\dagger_{\mathrm{rig}}(V))$.

Proof. see [H1] and [Liu], Theorem 1.1.

1.2.3. Iwasawa cohomology (see [CC2]). If V is a p-adic representation of $G_{\mathbb{Q}_p}$ and T is an O_L-lattice of V stable under $G_{\mathbb{Q}_p}$ we define

$$H^i_{\mathrm{Iw}}(\mathbb{Q}_p,T)=\varprojlim_{\mathrm{cor}_{K_n/K_{n-1}}} H^i(K_n,T)$$

and $H^i_{\mathrm{Iw}}(\mathbb{Q}_p,V)=H^i_{\mathrm{Iw}}(\mathbb{Q}_p,T)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Since $\mathbf{D}^\dagger(V)$ is etale, each $x\in \mathbf{D}^\dagger(V)$ can be written in the form $x=\sum_{i=1}^d a_i\varphi(e_i)$ where $\{e_i\}_{i=1}^d$ is a basis of $\mathbf{D}^\dagger(V)$ and $a_i\in \mathscr{E}^\dagger_L$. Therefore the formula

$$\psi\left(\sum_{i=1}^d a_i\,\varphi(e_i)\right)=\sum_{i=1}^d \psi(a_i)\,e_i$$

defines an operator $\psi\,:\,\mathbf{D}^\dagger(V)\to \mathbf{D}^\dagger(V)$ which is a left inverse for φ. The Iwasawa cohomology $H^*_{\mathrm{Iw}}(\mathbb{Q}_p,V)$ is canonically (up to the choice of γ) and functorially isomorphic to the cohomology of the complex

$$C^\dagger_{\mathrm{Iw},\psi}(V)\;:\;\mathbf{D}^\dagger(V)\xrightarrow{\psi-1}\mathbf{D}^\dagger(V).$$

The projection map $\mathrm{pr}_{V,n}\,:\,H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V)\to H^1(K_n,V)$ has the following description in terms of (φ,Γ)-modules. Let $x\in \mathbf{D}^\dagger(V)^{\psi=1}$. Then $(\varphi-1)\,x\in \mathbf{D}^\dagger(V)^{\psi=0}$ and by Lemma 1.5.1 of [CC1] there exists $y\in \mathbf{D}^\dagger(V)$ such that $(\gamma_n-1)\,y=(\varphi-1)\,x$. Then $\mathrm{pr}_{V,n}$ sends $\mathrm{cl}(x)$ to $\mathrm{cl}(y,x)$. This interpretation of the Iwasawa cohomology in terms of (φ,Γ)-modules was found by Fontaine (unpublished but see [CC2]).

1.2.4. The exponential map (see [BK], [Ne], [Ben2]). Let $\mathbf{D}$ be a (φ,Γ_n)-module. To any cocycle $\alpha=(a,b)\in Z^1(C_{\varphi,\gamma}(\mathbf{D}))$ one can associate the extension

$$0\to \mathbf{D}\to \mathbf{D}_\alpha\to \mathscr{R}_L\to 0$$

defined by

$$\mathbf{D}_\alpha=\mathbf{D}\oplus \mathscr{R}_L\,e,\qquad (\varphi-1)\,e=a,\quad (\gamma_n-1)\,e=b.$$

As usual, this gives rise to a canonical isomorphism $H^1(\mathbf{D})\simeq \mathrm{Ext}^1_{(\varphi,\Gamma_n)}(\mathscr{R}_L,\mathbf{D})$. We say that the class $\mathrm{cl}(\alpha)$ of α in $H^1(\mathbf{D})$ is crystalline if $\dim_L \mathscr{D}_{\mathrm{cris}}(\mathbf{D}_\alpha)=\dim_L \mathscr{D}_{\mathrm{cris}}(\mathbf{D})+1$ and define

$$H^1_f(\mathbf{D})\;=\;\{\mathrm{cl}(\alpha)\in H^1(\mathbf{D})\mid \mathrm{cl}(\alpha)\text{ is crystalline}\}$$

(see [Ben2], section 1.4). If M is a filtered Dieudonné nodule over K_n with coefficients in L we set

$$H^i(K_n,M)=\mathrm{Ext}^i_{\mathbf{MF}^{\varphi}_{K_n}}(\mathbf{1},M),\qquad i=0,1.$$

We remark that $H^*(K_n,M)$ can be computed explicitly as the cohomology of the complex

$$C^\bullet_{\mathrm{cris}}(M)\ :\ M \xrightarrow{f} (M_{K_n}/\mathrm{Fil}^0 M_{K_n})\oplus M$$

where the modules are placed in degrees 0 and 1 and $f(d)=(d \pmod{\mathrm{Fil}^0 M_{K_n}}, (1-\varphi)(d))$ (see [Ne], [FP]). Assume that $\mathbf{D}$ is a crystalline (φ,Γ_n)-module and define the tangent space of $\mathbf{D}$ over K_n by

$$t_{\mathbf{D}}(K_n)=\mathscr{D}_{\mathrm{cris}}(\mathbf{D})_{K_n}/\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{D})_{K_n}.$$

It follows from Proposition 1.1.4 that the functor $\mathscr{D}_{\mathrm{cris}}$ induces a canonical isomorphism

$$H^1(K_n,\mathscr{D}_{\mathrm{cris}}(\mathbf{D}))\to H^1_f(\mathbf{D}).$$

We define the exponential map

$$\exp_{\mathbf{D},K_n}\ :\ t_{\mathbf{D}}(K_n)\oplus\mathscr{D}_{\mathrm{cris}}(\mathbf{D})\to H^1(\mathbf{D})$$

as the composition of this isomorphism with the natural projection $t_{\mathbf{D}}(K_n)\oplus\mathscr{D}_{\mathrm{cris}}(\mathbf{D})\to H^1(K,\mathscr{D}_{\mathrm{cris}}(\mathbf{D}))$ and the embedding $H^1_f(\mathbf{D})\hookrightarrow H^1(\mathbf{D})$.

If V is a crystalline representation and $\mathbf{D}=\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ then the isomorphism $H^1(\mathbf{D})\simeq H^1(K_n,V)$ identifies $H^1_f(\mathbf{D})$ with $H^1_f(K_n,V)$ of Bloch-Kato (see [Ben2], Proposition 1.4.2). Let

$$t_V(K_n)=\big(\mathbf{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V)\big)\otimes_{\mathbb{Q}_p}K_n$$

denote the tangent space of V. By Proposition 1.21 of [Ne] the following diagram commutes and identifies our exponential map with the exponential map $\exp_{V,K_n}$ of Bloch-Kato ([BK], §4)

$$\begin{array}{ccc} t_{\mathbf{D}}(K_n) & \xrightarrow{\exp_{\mathbf{D},K_n}} & H^1(\mathbf{D}) \\ \downarrow & & \downarrow \\ t_V(K_n) & \xrightarrow{\exp_{V,K_n}} & H^1(K_n,V). \end{array}$$

Let

$$[\ ,\ ]\ :\ \mathscr{D}_{\mathrm{cris}}(\mathbf{D})_{K_n}\times\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))_{K_n}\to L\otimes_{\mathbb{Q}_p}K_n$$

be the canonical duality. The dual exponential map

$$\exp^*_{\mathbf{D}^*(\chi),K_n}\ :\ H^1(\mathbf{D}^*(\chi))\to\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))_{K_n}$$

is defined as the unique map such that

$$\exp_{\mathbf{D},K_n}(x)\cup y=\mathrm{Tr}_{K_n/\mathbb{Q}_p}[x,\exp^*_{\mathbf{D}^*(\chi),K_n}(y)]$$

for all $x\in\mathscr{D}_{\mathrm{cris}}(\mathbf{D})_{K_n}$, $y\in\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))_{K_n}$.

1.2.5. (φ,Γ)**-modules of rank** 1 (see [Cz5], [Ben2]). With each continuous character $\delta\ :\ \mathbb{Q}_p^*\to L^*$ one can associate the (φ,Γ)-module of rank one $\mathscr{R}_L(\delta)=\mathscr{R}_Le_\delta$ defined by $\gamma(e_\delta)=\delta(\chi(\gamma))e_\delta$ and $\varphi(e_\delta)=\delta(p)e_\delta$. Colmez proved that any (φ,Γ)-module of rank one over $\mathscr{R}_L$ is isomorphic to one and only one of $\mathscr{R}_L(\delta)$ ([Cz5], Proposition 3.1). It is easy to see that $\mathscr{R}_L(\delta)$ is crystalline if and only if there exists $k \in \mathbb{Z}$ such that $\delta(u) = u^k$ for all $u \in \mathbb{Z}_p^*$ ([Ben2], Lemma 1.5.2). In this case $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$ is the one-dimensional vector space generated by $t^{-k}e_\delta$ with Hodge-Tate weight equal to $-k$ and φ acts on $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$ as multiplication by $p^{-k}\delta(p)$. The computation of the cohomology of crystalline (φ,Γ)-modules of rank 1 reduces to the following four cases. (We refer to [Cz5], sections 2.3-2.5 and to [Ben2], Proposition 1.5.3 and Theorem 1.5.7 for proofs and more details).

• *$\delta(u) = u^{-m}$ ($u \in \mathbb{Z}_p^*$) for some $m \geqslant 0$ but $\delta(x) \neq x^{-m}$.* In this case $H^i(\mathscr{R}_L(\delta)) = 0$ for $i = 0, 2$, $H^1(\mathscr{R}_L(\delta))$ is a one-dimensional L-vector space and $H^1_f(\mathscr{R}_L(\delta)) = 0$.

• *$\delta(x) = x^{-m}$ for some $m \geqslant 0$.* In this case $H^0(\mathscr{R}_L(\delta)) = \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$ and $H^2(\mathscr{R}_L(\delta)) = 0$. The map

$$\begin{aligned} &i_\delta \,:\, \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta)) \oplus \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta)) \to H^1(\mathscr{R}_L(\delta)),\\ &i_\delta(x,y) = \mathrm{cl}(-x, \log\chi(\gamma)y) \end{aligned}$$

is an isomorphism. We let $i_{\delta,f}$ and $i_{\delta,c}$ denote its restrictions on the first and second direct summand respectively. Then $\mathrm{Im}(i_{\delta,f}) = H^1_f(\mathscr{R}_L(\delta))$ and we have a canonical decomposition

$$H^1(\mathscr{R}_L(\delta)) \simeq H^1_f(\mathscr{R}_L(\delta)) \oplus H^1_c(\mathscr{R}_L(\delta)) \tag{5}$$

where $H^1_c(\mathscr{R}_L(\delta)) = \mathrm{Im}(i_{\delta,c})$. Set

$$\begin{aligned} \mathbf{x}_m &= i_{\delta,f}(t^m e_\delta) = -\mathrm{cl}(t^m, 0)\, e_\delta,\\ \mathbf{y}_m &= i_{\delta,c}(t^m e_\delta) = \log\chi(\gamma)\,\mathrm{cl}(0, t^m)\, e_\delta. \end{aligned}$$

• *$\delta(u) = u^m$ ($u \in \mathbb{Z}_p^*$) for some $m \geqslant 1$ but $\delta(x) \neq |x|x^m$.* Then $H^i(\mathscr{R}_L(\delta)) = 0$ for $i = 0, 2$, $H^1(\mathscr{R}_L(\delta))$ is a one-dimensional L-vector space and $H^1_f(\mathscr{R}_L(\delta)) = H^1(\mathscr{R}_L(\delta))$.

• *$\delta(x) = |x|x^m$ for some $m \geqslant 1$.* Then $H^0(\mathscr{R}_L(\delta)) = 0$ and $H^2(\mathscr{R}_L(\delta))$ is a one-dimensional L-vector space. Moreover $\chi\delta^{-1}(x) = x^{1-m}$ and there exists a unique isomorphism

$$i_\delta \,:\, \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta)) \oplus \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta)) \to H^1(\mathscr{R}_L(\delta))$$

such that

$$i_\delta(\alpha,\beta) \cup i_{\chi\delta^{-1}}(x,y) = [\beta, x] - [\alpha, y]$$

where $[\ ,\ ] \,:\, \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta)) \times \mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\chi\delta^{-1})) \to L$ is the canonical pairing. Denote $i_{\delta,f}$ and $i_{\delta,c}$ the restrictions of i_δ on the first and second direct summand respectively. Then $\mathrm{Im}(i_{\delta,f}) = H^1_f(\mathscr{R}_L(\delta))$ and again we have a canonical decomposition

$$H^1(\mathscr{R}_L(\delta)) \simeq H^1_f(\mathscr{R}_L(\delta)) \oplus H^1_c(\mathscr{R}_L(\delta)) \tag{6}$$

where $H^1_c(\mathscr{R}_L(\delta)) = \mathrm{Im}(i_{\delta,c})$.

More explicitly, let $\boldsymbol{\alpha}_m = -\left(1 - \dfrac{1}{p}\right)\,\mathrm{cl}(\alpha_m)$ and $\boldsymbol{\beta}_m = \left(1 - \dfrac{1}{p}\right)\log\chi(\gamma)\,\mathrm{cl}(\beta_m)$ where

$$\begin{aligned} \alpha_m &= \frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(\frac{1}{X} + \frac{1}{2}, a\right) e_\delta, \qquad (1-\varphi)\,a = (1-\chi(\gamma)\gamma)\left(\frac{1}{X} + \frac{1}{2}\right),\\ \beta_m &= \frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(b, \frac{1}{X}\right) e_\delta, \qquad (1-\varphi)\left(\frac{1}{X}\right) \;=\; (1-\chi(\gamma)\,\gamma)\,b \end{aligned}$$

and $\partial = (1+X)\dfrac{d}{dX}$. Then $H^1_f(\mathscr{R}_L(\delta))$ and $H^1_c(\mathscr{R}_L(\delta))$ are generated by $\boldsymbol{\alpha}_m$ and $\boldsymbol{\beta}_m$ respectively and one has

$$\boldsymbol{\alpha}_m \cup \mathbf{x}_{m-1} = \boldsymbol{\beta}_m \cup \mathbf{y}_{m-1} = 0, \quad \boldsymbol{\alpha}_m \cup \mathbf{y}_{m-1} = -1, \quad \boldsymbol{\beta}_m \cup \mathbf{x}_{m-1} = 1. \tag{7}$$

Proposition 1.2.6. *Let $\delta(x)=|x|x^m$ where $m\geqslant 1$. Then $d_m=t^{-m}e_\delta$ is a basis of $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$ and the exponential map sends $(d_m,0)$ to $\boldsymbol{\alpha}_m$.*

Proof. See [Ben2], Theorem 1.5.7 ii).

1.3. The large exponential map.

1.3.1. The large exponential map (see [PR2], [Cz1], [Ben1], [Ber2]). In this section we review the construction and basic properties of Perrin-Riou's large exponential map [PR2]. Let p is an odd prime number. We let denote $\Lambda=O_L[[\Gamma]]$ the Iwasawa algebra of Γ over O_L. Define

$$\mathscr{H}\;=\;\{f(\gamma_1-1)\,|\,f\in\mathscr{R}_L^+\},\qquad \mathscr{H}(\Gamma)\;=\;\mathbb{Z}_p[\Delta]\otimes_{\mathbb{Z}_p}\mathscr{H}(\Gamma).$$

Thus $\mathscr{H}(\Gamma)=\underset{i=0}{\overset{p-2}{\oplus}}\mathscr{H}\delta_i$ where $\delta_i=\sum\limits_{g\in\Delta}\omega^{-i}(g)g$. We equip $\mathscr{H}(\Gamma)$ with twist operators $\mathrm{Tw}_m\,:\,\mathscr{H}(\Gamma)\to\mathscr{H}(\Gamma)$ defined by $\mathrm{Tw}_m(f(\gamma_1-1)\,\delta_i)=f(\chi(\gamma_1)^m\gamma_1-1)\,\delta_{i-m}$. The ring $\mathscr{H}(\Gamma)$ acts on $\mathscr{R}_L^+$ and $(\mathscr{R}_L^+)^{\psi=0}$ is the free $\mathscr{H}(\Gamma)$-module generated by $(1+X)$ ([PR2], Proposition 1.2.7). Let V be a crystalline representation of $G_{\mathbb{Q}_p}$. We will assume that $H^0(\mathbb{Q}_p,V(m))=0$ for all $m\in\mathbb{Z}$. As $H^0(K_\infty,V)=\underset{m\in\mathbb{Z}}{\oplus}V(m)^{G_{\mathbb{Q}_p}}(-m)$ this assumption implies that $H^0(K_\infty,V)=0$ and therefore that $H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V)$ is a free $\Lambda_{\mathbb{Q}_p}$- module of rank $d=\dim_L(V)$. In particular, for each n the map $H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V)_{\Gamma_n}\to H^1(K_n,V)$ is injective.

Set $\mathcal{D}(V)=(\mathscr{R}_L^+)^{\psi=0}\otimes_L\mathbf{D}_{\mathrm{cris}}(V)$ and define a map $\Xi_{V,n}\,:\,\mathcal{D}(V)\to H^1(K_n,\mathscr{D}_{\mathrm{cris}}(V))$ by

$$\Xi^{\varepsilon}_{V,n}(\alpha)\;=\;\begin{cases} p^{-n}\left(\sum_{k=1}^n(\sigma\otimes\varphi)^{-k}\alpha(\zeta_{p^k}-1),\,-\alpha(0)\right) & \text{if } n\geqslant 1,\\ -\big(0,(1-p^{-1}\varphi^{-1})\,\alpha(0)\big) & \text{if } n=0.\end{cases}$$

In particular, if $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}=0$ the operator $1-\varphi$ is invertible on $\mathbf{D}_{\mathrm{cris}}(V)$ and

$$\Xi^{\varepsilon}_{V,0}(\alpha)=\left(\frac{1-p^{-1}\varphi^{-1}}{1-\varphi}\,\alpha(0),0\right).$$

For any $m\in\mathbb{Z}$ let $\mathrm{Tw}^{\varepsilon}_{V,m}\,:\,H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V)\to H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V(m))$ denote the twist map $\mathrm{Tw}^{\varepsilon}_{V,m}(x)=x\otimes\varepsilon^{\otimes m}$.

Theorem 1.3.2. *Let V be a crystalline representation of $G_{\mathbb{Q}_p}$ such that $H^0(\mathbb{Q}_p,V(m))=0$ for all $m\in\mathbb{Z}$. Then for any integers h and m such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V)=\mathbf{D}_{\mathrm{cris}}(V)$ and $m+h\geqslant 1$ there exists a unique $\mathscr{H}(\Gamma)$-homomorphism*

$$\mathrm{Exp}^{\varepsilon}_{V(m),h}\;:\;\mathcal{D}(V(m))\to\mathscr{H}(\Gamma)\otimes_{\Lambda_{\mathbb{Q}_p}}H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V(m))$$

satisfying the following properties:

1) For any $n\geqslant 0$ the diagram

$$\begin{tikzcd}
\mathcal{D}(V(m)) \arrow[r, "\mathrm{Exp}^{\varepsilon}_{V(m),h}"] \arrow[d, "\Xi^{\varepsilon}_{V(m),n}"'] & \mathscr{H}(\Gamma)\otimes_{\Lambda_{\mathbb{Q}_p}}H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V(m)) \arrow[d, "\mathrm{pr}_{V(m),n}"'] \\
H^1(K_n,\mathbf{D}_{\mathrm{cris}}(V(m))) \arrow[r, "(h-1)!\,\exp_{V(m),K_n}"] & H^1(K_n,V(m))
\end{tikzcd}$$

commutes.

ii) Let $e_1=\varepsilon^{-1}\otimes t$ *denote the canonical generator of* $\mathbf{D}_{\mathrm{cris}}(\mathbb{Q}_p(-1))$. *Then*

$$\mathrm{Exp}^{\varepsilon}_{V(m+1),h+1}=-\mathrm{Tw}^{\varepsilon}_{V(m),1}\circ \mathrm{Exp}^{\varepsilon}_{V(m),h}\circ(\partial\otimes e_1).$$

iii) One has

$$\mathrm{Exp}^{\varepsilon}_{V(m),h+1}=\ell_h\mathrm{Exp}^{\varepsilon}_{V(m),h}$$

where $\ell_m=m-\frac{\log(\gamma_1)}{\log\chi(\gamma_1)}$.

Proof. The first proof of this theorem was given in [PR2] where $\mathrm{Exp}^{\varepsilon}_{V,h}(\alpha)$ was defined only for α such that $\partial^m\alpha(0)\in(1-p^m\varphi)\,\mathbf{D}_{\mathrm{cris}}(V)$ for all $m\in\mathbb{Z}$. We remark that this condition is not necessary (see [PR4] or [Ben1], section 5.1). Other proofs can be found in [Cz1], [Ben1] and [Ber2]. We recall here the construction of $\mathrm{Exp}^{\varepsilon}_{V,h}$ in terms of (φ,Γ)-modules found by Berger [Ber2] which will be used in the proof of Proposition 2.3.2 below. The action of $\mathscr{H}(\Gamma)$ on $\mathbf{D}^{\dagger}(V)^{\psi=1}$ induces an injection $\mathscr{H}(\Gamma)\otimes_{\Lambda_{\mathbb{Q}_p}}\mathbf{D}^{\dagger}(V)^{\psi=1}\hookrightarrow \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\psi=1}$. Composing this map with the canonical isomorphism $H^1_{\mathrm{Iw}}(K,V)\simeq \mathbf{D}^{\dagger}(V)^{\psi=1}$ we obtain a map $\mathscr{H}(\Gamma)\otimes_{\Lambda_{\mathbb{Q}_p}}H^1_{\mathrm{Iw}}(K,V)\hookrightarrow \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\psi=1}$. It is not difficult to check that ℓ_m acts on $\mathscr{R}_L$ as $m-t\partial$ and an easy induction shows that $\prod\limits_{k=0}^{h-1}\ell_k=(-1)^ht^h\partial^h$. Let $h\geqslant 1$ be such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V)=\mathbf{D}_{\mathrm{cris}}(V)$. To simplify the formulation, assume that $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}=0$. For any $\alpha\in\mathcal{D}(V)$ the equation

$$(\varphi-1)\,F=\alpha-\sum_{m=1}^{h}\frac{\partial^m\alpha(0)}{m!}\,t^m$$

has a solution in $\mathscr{R}^+_L\otimes\mathbf{D}_{\mathrm{cris}}(V)$ and we define

$$\Omega^{\varepsilon}_{V,h}(\alpha)\;=\;\frac{\log\chi(\gamma_1)}{p}\,\ell_{h-1}\ell_{h-2}\cdots\ell_0(F(X)).$$

It is easy to see that $\Omega^{\varepsilon}_{V,h}(\alpha)\in\mathbf{D}^{+}_{\mathrm{rig}}(V)^{\psi=1}$. In [Ber2], Theorem II.13 Berger shows that $\Omega^{\varepsilon}_{V,h}(\alpha)\in\mathscr{H}(\Gamma)\otimes_{\Lambda_{\mathbb{Q}_p}}\mathbf{D}^{\dagger}(V)^{\psi=1}$ and coincides with $\mathrm{Exp}^{\varepsilon}_{V,h}(\alpha)$.

1.3.3. The logarithmic maps. The Iwasawa algebra Λ is equipped with an involution $\iota:\Lambda\to\Lambda$ defined by $\iota(\tau)=\tau^{-1}$, $\tau\in\Gamma$. If M is a Λ-module we set $M^{\iota}=\Lambda\otimes_{\iota}M$ and denote by $m\mapsto m^{\iota}$ the canonical bijection of M onto M^{ι}. Thus $\lambda\,m^{\iota}=(\iota(\lambda)\,m)^{\iota}$ for all $\lambda\in\Lambda$, $m\in M$. Let T be a O_L-lattice of V stable under the action of $G_{\mathbb{Q}_p}$. The cohomological pairings

$$(\ ,\ )_{T,n}\;:\;H^1(K_n,T)\times H^1(K_n,T^*(1))\to O_L$$

give rise to a $\Lambda(\Gamma)$-bilinear pairing

$$\big\langle\ ,\ \big\rangle_T\;:\;H^1_{\mathrm{Iw}}(\mathbb{Q}_p,T)\times H^1_{\mathrm{Iw}}(\mathbb{Q}_p,T^*(1))^{\iota}\to\Lambda$$

defined by

$$\big\langle x,y^{\iota}\big\rangle_T\equiv\sum_{\tau\in\Gamma/\Gamma_n}(\tau^{-1}x_n,y_n)_{T,n}\tau\mod(\gamma_n-1),\qquad n\geqslant 1$$

(see [PR2], section 3.6.1). By linearity we extend this pairing to

$$\big\langle\ ,\ \big\rangle_V\;:\;\mathscr{H}(\Gamma)\otimes_{\Lambda}H^1_{\mathrm{Iw}}(\mathbb{Q}_p,T)\times\mathscr{H}(\Gamma)\otimes_{\Lambda}H^1_{\mathrm{Iw}}(\mathbb{Q}_p,T^*(1))^{\iota}\to\mathscr{H}(\Gamma).$$

For any $\eta\in\mathbf{D}_{\mathrm{cris}}(V^*(1))$ the element $\widetilde{\eta}=\eta\otimes(1+X)$ lies in $\mathcal{D}(V^*(1))$ and we define a map

$$\mathfrak{L}^{\varepsilon}_{V,1-h,\eta}\;:\;H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V)\to\mathscr{H}(\Gamma)$$

by

$$\mathfrak{L}^{\varepsilon}_{V,1-h,\eta}(x)=\big\langle x,\mathrm{Exp}^{\varepsilon^{-1}}_{V^*(1),h}(\tilde{\eta})^{\iota}\big\rangle_V.$$

Lemma 1.3.4. *For any $j\in\mathbb{Z}$ one has*

$$\mathfrak{L}^{\varepsilon}_{V(-1),-h,\eta\otimes e_1}(\mathrm{Tw}^{\varepsilon}_{-1}(x))\,=\,\mathrm{Tw}_1\left(\mathfrak{L}^{\varepsilon}_{V,1-h,\eta}(x)\right).$$

Proof. A short computation shows that $\left\langle \mathrm{Tw}^{\varepsilon}_{j}(x),\mathrm{Tw}^{\varepsilon}_{-j}(y)\right\rangle_V\,=\,\mathrm{Tw}_{-j}\left\langle x,y\right\rangle_{V(j)}$. Taking into account that $\mathrm{Tw}_1^{\varepsilon^{-1}}=-\mathrm{Tw}_1^{\varepsilon}$ we have

$$\begin{aligned}
&\mathfrak{L}^{\varepsilon}_{V(-1),-h,\eta\otimes e_1}(\mathrm{Tw}^{\varepsilon}_{-1}(x))=\left\langle \mathrm{Tw}^{\varepsilon}_{-1}(x),\mathrm{Exp}^{\varepsilon^{-1}}_{V^*(2),h+1}(\widetilde{\eta\otimes e_1})^{\iota}\right\rangle_{V(-1)}=\\
&\left\langle \mathrm{Tw}^{\varepsilon}_{-1}(x),-\mathrm{Tw}_1^{\varepsilon^{-1}}\left(\mathrm{Exp}^{\varepsilon^{-1}}_{V^*(1),h}(\widetilde{\eta})\right)^{\iota}\right\rangle_{V(-1)}=\left\langle \mathrm{Tw}^{\varepsilon}_{-1}(x),\mathrm{Tw}_1^{\varepsilon}\left(\mathrm{Exp}^{\varepsilon^{-1}}_{V^*(1),h}(\widetilde{\eta})\right)^{\iota}\right\rangle_{V(-1)}=\\
&\mathrm{Tw}_1\left\langle x,\mathrm{Exp}^{\varepsilon^{-1}}_{V^*(1),h}(\widetilde{\eta})^{\iota}\right\rangle_V=\mathrm{Tw}_1\left(\mathfrak{L}^{\varepsilon}_{V,1-h,\eta}(x)\right)
\end{aligned}$$

and the lemma is proved.

1.4. p-adic distributions (see [Cz6], chapter II, [PR2], sections 1.1-1.2). Let $\mathcal{D}(\mathbb{Z}_p^*,L)$ be the space of distributions on $\mathbb{Z}_p^*$ with values in a finite extensions L of $\mathbb{Q}_p$. To each $\mu\in\mathcal{D}(\mathbb{Z}_p^*,L)$ one can associate its Amice transform $\mathscr{A}_{\mu}(X)\in L[[X]]$ by

$$\mathscr{A}_{\mu}(X)=\int_{\mathbb{Z}_p^*}(1+X)^x\mu(x)=\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p^*}\binom{x}{n}\mu(x)\right)X^n.$$

The map $\mu\mapsto\mathscr{A}_{\mu}(X)$ establishes an isomorphism between $\mathcal{D}(\mathbb{Z}_p^*,L)$ and $(\mathscr{R}_L^+)^{\psi=0}$. We will denote by $\mathbf{M}(\mu)$ the unique element of $\mathscr{H}(\Gamma)$ such that

$$\mathbf{M}(\mu)\,(1+X)\,=\,\mathcal{A}_{\mu}(X).$$

For each $m\in\mathbb{Z}$ the character $\chi^m\,:\,\Gamma\to\mathbb{Z}_p^*$ can be extended to a unique continuous L-linear map $\chi^m\,:\,\mathscr{H}(\Gamma)\to L^*$. If $h=\sum\limits_{i=0}^{p-2}\delta_ih_i(\gamma_1-1)$, then $\chi^m(h)=h_i(\chi^m(\gamma_1)-1)$ with $i\equiv m\pmod{(p-1)}$. An easy computation shows that

$$\int_{\mathbb{Z}_p^*}x^m\mu(x)\,=\,\partial^m\mathscr{A}_{\mu}(0)\,=\,\chi^m(\mathbf{M}(\mu)).$$

If $x\in\mathbb{Z}_p^*$ we set $\langle x\rangle=\omega^{-1}(x)\,x$ where ω denotes the Teichmüller character. To any $\mu\in\mathcal{D}(\mathbb{Z}_p^*,L)$ we associate p-adic functions

$$L_p(\mu,\omega^i,s)=\int_{\mathbb{Z}_p^*}\omega^i(x)\left\langle x\right\rangle^s\mu(x),\qquad 0\leqslant i\leqslant p-2.$$

Write $\mathbf{M}(\mu)=\sum\limits_{i=0}^{p-2}\delta_ih_i(\gamma_1-1)$. Then

$$L_p(\mu,\omega^i,s)=h_i\left(\chi(\gamma_1)^s-1\right). \tag{8}$$

To prove this formula it is enough to compare the values of the both sides at the integers $s\equiv i\pmod{(p-1)}$.

We say that μ is of order $r>0$ if its Amice transform $\mathscr{A}_\mu(X)=\sum_{n=1}^{\infty} a_nX^n$ is of order r i.e. if the sequence $|a_n|_p/n^r$ is bounded above. A distribution of order r is completely determined by the values of the integrals

$$\int_{\mathbb{Z}_p^*} \zeta_{p^n}^x x^i \mu(x), \qquad n\in\mathbb{N}, \quad 0\leqslant i\leqslant [r]$$

where $[r]$ is the largest integer no greater then r.

Set $\hat{\mathbb{Z}}^{(p)}=\mathbb{Z}_p^*\times\prod_{l\neq p}\mathbb{Z}_l$. A locally analytic function on $\prod_{l\neq p}\mathbb{Z}_l$ is locally constant and we say that a distribution μ on $\hat{\mathbb{Z}}^{(p)}$ is of order r if for any locally constant function $g(y)$ on $\prod_{l\neq p}\mathbb{Z}_l$ the linear map $f\mapsto \int_{\hat{\mathbb{Z}}^{(p)}} f(x)g(y)\mu(x,y)$ is a distribution of order r on $\mathbb{Z}_p^*$.

§2. The $\mathscr{L}$-invariant

2.1. The $\mathscr{L}$-invariant (see [Ben2]).

2.1.1. Definition of the $\mathscr{L}$-invariant. In this section we review the definition of the $\mathscr{L}$-invariant proposed in our previous article [Ben2]. We consider only representations which are crystalline at p because it is sufficient for the goals of this paper. Namely let S be a finite set of primes and $\mathbb{Q}^{(S)}/\mathbb{Q}$ be the maximal Galois extension of $\mathbb{Q}$ unramified outside $S\cup\{\infty\}$. Fix a finite extension $L/\mathbb{Q}_p$. Let V be an L-adic representation of G_S i.e. a finite dimensional L-vector space equipped with a continuous linear action of G_S. We write $H^*_S(\mathbb{Q},V)$ for the continuous cohomology of G_S with coefficients in V. For any prime l Bloch and Kato [BK] defined a subgroup $H^1_f(\mathbb{Q}_l,V)$ of $H^1(\mathbb{Q}_l,V)$ by

$$H^1_f(\mathbb{Q}_l,V)=\begin{cases}\ker(H^1(\mathbb{Q}_l,V)\to H^1(\mathbb{Q}_l^{\mathrm{ur}},V)) & \text{if } l\neq p,\\ \ker(H^1(\mathbb{Q}_p,V)\to H^1(\mathbb{Q}_p,V\otimes\mathbf{B}_{\mathrm{cris}})) & \text{if } l=p\end{cases}$$

where $\mathbf{B}_{\mathrm{cris}}$ is the ring of crystalline periods [Fo2]. The Selmer group of V is defined as

$$H^1_f(\mathbb{Q},V)=\ker\left(H^1_S(\mathbb{Q},V)\to\bigoplus_{l\in S}\frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)}\right).$$

We also define

$$H^1_{f,\{p\}}(\mathbb{Q},V)=\ker\left(H^1_S(\mathbb{Q},V)\to\bigoplus_{l\in S-\{p\}}\frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)}\right). \tag{9}$$

Note that this definition does not depend on the choice of S. From now until the end of this §we assume that V satisfies the following conditions

1) $H^1_f(\mathbb{Q},V)=H^1_f(\mathbb{Q},V^*(1))=0$.

2) $H^0_S(\mathbb{Q},V)=H^0_S(\mathbb{Q},V^*(1))=0$.

3) V is crystalline at p and $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}=0$.

4) $\dim_L t_V(\mathbb{Q}_p)=1$.

We remark that the last condition is not necessary to define the $\mathscr{L}$-invariant but it will be used in section 2.2 below.

The condition **1)** together with the Poitou-Tate exact sequence (see [FP], Proposition 2.2.1)

$$\cdots \to H^1_f(\mathbb{Q},V) \to H^1_S(\mathbb{Q},V) \simeq \bigoplus_{l\in S} \frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)} \to H^1_f(\mathbb{Q},V_f^*(1))^* \to \cdots$$

gives an isomorphism

$$H^1_S(\mathbb{Q},V) \simeq \bigoplus_{l\in S} \frac{H^1(\mathbb{Q}_l,V)}{H^1_f(\mathbb{Q}_l,V)}. \tag{10}$$

Let D be a one-dimensional subspace of $\mathbf{D}_{\mathrm{cris}}(V)$ on which φ acts as multiplication by p^{-1}. Using the weak admissibility of $\mathbf{D}_{\mathrm{cris}}(V)$ it is easy to see that D is not contained in $\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V)$ and therefore

$$\mathbf{D}_{\mathrm{cris}}(V) = \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) \oplus D$$

as L-vector spaces. Let m denote the unique Hodge-Tate weight of D. By Berger's theory [Ber4] (see also [BC], section 2.4.2), the intersection $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \cap \left(D \otimes_L \mathscr{R}_L[1/t]\right)$ is a saturated (φ,Γ)-submodule of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ of rank 1 which is isomorphic to $\mathscr{R}_L(\delta)$ with $\delta(x) = |x|x^m$. Thus we have an exact sequence of (φ,Γ)-modules

$$0 \to \mathscr{R}_L(\delta) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \to \mathbf{D} \to 0 \tag{11}$$

where $\mathbf{D} = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/\mathscr{R}_L(\delta)$. Since $\mathscr{D}_{\mathrm{cris}}(\mathbf{D})^{\varphi=1} = 0$ we have $H^0(\mathbf{D}) = 0$ and therefore $H^1(\mathscr{R}_L(\delta))$ injects into $H^1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \simeq H^1(\mathbb{Q}_p,V)$. Moreover, from the computation

$$\dim_L H^1_f(\mathbb{Q}_p,V) = \dim_L t_V(\mathbb{Q}_p) + \dim_L H^0(\mathbb{Q}_p,V) = 1$$

and the fact that $\dim_L H^1_f(\mathscr{R}_L(\delta)) = 1$ (see section 1.2.5) it follows that $H^1_f(\mathbb{Q}_p,V) \simeq H^1_f(\mathscr{R}_L(\delta))$. Let $H^1_D(V)$ denote the inverse image of $H^1(\mathscr{R}_L(\delta))/H^1_f(\mathscr{R}_L(\delta))$ under the isomorphism (10). Then

$$H^1_D(V) \simeq \frac{H^1(\mathscr{R}_L(\delta))}{H^1_f(\mathscr{R}_L(\delta))} \tag{12}$$

and the localisation map $H^1_S(V) \to H^1(\mathbb{Q}_p,V)$ induces an injection $H^1_D(V) \to H^1(\mathscr{R}_L(\delta))$ which can be inserted in a commutative diagram

$$\begin{tikzcd}
D \arrow[r, "i_f", "\sim"'] & H^1_f(\mathscr{R}_L(\delta)) \\
H^1_D(V) \arrow[u, "\rho_f"] \arrow[r] \arrow[d, "\rho_c"'] & H^1(\mathscr{R}_L(\delta)) \arrow[u, "p_f"'] \arrow[d, "p_c"] \\
D \arrow[r, "i_c", "\sim"'] & H^1_c(\mathscr{R}_L(\delta)),
\end{tikzcd}$$

where ρ_f and ρ_c are defined as the unique maps making this diagram commute. We remark that from (12) it follows that ρ_c is an isomorphism.

Definition. *The $\mathscr{L}$-invariant $\mathscr{L}(V,D)$ is defined to be the unique element of L such that*

$$\rho_f \circ \rho_c^{-1}(x) = \mathscr{L}(V,D)\, x, \qquad x \in D.$$

2.1.2. Duality. Passing to duals in (11) and taking the long exact cohomology sequence we obtain an exact sequence

$$H^1(\mathbf{D}^*(\chi)) \to H^1(\mathbb{Q}_p, V^*(1)) \to H^1(\mathscr{R}_L(\chi\delta^{-1})) \to 0.$$

Moreover the map $H^1_f(\mathbb{Q}_p, V^*(1)) \to H^1_f(\mathscr{R}_L(\chi\delta^{-1}))$ is surjective ([Ben2], Corollary 1.4.6) and

$$\dim_L \left(\frac{H^1(\mathbb{Q}_p, V^*(1))}{H^1_f(\mathbb{Q}_p, V^*(1))} \right) = \dim_L H^1_f(\mathbb{Q}_p, V) = 1.$$

Therefore

$$\frac{H^1(\mathbb{Q}_p, V^*(1))}{H^1_f(\mathbb{Q}_p, V^*(1))} \simeq \frac{H^1(\mathscr{R}_L(\chi\delta^{-1}))}{H^1_f(\mathscr{R}_L(\chi\delta^{-1}))}$$

and the Poitou-Tate exacte sequence gives an isomorphism

$$H^1_{f,\{p\}}(\mathbb{Q}, V^*(1)) \simeq \frac{H^1(\mathbb{Q}_p, V^*(1))}{H^1_f(\mathbb{Q}_p, V^*(1))} \simeq \frac{H^1(\mathscr{R}_L(\chi\delta^{-1}))}{H^1_f(\mathscr{R}_L(\chi\delta^{-1}))}.$$

Now the decomposition (4) for the character $\delta^{-1}\chi$ provides a diagram

$$\begin{tikzcd}
\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\chi\delta^{-1})) \arrow[r, "i_f^*", "\sim"'] & H^1_f(\mathscr{R}_L(\chi\delta^{-1})) \\
H^1_{f,\{p\}}(\mathbb{Q}, V^*(1)) \arrow[r] \arrow[u, "\rho_f^*"] \arrow[d, "\rho_c^*"'] & H^1(\mathscr{R}_L(\chi\delta^{-1})) \arrow[u, "p_f^*"'] \arrow[d, "p_c^*"] \\
\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\chi\delta^{-1})) \arrow[r, "i_c^*", "\sim"'] & H^1_c(\mathscr{R}_L(\chi\delta^{-1})).
\end{tikzcd}$$

Let $D^* = \mathrm{Hom}(\mathbf{D}_{\mathrm{cris}}(V)/D, \mathbf{D}_{\mathrm{cris}}(L(\chi))$. Then D^* is a φ-submodule of $\mathbf{D}_{\mathrm{cris}}(V)$ such that $\mathbf{D}_{\mathrm{cris}}(V^*(1)) = D^* \oplus \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V^*(1))$ and we define the $\mathscr{L}$-invariant associated to $V^*(1)$ and D^* as the unique element $\mathscr{L}(V^*(1), D^*) \in L$ such that $\rho_f^* \circ (\rho_c^*)^{-1}(x) = \mathscr{L}(V^*(1), D^*)\, x$. Note that

$$\mathscr{L}(V^*(1), D^*) = -\mathscr{L}(V,D) \tag{13}$$

(see [Ben2], Proposition 2.2.7).

2.2. Derivative of the large exponential map.

2.2.1. In this section we interpret $\mathscr{L}(V,D)$ in terms of the Bockstein homomorphism associated to the large exponential map. This interpretation is crucial for the proof of the main theorem of this paper. We keep the notations and conventions of section 2.1. Recall (see 1.3.2) that $H^1(\mathbb{Q}_p, \mathscr{H}(\Gamma) \otimes_{\mathbb{Q}_p} V) = \mathscr{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$ injects into $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$. Set

$$H^1_\delta(\mathbb{Q}_p, \mathscr{H}(\Gamma) \otimes_{\mathbb{Q}_p} V) = \mathscr{R}_L(\delta) \cap H^1(\mathbb{Q}_p, \mathscr{H}(\Gamma) \otimes_{\mathbb{Q}_p} V).$$

The projection map induces a commutative diagram

$$\begin{tikzcd}
H^1_\delta(\mathbb{Q}_p, \mathscr{H}(\Gamma)\otimes_{\mathbb{Q}_p} V) \arrow[r] \arrow[d] & H^1(\mathbb{Q}_p, \mathscr{H}(\Gamma)\otimes_{\mathbb{Q}_p} V) \arrow[d, "\mathrm{pr}_0"'] \\
H^1(\mathscr{R}_L(\delta)) \arrow[r] & H^1(\mathbb{Q}_p, V)
\end{tikzcd}$$

where the bottom arrow is an injection. We fix a generator $\gamma \in \Gamma$ and an integer $h \geqslant 1$ such that $\mathrm{Fil}^{-h}\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$.

Proposition 2.2.2. *For any $a \in D$ let $x \in \mathcal{D}(V)$ be such that $x(0) = a$. Then*
i) There exists a unique $F \in H^1_\delta(\mathbb{Q}_p, \mathscr{H}(\Gamma)\otimes V)$ such that

$$(\gamma - 1)\, F = \mathrm{Exp}^{\varepsilon}_{V,h}(x).$$

ii) The composition map

$$\begin{aligned}
&\delta_D\,:\, D \to H^1_\delta(\mathbb{Q}_p, \mathscr{H}(\Gamma)\otimes V) \to H^1(\mathscr{R}_L(\delta))\\
&\delta_D(a) = \mathrm{pr}_0(F)
\end{aligned}$$

is well defined and is explicitly given by the following formula

$$\delta_D(a) \;=\; \Gamma\,(h)\,\left(1-\frac{1}{p}\right)^{-1} (\log\chi(\gamma))^{-1}\, i_c(a).$$

Proof. 1) Since $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} = 0$, the operator $1-\varphi$ is invertible on $\mathbf{D}_{\mathrm{cris}}(V)$ and we have a diagram

$$\begin{tikzcd}
\mathcal{D}(V) \arrow[r, "\mathrm{Exp}^{\varepsilon}_{V,h}"] \arrow[d, "\Xi^{\varepsilon}_{V,0}"] & H^1(\mathbb{Q}_p, \mathscr{H}(\Gamma)\otimes V) \arrow[d, "\mathrm{pr}_0"] \\
t_V(\mathbb{Q}_p)\otimes \mathbf{D}_{\mathrm{cris}}(V) \arrow[r, "(h-1)!\exp_V"] & H^1(\mathbb{Q}_p, V).
\end{tikzcd}$$

where $\Xi^{\varepsilon}_{V,0}(f) = \left(\dfrac{1-p^{-1}\varphi^{-1}}{1-\varphi}\, f(0), 0\right)$. If $x \in D\otimes \mathscr{R}_L^{\psi=0}$ then $\Xi^{\varepsilon}_{V,0}(x) = 0$ and $\mathrm{pr}_0\left(\mathrm{Exp}^{\varepsilon}_{V,h}(x)\right) \;=\; 0.$ On the other hand, as $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$ is $\Lambda_{\mathbb{Q}_p}$-free, the map $\left(\mathscr{H}(\Gamma)\otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)\right)_\Gamma \to H^1(\mathbb{Q}_p, V)$ is injective and therefore there exists a unique $F \in \mathscr{H}(\Gamma)\otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$ such that $\mathrm{Exp}^{\varepsilon}_{V,h}(x) \;=\; (\gamma-1)\,F$. Let $y \in D\otimes \mathscr{R}_L^{\psi=0}$ be another element such that $y(0) = a$ and let $\mathrm{Exp}^{\varepsilon}_{V,h}(y) = (\gamma-1)\,G$. Since $\mathscr{R}_L^{\psi=0} = \mathscr{H}(\Gamma)\,(1+X)$ we have $y = x + (\gamma-1)g$ for some $g \in D\otimes \mathscr{R}_L^{\psi=0}$. As $\mathrm{Exp}^{\varepsilon}_{V,h}(g) = 0$, we obtain immediately that $\mathrm{pr}_0(G) = \mathrm{pr}_0(F)$ and we proved that the map δ_D is well defined.

2) Take $a \in D$ and set

$$x = a\otimes \ell\left(\frac{(1+X)^{\chi(\gamma)}-1}{X}\right),$$

where $\ell(u) \;=\; \dfrac{1}{p}\log\left(\dfrac{u^p}{\varphi(u)}\right)$. An easy computation shows that

$$\sum_{\zeta^p=1} \ell\left(\frac{\zeta^{\chi(\gamma)}(1+X)^{\chi(\gamma)}-1}{\zeta(1+X)-1}\right) \;=\; 0.$$

Thus $x \in D \otimes O_L[[X]]^{\psi=0}$. Write x in the form $f = (1-\varphi)\,(\gamma-1)\,(a\otimes \log(X))$. Then

$$\Omega^{\varepsilon}_{V,h}(x) \;=\; (-1)^{h-1}\frac{\log\chi(\gamma_1)}{p}\,t^h\partial^h((\gamma-1)\,(a\log(X)) \;=\; \left(1-\frac{1}{p}\right)\log\chi(\gamma)\,(\gamma-1)\,F$$

where

$$F \;=\; (-1)^{h-1}t^h\partial^h(a\log(X)) \;=\; (-1)^{h-1}at^h\partial^{h-1}\left(\frac{1+X}{X}\right).$$

This implies immediately that $F \in H^1_\delta(\mathbb{Q}_p, \mathscr{H}(\Gamma)\otimes V)$. On the other hand, as $D = \mathcal{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$ without lost of generality we may assume that $a = t^{-m}e_\delta$ where $\delta(x) = |x|x^m$. Then

$$F = (-1)^{h-1}t^{h-m}\partial^h\log(X)\,e_\delta.$$

By 1.2.3 one has $\mathrm{pr}_0(F) = \mathrm{cl}(G,F)$ where $(1-\gamma)\,G \;=\; (1-\varphi)\,F$ and by Lemma 1.5.1 of [CC1] there exists a unique $b \in \mathscr{E}_L^{\dagger,\psi=0}$ such that $(1-\gamma)\,b = \ell(X)$. One has

$$(1-\gamma)\,\left(t^{h-m}\partial^h b e_\delta\right) \;=\; (1-\varphi)\,\left(t^{h-m}\partial^h\log(X)e_\delta\right) \;=\; (-1)^{h-1}(1-\varphi)\,F.$$

Thus $G = (-1)^{h-1}t^{h-m}\partial^h(b)e_\delta$ and $\mathrm{res}\left(G\,t^{m-1}dt\right) \;=\; (-1)^{h-1}\mathrm{res}\left(t^{h-1}\partial^h(b)dt\right)\,e_\delta = 0$. Next from the congruence $F \equiv (h-1)!\,t^{-m}e_\delta \pmod{\mathbb{Q}_p[[X]]\,e_\delta}$ it follows that $\mathrm{res}(Ft^{m-1}dt) \;=\; (h-1)!\,e_\delta$. Therefore by [Ben2], Corollary 1.5.5 we have

$$\left(1-\frac{1}{p}\right)\,(\log\chi(\gamma))\,\mathrm{cl}(G,F) \;=\; (h-1)!\,\mathrm{cl}(\boldsymbol{\beta}_m) \;=\; (h-1)!\,i_c(a). \tag{14}$$

On the other hand

$$x(0) \;=\; a\otimes \ell\left(\frac{(1+X)^{\chi(\gamma)}-1}{X}\right)\Bigg|_{X=0} \;=\; a\left(1-\frac{1}{p}\right)\,\log\chi(\gamma). \tag{15}$$

The formulas (14) and (15) imply that

$$\delta_D(a) = (h-1)!\,\left(1-\frac{1}{p}\right)^{-1}\,(\log\chi(\gamma))^{-1}\,i_c(a).$$

and the proposition is proved.

2.2.3. Fix a non-zero element $d \in D$ and consider the large logarithmic map

$$\mathfrak{L}^{\varepsilon}_{V^*(1),1-h,d}\;:\;H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V^*(1)) \to \mathscr{H}(\Gamma)$$

(see 1.3.3). Let

$$H^1_{\mathrm{Iw},S}(\mathbb{Q}, T_f^*(1)) = \varprojlim_{\mathrm{cores}} H^1_S(\mathbb{Q}(\zeta_{p^n}), T^*(1))$$

denote the global Iwasawa cohomology with coefficients in $T^*(1)$ and let

$$H^1_{\mathrm{Iw},S}(\mathbb{Q}, V^*(1)) = H^1_{\mathrm{Iw},S}(\mathbb{Q}, T^*(1))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

The main results of this paper will be directly deduced from the following statement.

Proposition 2.2.4. *Let* $\mathbf{z}\in H^1_{\mathrm{Iw},S}(\mathbb{Q},V^*(1))$. *Assume that* $\mathbf{z}_0=\mathrm{pr}_0(\mathbf{z})\in H^1_S(\mathbb{Q},V^*(1))$ *is non-zero and denote by* $\mu_{\mathbf{z}}\in\mathcal{D}(\mathbb{Z}_p^*,L)$ *the distribution defined by*

$$\mathbf{M}(\mu_{\mathbf{z}})=\mathfrak{L}^{\varepsilon}_{V^*(1),1-h,d}(\mathbf{z}).$$

Consider the p-adic function

$$L_p(\mu_{\mathbf{z}},s)=\int_{\mathbb{Z}_p^*}\langle x\rangle^s\mu_{\mathbf{z}}(x).$$

Then $L_p(\mu_{\mathbf{z}},0)=0$ *and*

$$L_p'(\mu_{\mathbf{z}},0)\;=\;-\mathscr{L}(V,D)\,\Gamma(h)\left(1-\frac{1}{p}\right)^{-1}\left[d,\exp^*_{V^*(1)}(\mathbf{z}_0)\right]_V$$

where $[\ ,\ ]_V\,:\,\mathbf{D}_{\mathrm{cris}}(V)\times\mathbf{D}_{\mathrm{cris}}(V^*(1))\to L$ *is the canonical duality.*

Proof. First note that by [PR1], section 2.1.7 for $l\neq p$ one has $H^1_{\mathrm{Iw}}(\mathbb{Q}_l,V^*(1))\simeq H^0(\mathbb{Q}_l(\zeta_{p^\infty}),V^*(1))$ and therefore $H^1_{\mathrm{Iw}}(\mathbb{Q}_l,V^*(1))_\Gamma$ is contained in $H^1_f(\mathbb{Q}_l,V^*(1))$. Thus $H^1_{\mathrm{Iw},S}(\mathbb{Q},V^*(1))_\Gamma$ injects into $H^1_{f,\{p\}}(\mathbb{Q},V^*(1))$ and $\mathbf{z}_0\in H^1_{f,\{p\}}(\mathbb{Q},V^*(1))$. Recall that we fixed a basis d of the one-dimensional L-vector space $D=\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\delta))$. Let d^* be the basis of $\mathscr{D}_{\mathrm{cris}}(\mathscr{R}_L(\chi\delta^{-1}))$ which is dual to d. Let $\tilde{\mathbf{z}}_0$ denote the image of $\mathbf{z}_0$ under the projection map $H^1(\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1)))\to H^1(\mathscr{R}_L(\chi\delta^{-1}))$. Write $\tilde{\mathbf{z}}_0=a\,i_f(d^*)+b\,i_c(d^*)$. Then $\mathscr{L}(V,D)=-a/b$ by (13). By Proposition 1.2.6 and (6) we have

$$\begin{aligned}(16)\qquad \left[d,\exp^*_{V^*(1)}(\mathbf{z}_0)\right]_V\;&=\;-\exp_V(d)\cup\mathbf{z}_0\;=\;-\exp_{\mathscr{R}_L(\delta)}(d)\cup\,\tilde{\mathbf{z}}_0\;=\\ &=-b\left(i_f(d)\cup i_c(d^*)\right)\;=\;-b\,\left(\boldsymbol{\alpha}_m\cup\mathbf{y}_{m-1}\right)\;=\;b.\end{aligned}$$

Let $\mathbf{M}(\mu_{\mathbf{z}})\;=\;\sum\limits_{i=0}^{p-2}\delta_ih_i(\gamma_1-1)$. Then $L_p(\mu_{\mathbf{z}},s)=h_0\left(\chi(\gamma_1)^s-1\right)$ by (7). From Proposition 2.2.2 it follows that there exists $F\in H^1_\delta(\mathbb{Q}_p,\mathscr{H}(\Gamma)\otimes V)$ such that $\mathrm{Exp}^{\varepsilon^{-1}}_{V,h}(d\otimes(1+X))=(\gamma-1)\,F$ and

$$\mathbf{M}(\mu_{\mathbf{z}})=\mathfrak{L}^{\varepsilon}_{V^*(1),1-h,d}(\mathbf{z})=\left\langle\mathbf{z}\,,\,\mathrm{Exp}^{\varepsilon^{-1}}_{V,h}(d\otimes(1+X)^\iota\right\rangle_V\;=\;(\gamma^{-1}-1)\,\langle\mathbf{z},F^\iota\rangle_V\,.$$

Put $\langle\mathbf{z},F^\iota\rangle_{V_f}=\sum\limits_{i=0}^{p-2}\delta_iH_i(\gamma_1-1)$. Then $L_p(\mu_{\mathbf{z}},s)=(\chi(\gamma)^{-s}-1)\,H_0(\chi(\gamma_1)^s-1)$. Since $\chi(\gamma_1)=\chi(\gamma)^{p-1}$ the last formula implies that $L_p(\mu_{\mathbf{z}},s)$ has a zero at $s=0$ and

$$(17)\qquad L_p'(\mu_{\mathbf{z}},0)=-(\log\chi(\gamma))\,H_0(0).$$

On the other hand, by Proposition 2.2.2

$$\begin{aligned}(18)\quad H_0(0)=\mathbf{z}_0\cup(\mathrm{pr}_0F)=\tilde{\mathbf{z}}_0\cup\delta_D(d)=\Gamma(h)\left(1-\frac{1}{p}\right)^{-1}(\log\chi(\gamma))^{-1}\left(\tilde{\mathbf{z}}_0\cup i_c(d)\right)\;=\\ =-\Gamma(h)\left(1-\frac{1}{p}\right)^{-1}(\log\chi(\gamma))^{-1}a.\end{aligned}$$

From (16), (17) and (18) we obtain that

$$L_p'(\mu_{\mathbf{z}},0)=\Gamma(h)\left(1-\frac{1}{p}\right)^{-1}a\;=\;-\mathscr{L}(V,D)\,\Gamma(h)\left(1-\frac{1}{p}\right)^{-1}\left[d_\alpha,\exp^*_{V^*(1)}(\mathbf{z}_0)\right]_V$$

and the proposition is proved.

§3. Trivial zeros of Dirichlet L-functions

3.1. Dirichlet L-functions. Let $\eta : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ be a Dirichlet character of conductor N. We fix a primitive N-th root of unity ζ_N and denote by $\tau(\eta) = \sum_{a \mod N} \eta(a)\, \zeta_N^a$ the Gauss sum. The Dirichlet L-function

$$L(\eta, s) = \sum_{n=1}^{\infty} \frac{\eta(n)}{n^s}, \qquad \mathrm{Re}(s) > 1$$

has a meromorphic continuation on the whole complex plane and satisfies the functional equation

$$\left(\frac{N}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\delta_\eta}{2}\right) L(\eta, s) \;=\; W_\eta \left(\frac{N}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{1-s+\delta_\eta}{2}\right) L(\bar{\eta}, 1-s)$$

where $W_\eta = i^{-\delta_\eta} N^{-1/2} \tau(\eta)$ and $\delta_\eta = \frac{1-\eta(-1)}{2}$. From now until the end of this § we assume that η is not trivial. Fix a primitive N-th root of unity ζ_N. Then for any $j \geqslant 0$ the special value $L(\eta, -j)$ is the algebraic integer given by

$$L(\eta, -j) = \frac{d^j F_\eta(0)}{dt^j} \tag{19}$$

where

$$F_\eta(t) = \frac{1}{\tau(\eta^{-1})} \sum_{a \mod N} \frac{\eta^{-1}(a)}{1 - \zeta_N^a e^t}$$

(see for example [PR3], proof of Proposition 3.1.4). In particular

$$L(\eta, 0) = \frac{1}{\tau(\eta^{-1})} \sum_{a \mod N} \frac{\eta^{-1}(a)}{1 - \zeta_N^a}. \tag{20}$$

Moreover $L(\eta, -j) = 0$ if and only if $j \equiv \delta_\eta \pmod 2$.

Let p be a prime number such that $(p, N) = 1$. We fix a finite extension L of $\mathbb{Q}_p$ containing the values of all Dirichlet characters η of conductor N. The power series

$$\mathscr{A}_{\mu_\eta}(X) = -\frac{1}{\tau(\eta^{-1})} \sum_{a \mod N} \left(\frac{\eta^{-1}(a)}{(1+X)\,\zeta_N^a - 1} - \frac{\eta^{-1}(a)}{(1+X)^p \zeta_N^{pa} - 1} \right)$$

lies in $O_L[[X]]^{\psi=0}$ and therefore can be viewed as the Amice transform of a unique mesure μ_η on $\mathbb{Z}_p^*$. The p-adic L-functions associated to η are defined to be

$$L_p(\eta\, \omega^m, s) = \int_{\mathbb{Z}_p^*} \omega^{m-1}(x) \langle x \rangle^{-s} \mu_\eta(x), \qquad 0 \leqslant m \leqslant p-2.$$

From (7) and (19) it follows that these functions satisfy the following interpolation property

$$L_p(\eta\, \omega^m, 1-j) = (1 - (\eta\, \omega^{m-j})(p) p^{1-j})\, L(\eta\, \omega^{m-j}, 1-j) \qquad j \geqslant 1$$

(Iwasawa theorem, see for example [PR3], Proposition 3.1.4). Note that the Euler factor $1-(\eta\, \omega^{m-n})(p)p^{1-j}$ vanishes if $m = j = 1$ and $\eta(p) = 1$ and that $L(\eta, 0)$ does not vanish if and only if η is odd i.e. $\eta(-1) = -1$.

3.2. p-adic representations associated to Dirichlet characters. We fix a conductor N and a prime number p such that $(p,N)=1$. Set $F=\mathbb{Q}(\zeta_N)$, $G=\mathrm{Gal}(F/\mathbb{Q})$ and let $\rho\,:\,G\simeq(\mathbb{Z}/N\mathbb{Z})^*$ denote the canonical isomorphism normalized by $g(\zeta_N)=\zeta_N^{\rho(g)^{-1}}$. Fix a finite extension $L/\mathbb{Q}_p$ containing the values of all Dirichlet characters modulo N. If η is such a character, we identify η with the character $\psi\circ\rho$ of the Galois group and denote by $L(\psi)$ the associated one-dimensional representation of $G_{\mathbb{Q}}$.

Let η be a non trivial character of conductor N. Let S denote the set of primes dividing N. We need the following well known results about the Galois cohomology of $L(\eta)$.

i) $H^*(\mathbb{Q}_l,L(\eta))=H^*(\mathbb{Q}_l,L(\chi\eta^{-1}))=0$ for $l\in S$.

ii) $H^1_f(\mathbb{Q},L(\eta))=0$ and $H^1_f(\mathbb{Q},L(\chi\eta^{-1}))\simeq(O_F^*\otimes_{\mathbb{Z}}L)^{(\eta)}$. In particular, $H^1_f(\mathbb{Q},L(\chi\eta^{-1}))=0$ if η is odd.

iii) The restriction of $L(\eta)$ on the decomposition group at p is crystalline. More precisely, φ acts on $\mathbf{D}_{\mathrm{cris}}(L(\eta))$ as multiplication by $\eta(p)$ and the unique Hodge-Tate weight of $L(\eta)$ is 0.

Note that $H^0(\mathbb{Q}_l,L(\eta))=0$ if $l|N$ because in this case the inertia group acts non-trivially on $L(\eta)$. Together with Poincaré duality and the Euler characteristic formula this gives i). To prove ii) it is enough to remark that $H^1_f(F,\mathbb{Q}_p(1))\simeq O_F^*\hat{\otimes}\mathbb{Q}_p$ (see for example [Ka1], §5). Finally iii) follows immediately from the definition of $\mathbf{D}_{\mathrm{cris}}$.

Assume now that η is odd and that $\eta(p)=1$. Then φ acts on $\mathbf{D}_{\mathrm{cris}}(L(\chi\eta^{-1}))$ as multiplication by p^{-1} and $D=\mathbf{D}_{\mathrm{cris}}(L(\chi\eta^{-1}))$ satisfies the conditions from section 2.1.1. The isomorphism (9) writes

$$H^1_S(\mathbb{Q},L(\chi\eta^{-1}))\simeq\frac{H^1(\mathbb{Q}_p,L(\chi))}{H^1_f(\mathbb{Q}_p,L(\chi))}.$$

We denote simply by $\mathscr{L}(\chi\eta^{-1})$ the associated $\mathscr{L}$-invariant.

3.3. Trivial zeros.

3.3.1. Cyclotomic units. Let $F_n=F(\zeta_{p^n})$. The collection $\mathbf{z}_{\mathrm{cycl}}=(1-\zeta_N^{p^{-n}}\zeta_{p^n})_{n\geqslant1}$ form a norm compartible system of units which can be viewed as an element of $H^1_{\mathrm{Iw},S}(F,L(\chi))$ using the Kummer maps $F_n^*\to H^1_S(F_n,L(\chi))$. Twisting by ε^{-1} we obtain an element $\mathbf{z}_{\mathrm{cycl}}(-1)\in H^1_{\mathrm{Iw},S}(F,L)$. Shapiro's lemma gives an isomorphism of G-modules $H^1_{\mathrm{Iw}}(F\otimes\mathbb{Q}_p,L)\simeq H^1_{\mathrm{Iw}}(\mathbb{Q}_p,L[G]^\iota)$. Let $e_\eta=\dfrac{1}{|G|}\sum\limits_{g\in G}\eta^{-1}(g)\,g$. Since $e_\eta L[G]^\iota=Le_{\eta^{-1}}$ is isomorphic to $L(\eta^{-1})$ we have an isomorphism

$$e_\eta H^1_{\mathrm{Iw}}(F\otimes\mathbb{Q}_p,L)\simeq H^1_{\mathrm{Iw}}(\mathbb{Q}_p,L(\eta^{-1})).$$

Moreover $\mathbf{D}_{\mathrm{cris}}(L[G])\simeq(L[G]\otimes F)^G\simeq L\otimes F$. The isomorphism $\mathbb{Q}[G]\simeq F$ defined by $\lambda\mapsto\lambda(\zeta_N)$ induces an isomorphism $\mathbf{D}_{\mathrm{cris}}(L[G])\simeq L[G]$ and therefore we can consider e_η as a basis of $\mathbf{D}_{\mathrm{cris}}(L(\eta^{-1}))$. Let $\mathbf{z}^\eta_{\mathrm{cycl}}(-1)$ denote the image of $\mathbf{z}_{\mathrm{cycl}}(-1)$ in $H^1_{\mathrm{Iw}}(\mathbb{Q}_p,L(\eta^{-1}))$. We need the following properties of these elements:

1) Relation to the complex L-function. Let $\mathbf{z}^{\eta^{-1}}_{\mathrm{cycl}}(-1)_0$ denote the projection of $\mathbf{z}^{\eta^{-1}}_{\mathrm{cycl}}(-1)$ on $H^1(\mathbb{Q}_p,L(\eta))$. Then

$$\exp^*_{L(\eta)}(\mathbf{z}^{\eta^{-1}}_{\mathrm{cycl}}(-1)_0)=-\left(1-\frac{\eta^{-1}(p)}{p}\right)\,L(\eta,0)\,e_{\eta^{-1}}$$

2) Relation to the p-adic L-function. Let $e^*_{\eta^{-1}}\in\mathbf{D}_{\mathrm{cris}}(L(\chi\eta^{-1}))$ be the basis which is dual to $e_{\eta^{-1}}$ and let $\mathfrak{L}^{(\varepsilon)}_{L(\eta),0}\,:\,H^1_{\mathrm{Iw}}(\mathbb{Q}_p,L(\eta))\to\mathscr{H}(\Gamma)$ denote the associated logarithmic map. Then

$$\mathfrak{L}^{\varepsilon}_{L(\eta),0}(\mathbf{z}^{\eta^{-1}}_{\mathrm{cycl}}(-1))=-\mathbf{M}(\mu_\eta).$$

We remark that 1) follows from the explicit reciprocity law of Iwasawa [Iw] together with (20). See also [Ka1], Theorem 5.12 and [HK], Corollary 3.2.7 where a more general statement is proved using the explicit reciprocity law for $\mathbb{Q}_p(r)$. The statement 2) is a reformulation of Coleman's construction of p-adic L-functions in terms of the large logarithmic map ([PR3], Proposition 3.1.4).

Theorem 3.3.2. *Let η be an odd character of conductor N. Assume that p is a prime odd number such that $p \nmid N$ and $\eta(p) = 1$. Then*

$$L'(\eta\,\omega, 0) \;=\; \mathscr{L}(\chi\eta^{-1})\, L(\eta, 0).$$

Proof. Applying Proposition 2.3.4 to $V = L(\chi\eta^{-1})$, $D = \mathbf{D}_{\mathrm{cris}}(L(\chi\eta^{-1}))$ and $\mathbf{z} = \mathbf{z}_{\mathrm{cycl}}^{\eta^{-1}}(-1)$ and taking into account 1-2) above we obtain that

$$L_p'(\eta\,\omega, 0) = L_p'(\mu_z, 0) = -\mathscr{L}(\chi\eta^{-1}) \left(1 - \frac{1}{p}\right)^{-1} \left[e^*_{\eta^{-1}}, \exp^*_{L(\eta)}(\mathbf{z}_0)\right] \;=\; \mathscr{L}(\chi\eta^{-1})\, L(\eta, 0)$$

and the theorem is proved.

Corollary 3.3.3. *Theorem 3.3.3 can be written in the form*

$$L'(\eta\,\omega, 0) \;=\; -\mathscr{L}(\eta)\, L(\eta, 0).$$

where $\mathscr{L}(\eta)$ is the invariant defined by (3).

Proof. It is easy to see that $\mathscr{L}(\eta)$ coincides with the $\mathscr{L}$-invariant $\mathscr{L}(L(\eta), D^*)$ where $D^* = \{0\}$ and therefore $\mathscr{L}(\eta) = -\mathscr{L}(\chi\eta^{-1})$ by (13).

§4. Trivial zeros of modular forms

4.1. The $\mathscr{L}$-invariant at near central points.

4.1.1. p-adic representations associated to modular forms. Let $f = \sum\limits_{n=1}^{\infty} a_n q^n$ be a normalized newform on $\Gamma_0(N)$ of an odd weight $k \geqslant 2$ and character ε and let p be a fixed prime. Deligne [D1] associated to f a p-adic representation

$$\rho_f \;:\; \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(W_f).$$

with coefficients in a finite extension L of $\mathbb{Q}_p$. This representation has the following properties:

i) $\det \rho_f$ is isomorphic to $\varepsilon\chi^{k-1}$ where χ is the cyclotomic character.

ii) If $l \nmid Np$ then the restriction of ρ_f on the decomposition group at l is unramified and

$$\det(1 - \mathrm{Fr}_l X \mid W_f) = 1 - a_l X + \varepsilon(l)\, l^{k-1} X^2$$

(Deligne-Langlands-Carayol theorem [Ca], [La]).

iii) The restriction of ρ_f on the decomposition group at p is potentially semistable with Hodge-Tate weights $(0, k-1)$ [Fa1]. It is crystalline if $p \nmid N$ and semistable non-crystalline if $p \parallel N$ and $(p, \mathrm{cond}(\varepsilon)) = 1$. In all cases

$$\det(1 - \varphi X \mid \mathbf{D}_{\mathrm{cris}}(W_f)) = 1 - a_p X + \varepsilon(p)\, p^{k-1} X^2$$

(Saito theorem [Sa], see also [Fa2], [Ts]).

Let $L(f,s)=\sum_{n=1}^{\infty} a_n n^{-s}$ be the complex L-function associated to f. It is well known that $L(f,s)$ decomposes into an Euler product

$$L(f,s)=\prod_{l} E_l(f,l^{-s})^{-1}$$

with $E_l(f,X)=1-a_lX+\varepsilon(l)l^{k-1}X^2$.

4.1.2. The Selmer group. From now until the end of this § we assume that k is odd. Thus $\frac{k-1}{2}$ and $\frac{k+1}{2}$ are near critical points for $L(f,s)$. Set $V_f=W_f\left(\frac{k+1}{2}\right)$. Let f^* denote the complex conjugation of f i.e. $f^*=\sum_{n=1}^{\infty}\bar a_n q^n$. The canonical pairing $W_f\times W_{f^*}\to L(1-k)$ induces an isomorphism $W_{f^*}\left(\frac{k-1}{2}\right)\simeq V_f^*(1)$. We need the following basic results about the Galois cohomology of V_f:

i) $H^0(\mathbb{Q}_p,V_f)=H^0(\mathbb{Q}_p,V_f^*(1))=0$ and $\dim_L H^1(\mathbb{Q}_p,V_f)=\dim_L H^1(\mathbb{Q}_p,V_f^*(1))=2$.

ii) $H^1_f(\mathbb{Q}_p,V_f)$ and $H^1_f(\mathbb{Q}_p,V_f^*(1))$ are one-dimensional L-vector spaces.

iii) $H^1_f(\mathbb{Q},V_f)=H^1_f(\mathbb{Q},V_f^*(1))=0$.

We remark that i) follows from the fact that 0 is not a Hodge-Tate number of V_f and $V_f^*(1)$ and from the Euler-Poincaré characteristic formula. Next ii) follows from i) together with the formula

$$\dim_L H^1_f(\mathbb{Q}_p,V_f)=\dim_L t_{V_f}(L)+\dim_L H^0(\mathbb{Q}_p,V_f).$$

Finally iii) is a deep result of Kato ([Ka2], theorem 14.2).

4.1.3. The $\mathscr{L}$-invariant. We keep previous convention and notation and assume in addition that W_f is crystalline at p. Then the associated Dieudonné module $\mathbf{D}_{\mathrm{cris}}(W_f)$ is a two-dimensional L-vector space and the eigenvalues α and β of φ acting on $\mathbf{D}_{\mathrm{cris}}(W_f)$ are the roots of $X^2-a_pX+\varepsilon(p)p^{k-1}$. Moreover $|\alpha|=|\beta|=p^{\frac{k-1}{2}}$ by Deligne [D2].

We will assume that φ acts semisimply on $\mathbf{D}_{\mathrm{cris}}(W_f)$ (which conjecturally always holds). Together with the admissibility of $\mathbf{D}_{\mathrm{cris}}(W_f)$ this imply that $\alpha\neq\beta$ (see [Cz3], section 4.4). From the isomorphism $\mathbf{D}_{\mathrm{cris}}(V_f)=\mathbf{D}_{\mathrm{cris}}(W_f)\otimes\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\frac{k+1}{2})\big)$ it follows that $\mathbf{D}_{\mathrm{cris}}(V_f)$ has a basis d_α,d_β such that

$$\varphi(d_\alpha)=\alpha\,p^{-\frac{k+1}{2}}d_\alpha,\quad \varphi(d_\beta)=\beta\,p^{-\frac{k+1}{2}}d_\beta.$$

Moreover

$$\dim_L \mathrm{Fil}^i\mathbf{D}_{\mathrm{cris}}(V_f)=\begin{cases}2 & \text{if } i\leqslant -\frac{k+1}{2},\\ 1 & \text{if } -\frac{k-1}{2}\leqslant i\leqslant \frac{k-3}{2},\\ 0 & \text{if } i\geqslant \frac{k-1}{2}.\end{cases}$$

Assume that $\alpha=p^{\frac{k-1}{2}}$. Then $\varphi(d_\alpha)=p^{-1}d_\alpha$, $\varphi(d_\beta)\neq p^{-1}d_\beta$ and therefore $D_\alpha=\mathbf{D}_{\mathrm{cris}}(V_f)^{\varphi=p^{-1}}$ is a one-dimensional filtered submodule of $\mathbf{D}_{\mathrm{cris}}(V_f)$ of Hodge-Tate weight $-\frac{k+1}{2}$. The results of section 4.1.2 imply that V_f satisfies the conditions 1-4) of section 2.1.1 and therefore the $\mathscr{L}$-invariant $\mathscr{L}(V_f,D_\alpha)$ is defined. To simplify notation we set $\mathscr{L}_\alpha(f)=\mathscr{L}(V_f,D_\alpha)$. More precisely, the intersection $\mathbf{D}^\dagger_{\mathrm{rig}}(V_f)\cap\big(D_\alpha\otimes_L\mathscr{R}_L[1/t]\big)$ is isomorphic to $\mathscr{R}_L(\delta)$ with $\delta(x)=|x|x^{\frac{k+1}{2}}$ and we have an exact sequence of (φ,Γ)-modules

$$0\to\mathscr{R}_L(\delta)\to\mathbf{D}^\dagger_{\mathrm{rig}}(V_f)\to\mathscr{R}_L(\delta')\to 0$$

where $\delta'\,:\,\mathbb{Q}_p^*\to L^*$ is such that $\delta'(p)=\beta p^{1-k}$ and $\delta'(u)=u^{-\frac{k-3}{2}}$. Since $\dim_L H^1(\mathscr{R}_L(\delta))=2$ (see section 1.2.5) we obtain that $H^1(\mathbb{Q}_p,V_f)\simeq H^1(\mathscr{R}_L(\delta))$. In particular $H^1_{D_\alpha}(V_f)=H^1_{f,\{p\}}(\mathbb{Q}_p,V_f)$ and

$\mathscr{L}_\alpha(f)$ is the slope of the image of the localization map $H^1_{f,\{p\}}(\mathbb{Q}_p,V_f)\to H^1(\mathbb{Q}_p,V_f)$ under the canonical decomposition $H^1(\mathbb{Q}_p,V_f)\simeq H^1_f(\mathscr{R}_L(\delta))\times H^1_c(\mathscr{R}_L(\delta))$ (see (6)). Set $\alpha^*=\varepsilon^{-1}(p)\,\alpha$ and $\beta^*=\varepsilon^{-1}(p)\,\beta$. An easy computation shows that

$$\det(1-\varphi X\mid \mathbf{D}_{\mathrm{cris}}(W_{f^*}))=(1-\alpha^*X)(1-\beta^*X)$$

i.e. that α^* and β^* are the roots of $X^2-\bar a_pX+\varepsilon^{-1}(p)\,p^{k-1}$. From (13) we obtain that $\mathscr{L}_{\alpha^*}(f^*)=-\mathscr{L}_\alpha(f)$.

4.2. p-adic L-functions of modular forms.

4.2.1. Construction of p-adic L-functions (see [AV],[Mn], [Vi],[MTT]). We conserve notations and conventions of section 2.1. Let f be a normalized newform on $\Gamma_0(N)$ of weight k and character ε. Let $p>2$ be a prime number such that the Euler factor $E_p(f,X)$ is not equal to 1 and let $\alpha\in\overline{\mathbb{Q}}_p$ be a root of the polynomial $X^2-a_pX+\varepsilon(p)p^{k-1}$. Assume that α is not critical i.e. that $v_p(\alpha)<k-1$. Manin-Vishik [Mn], [Vi], and independently Amice-Velu [AV] proved that there exists a unique distribution $\mu_{f,\alpha}$ on $\hat{\mathbb{Z}}^{(p)}$ of order $v_p(\alpha)$ such that for any Dirichlet caracter η of conductor M prime to p and any Dirichlet character ξ of conductor p^m

$$\int_{\hat{\mathbb{Z}}^{(p)}}\eta(x)\xi(x)x^{j-1}\mu_{f,\alpha}(x)=\begin{cases}\left(1-\dfrac{\bar\eta(p)p^{j-1}}{\alpha}\right)\left(1-\dfrac{\beta\eta(p)}{p^j}\right)\widetilde{L}(f,\eta,j) & \text{if } 1\leqslant j\leqslant k-1 \text{ and } m=0,\\ \dfrac{p^{mj}\,\bar\eta(p^m)}{\alpha^m\tau(\bar\xi)}\,\widetilde{L}(f,\eta\xi^{-1},j) & \text{if } 1\leqslant j\leqslant k-1 \text{ and } m\geqslant 1\end{cases}$$

where $\tau(\bar\xi)=\sum_{a=1}^{p^m-1}\bar\xi(a)\zeta_{p^m}^a$ and $\widetilde{L}(f,\eta,j)$ is the algebraic part of $L(f,\eta,j)$ (see (1)). For us it will be more convenient to work with the distribution $\lambda_{f,\alpha}=x^{-1}\mu_{f,\alpha}$. The p-adic L-functions associated to $\eta\,:\,(\mathbb{Z}/M\mathbb{Z})^*\to\overline{\mathbb{Q}}_p^*$ are defined by[2]

$$L_{p,\alpha}(f,\eta\omega^m,s)=\int_{\hat{\mathbb{Z}}^{(p)}}\eta\omega^m(x)\langle x\rangle^s\lambda_{f,\alpha}(x)\qquad 0\leqslant m\leqslant p-2. \tag{21}$$

It is easy to see that $L_{p,\alpha}(f,\eta\omega^m,s)$ is a p-adic analytic function which satisfies the following interpolation property

$$L_{p,\alpha}(f,\eta\omega^m,j)=\mathcal{E}_\alpha(f,\eta\omega^m,j)\ \widetilde{L}(f,\eta\omega^{j-m},j),\qquad 1\leqslant j\leqslant k-1$$

where

$$\mathcal{E}_\alpha(f,\eta\omega^m,j)=\begin{cases}\left(1-\dfrac{\bar\eta(p)p^{j-1}}{\alpha}\right)\left(1-\dfrac{\eta(p)\varepsilon(p)p^{k-j-1}}{\alpha}\right) & \text{if } j\equiv m \pmod{p-1}\\ \dfrac{\bar\eta(p)p^j}{\alpha\,\tau(\omega^{j-m})} & \text{if } j\not\equiv m \pmod{p-1}.\end{cases} \tag{22}$$

4.2.2. Trivial zeros (see [MTT]). We say that $L_{p,\alpha}(f,\eta\omega^m,s)$ has a trivial zero at $s=j$ if

$$\mathcal{E}_\alpha(f,\eta\omega^m,j)=0.$$

[2]Our $L_{p,\alpha}(f,\psi\omega^m,s)$ coincides with $L_p(f,\alpha,\bar\eta\omega^m,s-1)$ of [MTT]

From (22) it is not difficult to deduce that this occurs in the following three cases (for proofs, see [MTT], §15).

• *The semistable case: $p \parallel N$, k is even and* $(p, \mathrm{cond}(\varepsilon)) = 1$. Thus $\varepsilon(p) = 0$, $E_p(f, X) = 1 - a_p X$ and $\alpha = a_p$ is the unique non-critical root of $X^2 - a_p X$. The restriction of the p-adic representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(W_f)$ on the decomposition group at p is semistable [Sa]. Let $\tilde{\varepsilon}$ be the primitive character associated to ε. Then $\tilde{\varepsilon}(p) \neq 0$ and $a_p^2 = \tilde{\varepsilon}(p) p^{k-2}$ (see [Li], theorem 3). Write $a_p = \xi p^{k/2-1}$ where ξ is a root of unity. Then $\mathcal{E}_\alpha(f, \eta\omega^m, j) = 0$ if and only if $j = k/2$, $m \equiv k/2 \mod (p-1)$ and $\bar{\eta}(p) = \xi$. Therefore the p-adic L-function $L_{p,\alpha}(f, \eta\omega^{k/2}, s)$ has a trivial zero at the central point $s = k/2$ if and only if $\bar{\eta}(p) = \xi$.

• *The crystalline case: $p \nmid N$.* The restriction of ρ_f on the decomposition group at p is crystalline [Fa2]. By Deligne [D2], $|\alpha| = p^{(k-1)/2}$ and we can write $\alpha = \xi p^{\frac{k-1}{2}}$ with $|\xi| = 1$. Thus $\mathcal{E}_\alpha(f, \eta\omega^m, j)$ vanishes if and only if $m \equiv j \pmod{p-1}$, k is odd, and either $j = \dfrac{k+1}{2}$ and $\bar{\eta}(p) = \xi$ or $j = \dfrac{k-1}{2}$ and $\eta(p)\varepsilon(p) = \xi$. The p-adic L-function $L_{p,\alpha}(f, \eta\omega^{\frac{k+1}{2}}, s)$ has a trivial zero at the near-central point $s = \frac{k+1}{2}$ if and only if $\alpha = \bar{\eta}(p)\, p^{\frac{k-1}{2}}$ and $L_{p,\alpha}(f, \eta\omega^{\frac{k-1}{2}}, s)$ has a trivial zero at $s = \frac{k-1}{2}$ if and only if $\alpha = \eta(p)\, \varepsilon(p)\, p^{\frac{k-1}{2}}$.

• *The potentially crystalline case: $p \mid N$ and* $\mathrm{ord}_p(N) = \mathrm{ord}_p(\mathrm{cond}(\varepsilon))$. Then the restriction of ρ_f on the decomposition group at p is potentially crystalline and $\mathbf{D}_{\mathrm{cris}}(W_f) = \mathbf{D}_{\mathrm{pcris}}(W_f)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is one-dimensional [Sa]. One has $E_p(f, X) = 1 - a_p X$ and $\alpha = a_p$ is the unique non-critical root of $X^2 - a_p X$. Moreover $\tilde{\varepsilon}(p) = 0$ and it can be shown that $|a_p| = p^{\frac{k-1}{2}}$ (see [O], [Li]). Thus $\mathcal{E}_\alpha(f, \eta\omega^m, j)$ vanishes if and only if k is odd, $j = m = \frac{k-1}{2}$ and $a_p = \bar{\eta}(p) p^{\frac{k-1}{2}}$. Therefore the p-adic L-function $L_{\alpha,p}(f, \eta\omega^{\frac{k-1}{2}}, s)$ has a trivial zero at the near-central point $s = \frac{k-1}{2}$ if and only if $a_p = \eta(p) p^{\frac{k-1}{2}}$.

If η is a Dirichlet character of conductor M, the twisted modular form $f_\eta = \sum\limits_{n=1}^{\infty} \eta(n)\, a_n q^n$ is not necessarily primitive, but there exists a unique normalized newform $f \otimes \eta$ such that

$$L(f, \eta, s) = L(f \otimes \eta, s) \prod_{l \mid M} E_l(f \otimes \eta, l^{-s})$$

(see [AL], Theorem 3.2 for more information and details). Write $L(f \otimes \eta, s) = \sum\limits_{n=1}^{\infty} \dfrac{a_{\eta,n}}{n^s}$. If $p \nmid M$, the Euler factors at p of $L_M(f \otimes \eta, s)$ and $L(f, \eta, s)$ coincide and $\alpha_\eta = \alpha\eta(p)$ is a root of $X^2 - a_{\eta,p} X + \varepsilon(p) p^{k-1}$. It is easy to see that $\mathcal{E}_{\alpha_\eta}(f_\eta, \omega^m, j) = \mathcal{E}_\alpha(f, \eta\omega^m, j)$ and from the interpolation formula (22) it follows immediately that

$$L_{p,\alpha}(f, \eta, s) = L_{p,\alpha_\eta}(f \otimes \eta, \omega^m, s) \prod_{l \mid M} \left(1 - a_{\eta,l}\omega^{-m}(l) \langle l \rangle^{-s} + \omega^{-2m}(l)\, \varepsilon(l)\, l^{k-1} \langle l \rangle^{-2s}\right).$$

Therefore the behavior of $L_{p,\alpha}(f, \eta\omega^m, s)$ and $L_{p,\alpha_\eta}(f \otimes \eta, \omega^m, s)$ is essentially the same and the general case reduces to the case of the trivial character η.

4.3. The main result.

4.3.1. Kato's Euler systems. Using the theory of modular units Kato [Ka2] constructed an element $\mathbf{z}_{\mathrm{Kato}} \in H^1_{\mathrm{Iw},S}(W_{f^*})$ which is closely related to the complex and the p-adic L-functions via the Bloch-Kato exponential map. (See also [Ru] for the CM-case.) Set

$$\mathbf{z}_{\mathrm{Kato}}(j) = \mathrm{Tw}_j^{\varepsilon}(\mathbf{z}_{\mathrm{Kato}}) \in H^1_{\mathrm{Iw},S}(W_{f^*}(j))$$

and denote by $\mathbf{z}_{\mathrm{Kato}}(j)_0 = \mathrm{pr}_0(\mathbf{z}_{\mathrm{Kato}}(j))$ the projection of $\mathbf{z}_{\mathrm{Kato}}(j)$ on $H^1_S(W_{f^*}(j))$. The following statements are direct analogues of properties 1-2) of cyclotomic units from section 3.3.1:

1) Relation to the complex L-function. One has

$$\exp^*_{W_{f^*}(j)}(\mathbf{z}_{\mathrm{Kato}}(j)_0) = \Gamma(k-j)^{-1} E_p\left(f, p^{k-j}\right) \widetilde{L}\,(f, k-j)\,\omega_j^*, \qquad 1 \leqslant j \leqslant k-1 \tag{23}$$

for some canonical basis ω_j^* of $\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(W_f^*(j))$ (see [Ka2], Theorem 12.5). Note that $\omega_{j+1}^* = \omega_j^* \otimes e_1$ where $e_1 = \varepsilon^{-1} \otimes t$.

2) Relation to the p-adic L-function. Let $\mathfrak{L}^{(\alpha),\varepsilon}_{W_{f^*}(k),1}$ denote the large logarithmic map $\mathfrak{L}^{(\varepsilon)}_{W_{f^*}(k),1,\eta}$ associated to $\eta = d_\alpha \otimes e_{\frac{k+1}{2}} \in \mathbf{D}_{\mathrm{cris}}(W_f)$. Then

$$\mathfrak{L}^{(\alpha),\varepsilon}_{W_{f^*}(k),1}(\mathbf{z}_{\mathrm{Kato}}(k)) = \mathbf{M}(\lambda_{f,\alpha}) \left[d_\alpha \otimes e_{\frac{k+1}{2}}\,,\, \omega_k^*\right]_{W_f} \tag{24}$$

([Ka2], Theorem 16.2).

We can now prove the main result of this paper.

Theorem 4.3.2. *Let f be a newform on $\Gamma_0(N)$ of character ε and odd weight k. Assume that p is an odd prime such that $p \nmid N$. Then p-adic L-function $L_{p,\alpha}\left(f, \omega^{\frac{k+1}{2}}, s\right)$ vanishes at $s = \frac{k+1}{2}$ if and only if $\alpha = p^{\frac{k-1}{2}}$. If φ acts semisimply on $\mathbf{D}_{\mathrm{cris}}(W_f)$ then*

$$L'_{p,\alpha}\left(f, \omega^{\frac{k+1}{2}}, \frac{k+1}{2}\right) = -\mathscr{L}_\alpha(f) \left(1 - \frac{\varepsilon(p)}{p}\right) \widetilde{L}\left(f, \frac{k+1}{2}\right).$$

Proof. To simplify notation set $k_0 = \frac{k+1}{2}$ and $\mathbf{z} = \mathbf{z}_{\mathrm{Kato}}(k_0 - 1)$. By Lemma 1.3.4 one has

$$\mathfrak{L}^{(\alpha),\varepsilon}_{V_f^*(1),1-k_0}(\mathbf{z}) = \mathrm{Tw}_{k_0}\left(\mathfrak{L}^{(\alpha),\varepsilon}_{W_{f^*(k)},1}(\mathbf{z}_{\mathrm{Kato}}(k))\right).$$

Let $\mu_{\mathbf{z}}$ be the distribution defined by $\mathbf{M}(\mu_{\mathbf{z}}) = \mathfrak{L}^{(\alpha),\varepsilon}_{V_f^*(1),1-k_0}(\mathbf{z})$. Then (23) gives

$$\mathbf{M}(\mu_{\mathbf{z}}) = \mathrm{Tw}_{k_0}\left(\mathbf{M}(\lambda_{f,\alpha})\right) [d_\alpha \otimes e_{k_0}, \omega_k^*]_{W_f} \;=\; \mathrm{Tw}_{k_0}\left(\mathbf{M}(\lambda_{f,\alpha})\right) [d_\alpha, \omega_{k_0-1}^*]_{V_f}$$

and from (8) and (21) it follows that

$$L_p(\mu_{\mathbf{z}}, s) = L_{p,\alpha}\left(f, \omega^{k_0}, s + k_0\right) \left[d_\alpha, \omega_{k_0-1}^*\right]_{V_f}.$$

Now, applying Proposition 2.2.4 we obtain

$$L'_{p,\alpha}\left(f, \omega^{k_0}, k_0\right) \left[d_\alpha, \omega_{k_0-1}^*\right]_{V_f} \;=\; -\mathscr{L}_\alpha(f)\,\Gamma\left(k_0\right) \left(1 - \frac{1}{p}\right)^{-1} \left[d_\alpha, \exp^*_{V_f^*(1)}(\mathbf{z}_0)\right]_{V_f}. \tag{25}$$

On the other hand, for $j = \frac{k-1}{2}$ the formula (23) gives

$$\exp^*_{V_f^*(1)}(\mathbf{z}_0) \;=\; \Gamma\left(k_0\right)^{-1} E_p\left(f, p^{k_0}\right) \widetilde{L}\left(f, k_0\right) \omega_{k_0-1}^*. \tag{26}$$

Since $E_p\left(f, p^{k_0}\right) \;=\; \left(1 - \dfrac{1}{p}\right)\left(1 - \dfrac{\varepsilon(p)}{p}\right)$ and $\left[d_\alpha, \omega_{k_0-1}^*\right]_{V_f} \neq 0$, from (25) and (26) we obtain that

$$L'_{p,\alpha}\left(f, \omega^{k_0}, k_0\right) = -\mathscr{L}_\alpha(f) \left(1 - \frac{\varepsilon(p)}{p}\right) \widetilde{L}\left(f, k_0\right)$$

and the theorem is proved.

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Mathématiques et Informatique, Université Bordeaux I, 351 cours de la Libération, 33405, Talence Cedex, France

E-mail address: `denis.benois@math.u-bordeaux1.fr`