SOME QUOTIENTS OF THE BOOLEAN LATTICE ARE SYMMETRIC CHAIN ORDERS

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ABSTRACT. R. Canfield has conjectured that for all subgroups G of the automorphism group of the Boolean lattice B_n (which can be regarded as the symmetric group S_n) the quotient order $B(n)/G$ is a symmetric chain order. We provide a straightforward proof of a generalization of a result of K. K. Jordan: namely, $B(n)/G$ is an SCO whenever G is generated by powers of disjoint cycles. The symmetric chain decompositions of Greene and Kleitman provide the basis for partitions of these quotients.

1. INTRODUCTION

There are several familiar notions of symmetry in finite partially ordered sets, usually in the family of ranked orders. A finite ordered set P in which all maximal chains have the same length admits a rank function $r = r_P$, namely, for all $x \in P$, $r(x)$ is the maximum of all lengths $l(C) = |C| - 1$ over all chains $C \subseteq P$ with maximum element x. If P has a minimum element, denoted by 0_P, then $r(x)$ is the maximum length of chains in the interval $[0_P, x]$. We let the *rank* $r(P)$ of P be maximum $r(x)$ over all $x \in P$. Note that a ranked ordered set satisfies the Jordan-Dedekind chain condition: for all $x < y$ in P, all saturated, or covering, chains in the interval $[x, y]$ have the same length.

In a ranked order P the chain $x_1 < x_2 < \cdots < x_k$ is a *symmetric chain* if it is saturated and if $r(x_1) + r(x_k) = r(P)$. A *symmetric chain decomposition* or *SCD* of P is a partition of P into symmetric chains $C_1, C_2, \ldots, C_k$. If P has an SCD, call P a *symmetric chain order*, or *SCO*. Here, we are concerned with ordered sets based on the *Boolean lattice*, denoted B_n, which is the power set of $[n] = \{1, 2, \ldots, n\}$ ordered by containment. Clearly B_n is a ranked poset, with $\emptyset$ being the minimum element, and $r(A) = |A|$ for all $A \subseteq [n]$. In fact, it is an SCO [2].

We are interested in ordered sets induced by actions of the automorphism group of B_n. It is well-known that this group is faithfully induced by the symmetric group S_n of all permutations on the underlying set $[n]$, so we will refer to S_n as the automorphism group of B_n. Given any subgroup G of S_n, the *quotient* B_n/G has

as its elements the equivalence classes

$$[A] = \{B \mid B = \sigma(A), \text{for some } \sigma \in G\},$$

$A \in B_n$, ordered by

$$[A] \le [B] \iff X \subseteq Y \text{ for some } X \in [A] \text{ and } Y \in [B].$$

In studying Venn diagrams, Griggs, Killian and Savage [8] explicitly constructed an SCD of the quotient B_n/G for n prime and given that G is generated by a single n-cycle. They asked if this *necklace* poset is an SCO for arbitrary n. Canfield and Mason [3] made a much more general conjecture: for all subgroups G of S_n, B_n/G is a symmetric chain order.

Jordan [9] gave a positive answer to the question of Griggs, Killian and Savage, basing the SCD of the quotient on the explicit construction of an SCD in B_n by Greene and Kleitman [6]. The construction in [9] requires an intermediate equivalence relation and some careful analysis. Here we provide a more direct proof of a generalization of Jordan's theorem by "pruning" the Greene-Kleitman SCD. More generally, we show that B_n/G is an SCO provided that G is generated by powers of disjoint cycles (see Theorem 1). We also provide a different proof that B_n/G is an SCO when G is a 2-element subgroup generated by a reflection, based on an SCD of $B_{\lfloor n/2 \rfloor}$.

The ordered sets B_n/G do share several forms of symmetry or regularity with the Boolean lattice. A SCO P is necessarily rank-symmetric, rank-unimodal, and strongly Sperner (see, for instance, [9] for definitions). A result of Stanley [11] shows that B_n/G has these three properties for all subgroups G of S_n. However, these three conditions are not sufficient to yield symmetric chain decompositions.

On the other hand, Griggs [7] showed that a ranked ordered set with the LYM property, rank-symmetry and rank-unimodality is an SCO. It is not known whether the quotients B_n/G have the LYM property in general, though they do if n is prime and G is generated by an n-cycle, which gives the Griggs, Killian and Savage result. (A SCO need not satisfy the LYM property – see [7] for examples.) Pouzet and Rosenberg [10] obtain Stanley's results and "local" families of symmetric chains for more general structures than the quotients B_n/G, but their results do not show that B_n/G is an SCO.

2. The Main Results

There are two results - for groups generated by powers of disjoint cycles and for a particular 2-element group.

Theorem 1. *Let G be a subgroup of S_n generated by powers of disjoint cycles. Then the partially ordered set B_n/G is a symmetric chain order.*

The proof of Theorem 1 follows from this sequence of results. The new proof of Lemma 1, which is a modest generalization of Jordan's result, is given in Section 3.

Lemma 1. *Let σ be an n-cycle in S_n and let H be a subgroup of the cyclic group generated by σ. Then B_n/H is a symmetric chain order.*

The following fact is well-known and can be proved by an argument much like the original proof in [2] that the divisor lattice of an integer is an SCO. (In [5], this credited to Alekseev [1].)

Lemma 2. *Let P and Q be partially ordered sets. If P and Q are symmetric chain orders then so is $P \times Q$.*

In the following lemma, use this notation. Suppose that σ_j $(j = 1, 2, \ldots, t)$ are disjoint cycles in S_n and that $\rho_j = \sigma_j^{r_j}$, for integers $r_1, r_2, \ldots, r_t$. Let X_j be the subset of $[n]$ of elements moved by ρ_j $(j = 1, 2, \ldots, t)$, and let X_0 be all elements of $[n]$ fixed by all the ρ_j's. Let $B(X)$ denote the Boolean lattice of all subsets of a set X.

Lemma 3. *Let H_j be the subgroup of S_n generated by ρ_j $(j = 1, 2, \ldots, t)$ and let G be the subgroup generated by $\{\rho_1, \rho_2, \ldots, \rho_t\}$. Then*

$$B(n)/G \;\cong\; B(X_0) \times B(X_1)/H_1 \times \cdots \times B(X_t)/H_t.$$

Proof. For any $A \subseteq [n]$, let $[A]$ denote its equivalence class in $B(n)/G$, and let $A_j = A \cap X_j$, $j = 0, 1, \ldots, t$. Define a map Φ on $B(n)/G$ by $\Phi([A]) = (A_0, [A_1], \ldots [A_t])$. From the definition of the ordering of the quotient, $[A] \leq [B]$ in $B(n)/G$ if and only if there is some $\tau \in G$ such that $A \subseteq \tau(B)$. Then $\tau = \rho_1^{i_1} \rho_2^{i_2} \cdots \rho_t^{i_t}$ for nonnegative integers $i_1, i_2, \ldots, i_t$. The claimed isomorphism follows from this fact:

$$A \subseteq \tau(B) \text{ if and only if } A_0 \subseteq B_0 \text{ and } A_j \subseteq \sigma^{i_j}(B_j) \text{ for } j = 1, 2, \ldots, t \ .$$

$\square$

The following is a corollary of Theorem 1. Indeed, a proof based on an approach like that used in the proof of Theorem 1 – a greedy pruning of a Greene-Kleitman SCD – can be shown to provide a basis for the proof offered in Section 4. However, the proof in Section 4 provides some insight into the Greene-Kleitman SCD and may be of use for other choices for the group of permutations, such as the dihedral group.

Corollary 1. *Let G be a 2-element subgroup with non-unit element a product of disjoint transpositions. Then the partially ordered set B_n/G is a symmetric chain order.*

3. PROOF OF LEMMA 1

We fix the natural order on $[n]$ and may assume that the n-cycle σ is $(1 \ 2 \ \cdots \ n)$. This is valid because any n-cycle ρ is a conjugate of $(1 \ 2 \ \cdots \ n)$ and for any subgroup

K of S_n and any $\pi \in S_n$, $B_n/K \cong B_n/\pi^{-1}K\pi$ via $[A] \mapsto [\pi(A)]$. Let H denote the subgroup of S_n generated by σ^s.

We first describe the procedure for obaining an SCD of B_n/H based on the Greene-Kleitman SCD of B_n then verify that the procedure yields the claimed SCD.

Let $C_1, C_2, \ldots, C_t$, where $t = \binom{n}{\lfloor n/2 \rfloor}$, be the symmetric chains in the Greene-Kleitman decomposition, ordered by decreasing length. For all $A \in B_n$, $[A]$ is the equivalence class containing A in B_n/H where H is the subgroup of S_n generated by $\rho = \sigma^m$.

Claim: There is a family $\mathcal{C} = \{C'_{i_1}, C'_{i_2}, \ldots C'_{i_m}\}$, with $(i_1, i_2, \ldots, i_m)$ a subsequence of $(1, 2, \ldots, t)$, that satisfies these conditions:

(3.1): for all $1 \le j \le m$, $C'_{i_j} \subseteq C_{i_j}$ and is a symmetric chain in B_n;

(3.2): for all $1 \le r < s \le m$ and for all $A \in C'_{i_r}, B \in C'_{i_s}$, $A \notin [B]$; and,

(3.3): for all $[X]$ there is some $Y \in [X]$ such that $Y \in C'_{i_j}$ for some j.

For $j = 1, 2, \ldots, m$, let $\widehat{C}_j = \{[A] \mid A \in C'_{i_j}\}$. Then the chains $\widehat{C}_1, \widehat{C}_2, \ldots, \widehat{C}_m$ cover B_n/H (by **(3.3)**), the sets are disjoint (by **(3.2)**), and form symmetric chains (by **(3.1)**). Thus, it is enough to verify the **Claim** in order to prove Lemma 1.

Several properties of the Greene-Kleitman SCD of B_n are needed. For the most part, these are well-known – see, for instance, the descriptions in [5] and [9]. It is useful to regard members of B_n both as subsets of $[n]$ and as binary sequences of length n, defined with respect to the natural order. (Indeed, one needs to fix an order to speak of *the* Greene-Kleitman SCD.) The SCD is obtained by a "bracketing" or pairing procedure that has several equivalent descriptions. Here are two that are useful to us. Let $A \subseteq [n]$.

(3.4): If $1 \notin A$ and $2 \in A$, pair 1 and 2; define $p_A(2) = 1$. Suppose that we have considered $1, 2, \ldots k - 1$. If $k \in A$ and there is some $j < k$, $j \notin A$ such that j is unpaired, then let $p_A(k)$ be the maximum such j and say $p_A(k)$ and k are *paired*. Continue for all k in $[n]$.

(3.5): For all $x \in A$ such that precisely half of the elements of the interval $[y, x]$ are members of A, for some $1 \le y < x$, let $p_A(x)$ be the maximum such y.

Let $R(A)$ be the set of all x for which $p_A(x)$ is defined, let $L(A) = \{p_A(x) \mid x \in R(A)\}$, and let $P(A) = L(A) \cup R(A)$. Now set

$$f(A) = A \cup \{z\}, \; z = \min([n] - (A \cup L(A))),$$

if $[n] - (A \cup P(A)) \neq \emptyset$; otherwise $f(A)$ is undefined. Then this rule inverts f:

$$f^{-1}(B) = B - \{z\}, \quad z = \max(B - R(B)).$$

Let $\mathcal{C}(A) = \{f^k(A) \mid k \in \mathbb{Z}\}$. As A runs over all of B_n, the distinct $\mathcal{C}(A)$'s provide the Greene-Kleitman SCD of B_n.

Then the following hold for all $A \in B_n$.

(3.6): For all $x \in R(A)$, $[p_A(x), x] \subseteq P(A)$.

(3.7): $\mathcal{C}(A) = \{X \in B_n \mid R(X) = R(A)\}$ and $p_X(a) = p_A(a)$ for all $X \in \mathcal{C}(A)$ and for all $a \in R(A)$.

(3.8): $\min(\mathcal{C}(A)) = R(A)$, $\max(\mathcal{C}(A)) = [n] - L(A)$; in fact, $\mathcal{C}(A)$ is the chain
$$R(A) \subset R(A) \cup \{a_1\} \subset R(A) \cup \{a_1, a_2\} \subset \ldots \subset R(A) \cup \{a_1, a_2, \ldots a_t\} = [n] - L(A),$$
where $[n] - (R(A) \cup L(A)) = \{a_1 < a_2 < \ldots < a_t\}$.

Here are the properties of this SCD that allow proof of the **Claim**. Given a symmetric chain C in B_n and $X \in C$ with $|X| \leq \lfloor n/2 \rfloor$, let X^* to be the member of C with $|X^*| = n - |X|$.

Lemma 4. *For $i = 1, 2, \ldots, t$ and for all $X \in C_i$ with $|X| \leq \lfloor n/2 \rfloor$, $(\sigma(X))^* = \sigma(X^*)$. Thus $(\sigma^j(X))^* = \sigma^j(X^*)$ for all integers j, so $(\rho(X))^* = \rho(X^*)$ for all $\rho \in H$.*

A special case of the preceding lemma is in [4]. Since this reference is a technical report and the result does not appear to be available in the literature, we prove this below. The following is, to our knowledge, new and a proof is provided as well.

Lemma 5. *Let $w \in \{1, 2, \ldots, t\}$, and let $A \in C_w$ with $|A| \leq \lceil n/2 \rceil$. Suppose that there is some $B \in [A]$ such that $B \in C_j$ for some $j < w$. Then there is some $k < w$ and $D \in C_k$ such that $D \in [f^{-1}(A)]$, provided that $f^{-1}(A)$ is defined.*

To prove the **Claim** from these facts, define $\mathcal{C}$ inductively.

First, let $i_1 = 1$ and $C'_{i_1} = C_1$. Suppose that $C'_{i_1}, C'_{i_2}, \ldots, C'_{i_k}$ are defined. If there exists $i \in \{i_k + 1, \ldots, t\}$ such that for some $X \in C_i$,

$$(1) \qquad\qquad [X] \cap \left(\bigcup_{j=1}^{k} C_{i_j} \right) = \emptyset$$

let i_{k+1} be the least such i and let

$$(2) \qquad\qquad C'_{i_{k+1}} = \{Y \in C_{i_{k+1}} \mid [Y] \cap \left(\bigcup_{j=1}^{k} C'_{i_j} \right) = \emptyset\}.$$

If there is no such i then $m = k$ and the procedure is complete.

If $Y \in C'_{i_{k+1}}$, with $|Y| \leq \lfloor n/2 \rfloor$ then $Y^* \in C'_{i_{k+1}}$, by Lemma 4. Also, if $Z \in C_{i_{k+1}}$ and $Y \subseteq Z \subseteq Y^*$ where $Y \in C'_{i_{k+1}}$ then $Z \in C'_{i_{k+1}}$ by Lemma 5 and Lemma 4. Thus, $C'_{i_{k+1}}$ is symmetric in B_n and **(3.1)** holds. Equation (2) verifies **(3.2)**; **(3.3)** follows from (1) and (2).

Proof of Lemma 4. The proof is divided into cases depending upon which of $R(X) \subseteq X \subseteq X^*$ contain n. It is not possible that $n \in X - R(X)$, because $|X| \leq \lfloor n/2 \rfloor$ means that for some $y < n$ precisely half the elements of $[y, n]$ are in X, and, hence, $n \in R(X)$ by **(3.5)**. Consequently, there are three cases. In each case, we show that

$$R(\sigma(X^*)) = R(\sigma(X)),$$

apply **(3.7)** to see that $\sigma(X^*)$ and $(\sigma(X))^*$ are both members of $\mathcal{C}(\sigma(X))$, and conclude that $\sigma(X^*) = (\sigma(X))^*$ since these sets both have cardinality $n - |\sigma(X)|$.

Case 1: $n \notin X^*$

Since $n \in [n] - L(X) = \max(\mathcal{C}(X))$ and $n \notin X^*$, $X \neq \min(\mathcal{C}(X)) = R(X)$. Thus, there exists $y = \min(X - R(X))$. If $y = 1$ then $p_{\sigma(X)}(2) = 1 = p_{\sigma(X^*)}(2)$. For each $z \in R(X)$, $\sigma(z) = z + 1 \in R(\sigma(X))$ and each $z + 1 \in R(\sigma(X))$ has $z \in R(X)$ apart from $z + 1 = 2$. Thus,

$$\begin{aligned} R(\sigma(X)) &= \sigma(R(X) \cup \{2\} && \text{since } p_{\sigma(X)}(2) = 1, \\ &= \sigma(R(X^*)) \cup \{2\} && \text{by } \mathbf{(3.7)}, \\ &= R(\sigma(X^*)) && \text{since } p_{\sigma(X^*)}(2) = 1. \end{aligned}$$

If $y > 1$ we claim that $[1, y - 1] \subseteq P(X)$. Note that $y - 1 \in X$ as otherwise $y \in R(X)$ with $p_X(y) = y - 1$, contradicting the choice of y. By the minimality of y, $y - 1 \in R(X)$ and, by **(3.7)**, $[p_X(y - 1), y - 1] \subseteq P(X)$. Continue in the same manner, with $p_X(y - 1) - 1$ in place of y, and thereby verify the claim that $[1, y - 1] \subseteq P(X)$. The argument is just about the same as when $y = 1$ except we use the fact that $[1, y - 1] \subseteq P(X)$ and $1 \notin X$, so, $p_{\sigma(X)}(y + 1) = 1$:

$$R(\sigma(X^*)) = \sigma(R(X^*)) \cup \{y + 1\} = \sigma(R(X)) \cup \{y + 1\} = R(\sigma(X)).$$

Case 2: $n \in R(X)$

Every element of $R(\sigma(X))$ is in $\sigma(R(X))$ and every element of $\sigma(R(X))$, except for 1, is in $R(\sigma(X))$. Thus,

$$R(\sigma(X)) = \sigma(R(X)) - \{1\} = \sigma(R(X^*)) - \{1\} = R(\sigma(X^*)).$$

Case 3: $n \in X^* - X$

Since $n \in X^* - X$, **(3.8)** shows that $X^* = \max(\mathcal{C}(X)) = [n] - L(X)$ and, thus, $X = \min(\mathcal{C}(X)) = R(X)$.

If $z + 1 \in R(\sigma(X))$ then $z \in X = R(X)$, so $R(\sigma(X)) \subseteq \sigma(R(X))$. Conversely, $1 \notin \sigma(R(X))$, and any $z + 1 \in \sigma(R(X))$ is obviously a member of $R(\sigma(X))$. Thus,

$R(\sigma(X)) = \sigma(R(X))$. Similarly, since $n \notin R(X)$, it follows that $R(\sigma(X^*)) = \sigma(R(X^*))$. Hence, $R(\sigma(X^*)) = R(\sigma(X))$.

Since $(\sigma(X))^* = \sigma(X^*)$ for all X with $|X| \leq \lfloor n/2 \rfloor$, we can apply induction on j to conclude that $(\sigma^j(X))^* = \sigma^j(X^*)$:

$$(\sigma^j(X))^* = (\sigma(\sigma^{j-1}(X)))^* = \sigma((\sigma^{j-1}(X))^*) = \sigma(\sigma^{j-1}(X^*)) = \sigma^j(X^*). \quad \square$$

Proof of Lemma 5. As before, let $C_1, C_2, \ldots, C_t$, where $t = \binom{n}{\lfloor n/2 \rfloor}$, be the symmetric chains in the Greene-Kleitman decomposition, ordered by decreasing length, and let $\sigma = (1 \ 2 \ \cdots \ n)$. Let $A \in C_w$ with $|A| \leq \lceil \frac{n}{2} \rceil$. Suppose that there exists a $j < w$ such that $B \in C_j$ and $B \in [A]$. Hence there is an integer m such that $B = \sigma^m(A)$.

Assume that $f^{-1}(A)$ is defined. We show that there is a $k < w$ such that $D \in C_k$ and $D \in [f^{-1}(A)]$. Since $f^{-1}(A)$ is defined, $A - R(A) \neq \emptyset$. Let $y = \max(A - R(A))$. We may assume that $-(y - 1) \leq m \leq n - y$, $m \neq 0$. We consider two cases:

Case 1: $y + m \in R(B)$

Then $m > 0$ since otherwise y would also be paired in A, contrary to its choice. Each $z \in B$ with $y + m < z$ must be in $R(B)$ since $y < z - m$ so, by the choice of y, $z - m \in R(A)$. Now consider the binary sequence $\sigma^m(f^{-1}(A))$ contained in some chain C_k. Recall that $f^{-1}(A) = A - \{y\}$ and note that $\sigma^m(f^{-1}(A)) = B - \{y + m\}$. It follows from this that $\sigma^m(f^{-1}(A))$ must have one fewer pairs than B, since $y + m$ will be unpaired in $\sigma^m(f^{-1}(A))$ while $y + m$ is paired in B, and there are no other differences in the pairings. By **(3.8)**, $|C_k| > |C_j|$, so $k < j < w$, as desired.

Case 2: $y + m \in B - R(B)$

If $y + m = \max(B - R(B))$ then we are done, since then we have $f^{-1}(B) = \sigma^m(f^{-1}(A))$. So suppose instead $z = \max(B - R(B))$ where $y + m < z$. If $m > 0$ then $z - m \in (A - R(A))$, contrary to the choice of y. Thus $m < 0$. If $z - m \leq n$ then $z - m$ would be an unpaired element of A, since $y < z - m$ remains unpaired in A. This would contradict the choice of y. Thus $n < z - m$.

We now prove that for some p, $\sigma^p(B)$ is the maximum element of its chain, and its chain is not a singleton. This will contradict the fact that $|A| \leq \lceil \frac{n}{2} \rceil$, since $|\sigma^p(B)| = |A|$.

Since $\sigma^{-m}(B) = A$, $-m > 0$, we obtain A from B by applying σ $-m$ times. Since $n < z - m$ there is some p such that $\sigma^p(z) = n$. Let $X = \sigma^p(B)$. Then, $\sigma^p(z) \in X - R(X)$. Because $n \in X - R(X)$, $X = [n] - L(X)$. By **(3.8)**, X is the maximum element of its chain. The chain containing X is not a singleton, since $X - R(X) \neq \emptyset$. $\square$

4. The proof of Corollary 1

Let $\rho = (i_1 j_i)(i_2 j_2) \cdots (i_k j_k)$, where the transpositions are pairwise disjoint, let $X = \bigcup_{r=1}^{k} \{i_r, j_r\}$, and let $G = \{1, \rho\}$. Then

$$B_n/G \cong B(X)/G \times B([n] - X)$$

via the mapping $[A] \mapsto ([A \cap X], A - X)$ for all $A \subseteq [n]$. By Lemma 2, we may assume that n is even and that $n = 2k$. Using the remark about conjugation at the beginning of Section 3, we may assume that $\rho = (1 \ 2k)(2 \ 2k-1) \cdots (k \ k+1)$. As noted in the introduction, G can be generated by a power of a $2k$-cycle, so Theorem 1 applies. (In fact, $\rho = \tau^{-1} \sigma \tau$ where $\sigma = (1 \ 2 \cdots 2k)$ and $\tau = (k+1 \ 2k)(k+2 \ 2k-1) \cdots$.) And, as we shall see, its proof method can be adapted to give the proof we offer here. However, the argument below might help with the most interesting open case, namely, showing that B_n/D_{2n} is an SCO for the dihedral group D_{2n}.

Regard each $A \in B_{2k}$ as a concatenated pair of binary strings of length k. That is, $A = \mathbf{b}_1 \mathbf{b}_2^r$, where $\mathbf{b}_1, \mathbf{b}_2 \in \{0, 1\}^k$ and $\mathbf{b}^r$ is the reverse of the binary k-sequence $\mathbf{b}$. Then the equivalence classes in B_n/G are the sets $\{\mathbf{b}_1 \mathbf{b}_2^r, \mathbf{b}_2 \mathbf{b}_1^r\}$; these sets have 2 elements except in the case that $\mathbf{b}_1 = \mathbf{b}_2$.

Let $C_1, C_2, \ldots, C_t$, where $t = \binom{k}{\lfloor k/2 \rfloor}$, be any symmetric chain decomposition of B_k, ordered by decreasing length. We define a total ordering $\preccurlyeq$ on $B_k = \{0, 1\}^k$ as follows:

(3) $\mathbf{b}_r \preccurlyeq \mathbf{b}_s$ if $\mathbf{b}_r \in C_i, \mathbf{b}_s \in C_j, \ i < j,$ or if $\mathbf{b}_r \subseteq \mathbf{b}_s$ in C_i for some i.

For $1 \le i < j \le t$, let $P_{ij} = C_i \times C_j$, with the coordinate-wise ordering induced by the containment order on B_k, for each $i = 1, 2, \ldots, t$, let

$$P_{ii} = \{(\mathbf{b}_r, \mathbf{b}_s) \in C_i \times C_i \mid \mathbf{b}_r \subseteq \mathbf{b}_s\},$$

ordered coordinate-wise, and let

$$P = \bigcup_{1 \le i \le j \le t} P_{ij},$$

again, ordered coordinate-wise. Thus, P is a subset of B_{2k} with the exactly the ordering inherited from the Boolean lattice.

In fact, with r_P and r_{B_k} as the rank functions in P and B_k, respectively, then for $1 \le i \le j \le t$ and with $r_{B_k}(\min C_i) = r_i$, and $r_{B_k}(\max C_i) = k - r_i$,

$$r_P(\min P_{ij}) = r_i + r_j, \ r_P(\max P_{ij}) = 2k - (r_i + r_j), \text{ and } l(P_{ij}) = 2k - 2(r_i + r_j).$$

We see that each P_{ij} is a symmetric subset of B_{2k} in which the covering relation is preserved, that is, $(\mathbf{b}_p, \mathbf{b}_q)$ is covered by $(\mathbf{b}_u, \mathbf{b}_v)$ in some P_{ij} if and only if $\mathbf{b}_p \mathbf{b}_q^r$ is covered by $\mathbf{b}_u \mathbf{b}_v^r$ in B_{2k}.

Consider the map ϕ of P to B_{2k}/G defined by $\phi((\mathbf{b}_1, \mathbf{b}_2)) = \{\mathbf{b}_1\mathbf{b}_2^r, \mathbf{b}_2\mathbf{b}_1^r\}$. Since $\mathbf{b}_1 \preccurlyeq \mathbf{b}_2$ for all $(\mathbf{b}_1, \mathbf{b}_2) \in P$, ϕ is injective. It is obviously a surjection. It is also order-preserving: if $(\mathbf{b}_p, \mathbf{b}_q) \le (\mathbf{b}_u, \mathbf{b}_v)$ in P then $\mathbf{b}_p\mathbf{b}_q^r \le \mathbf{b}_u\mathbf{b}_v^r$ in B_{2k}.

Since the rank of an equivalence class in B_{2k}/G is the rank of its members in B_{2k}, it follows that a symmetric chain in P is a symmetric chain in B_{2k}/G. Thus, it is enough to proof the following.

Claim: P has a symmetric chain decomposition.

Since P is partitioned by P_{ij}, $1 \le i \le j \le t$, each of which preserve the covering relation in P, it is enough to prove that each P_{ij} has a partition into chains, each of which is symmetric in P.

For $1 \le i < j \le t$, $P_{ij} = C_i \times C_j$ is a cover-preserving subset of P, with minimum element at level $r_i + r_j$ and maximum element at level $2k - (r_i + r_l)$ in P, a partially ordered set of length $2k$. Then the "standard" symmetric partition of a product of two chains (the original partition in [2]) provides symmetric chains in P. (See Figure **??**.)

For $i = 1, 2, \ldots, t$, $P_{ii} = \{(\mathbf{b}_r, \mathbf{b}_s) \in C_i \times C_i \mid \mathbf{b}_r \subseteq \mathbf{b}_s\}$, where C_i is the chain of binary strings $\mathbf{b}_{r_i} \subset \mathbf{b}_{r_i+1} \subset \cdots \subset \mathbf{b}_{k-r_i}$, where $r_{B_k}\mathbf{b}_s = s$ in B_k, that is, is an s-element set, for $s = r_i, r_I + 1, \ldots, k - r_i$. Then P_{ii} is an interval in P with minimum element at level $2r_i$ and maximum element at level $2k - 2r_i$ in P. Also,

$$(\mathbf{b}_{r_i}, \mathbf{b}_{r_i}) < (\mathbf{b}_{r_i}, \mathbf{b}_{r_i+1}) < \cdots < (\mathbf{b}_{r_i}, \mathbf{b}_{k-r_i}) < (\mathbf{b}_{r_i+1}, \mathbf{b}_{k-r_i}) < \cdots < (\mathbf{b}_{k-r_i}, \mathbf{b}_{k-r_i})$$

is a symmetric chain in P and $P_{ii} - C$ is a cover-preserving subset of P, isomorphic to the product of two chains, with minimum element $(\mathbf{b}_{r_i+1}, \mathbf{b}_{r_i+1})$ at level $2r_i + 2$ and maximum element $(\mathbf{b}_{k-r_i-1}, \mathbf{b}_{k-r_i-1})$ at level $2k - 2r_i - 2$. By induction, we have a decomposition of P_{ii} by chains symmetric in P. This verifies the **Claim** and completes the proof of Corollary 1.

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