Large Eddy Simulation for Turbulent Flows with Critical Regularization

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Abstract

In this paper, we establish the existence of a unique “regular” weak solution to the Large Eddy Simulation (LES) models of turbulence with critical regularization. We first consider the critical LES for the Navier-Stokes equations and we show that its solution converges to a solution of the Navier-Stokes equations as the averaging radii converge to zero. Then we extend the study to the critical LES for Magnetohydrodynamics equations.

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1 Introduction

Let us consider the Navier-Stokes equations in a three dimensional torus $\mathbb{T}_3$,

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.1}$$

$$\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \nu \Delta \boldsymbol{v} + \nabla p = \boldsymbol{f}, \tag{1.2}$$

subject to $\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x})$ and periodic boundary conditions. Here, $\boldsymbol{v}$ is the fluid velocity field, p is the pressure, $\boldsymbol{f}$ is the external body forces, ν stands for the viscosity .

Equations (1.1)-(1.2) are known to be the idealized physical model to compute Newtonian fluid flows. They are also known to be unstable in numerical simulations when the Reynolds number is high, thus when the flow is turbulent. Therefore, numerical turbulent models are needed for real simulations of turbulent flows. In many practical applications, knowing the mean characteristics of the flow by averaging techniques is sufficient. However, averaging the nonlinear term in NSE leads to the well-known closure problem. To be more precise, if denotes the filtered/averaged velocity field then the Reynolds averaged NSE (RANS)

$$\overline{\boldsymbol{v}}_{,t} + \operatorname{div}(\overline{\boldsymbol{v}} \otimes \overline{\boldsymbol{v}}) - \nu \Delta \overline{\boldsymbol{v}} + \nabla \overline{p} + \operatorname{div} \mathcal{R}(\boldsymbol{v}, \boldsymbol{v}) = \overline{\boldsymbol{f}}, \tag{1.3}$$

where $\mathcal{R}(\boldsymbol{v}, \boldsymbol{v}) = \overline{\boldsymbol{v} \otimes \boldsymbol{v}} - \overline{\boldsymbol{v}} \otimes \overline{\boldsymbol{v}}$ is the Reynolds stress tensor, is not closed because we cannot write it in terms of $\overline{\boldsymbol{v}}$ alone. The main essence of turbulence

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modeling is to derive simplified, reliable and computationally realizable closure models. In [17] and [18] Layton and Lewandowski suggested an approximation of the Reynolds stress tensor, given by

$$\mathcal{R}(\boldsymbol{v},\boldsymbol{v}) = \overline{\overline{\boldsymbol{v}} \otimes \overline{\boldsymbol{v}}} - \overline{\boldsymbol{v}} \otimes \overline{\boldsymbol{v}}. \tag{1.4}$$

This is equivalent form to the approximation

$$\operatorname{div}(\overline{\boldsymbol{v} \otimes \boldsymbol{v}}) \approx \operatorname{div}(\overline{\overline{\boldsymbol{v}} \otimes \overline{\boldsymbol{v}}}). \tag{1.5}$$

Hence, Layton and Lewandowski studied the following Large Scale Model considered as a Large Eddy Simulation (LES) model:

$$\operatorname{div} \boldsymbol{w} = 0, \tag{1.6}$$

$$\boldsymbol{w}_{,t} + \operatorname{div}(\overline{\boldsymbol{w} \otimes \boldsymbol{w}}) - \nu \Delta \boldsymbol{w} + \nabla q = \overline{\boldsymbol{f}}, \tag{1.7}$$

considered in $(0,T) \times \mathbb{T}_3$ and subject to $\boldsymbol{w}(\boldsymbol{x},0) = \boldsymbol{w}_0(\boldsymbol{x}) = \overline{\boldsymbol{v}_0}$ and periodic boundary conditions with mean value equal to zero. Where they denoted $(\boldsymbol{w}, q)$ the approximation of $(\overline{\boldsymbol{v}}, \overline{p})$.
The averaging operator chosen in (1.7) is a differential filter, [11], [12], [8], [17], [9], [7], that commutes with differentiation under periodic boundary conditions and is defined as follows. Let $\alpha > 0$, given a periodic function $\varphi \in L^2(\mathbb{T}_3)$, define its average $\overline{\varphi}$ to be the unique solution of

$$-\alpha^2 \Delta \overline{\varphi} + \overline{\varphi} = \varphi, \tag{1.8}$$

with periodic conditions, and fields with mean value equal to zero.

The main goal in using such a model is to filter eddies of scale less than the numerical grid size α in numerical simulations. For a general overview of LES models, the readers are refered to Berselli et al. [5] and references cited therein. Notice that the Layton-Lewandowski model (1.6)-(1.7) differs from the one introduced by Bardina et al. [4] where the following approximation of the Reynolds stress tensor is used:

$$\mathcal{R}(\boldsymbol{v},\boldsymbol{v}) = \overline{\overline{\boldsymbol{v}} \otimes \overline{\boldsymbol{v}}} - \overline{\overline{\boldsymbol{v}}} \otimes \overline{\overline{\boldsymbol{v}}}. \tag{1.9}$$

In [17] and [18] Layton and Lewandowski have proved that (1.6)-(1.7) have a unique regular solution . They have aslo shown that there exists a sequence α_j which converges to zero and such that the sequence $(\boldsymbol{w}_{\alpha_j}, q_{\alpha_j})$ converges to a distributional solution $(\boldsymbol{v}, p)$ of the Navier-Stokes equations .
We remark that many of these results established in the above cited papers have been extended to the following three dimensional magnetohydrodynamic equations (MHD):

$$\partial_t \boldsymbol{v} - \nu_1 \Delta \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div}(\mathcal{B} \otimes \mathcal{B}) + \nabla p = 0, \tag{1.10}$$

$$\partial_t \mathcal{B} - \nu_2 \Delta \mathcal{B} + \operatorname{div}(\boldsymbol{v} \otimes \mathcal{B}) - \operatorname{div}(\mathcal{B} \otimes \boldsymbol{v}) = 0, \tag{1.11}$$

$$\int_{\mathbb{T}_3} \mathcal{B}\, d\boldsymbol{x} = \int_{\mathbb{T}_3} \boldsymbol{v}\, d\boldsymbol{x} = 0, \quad \operatorname{div} \mathcal{B} = \operatorname{div} \mathcal{B} = 0, \tag{1.12}$$

$$\mathcal{B}(0) = \mathcal{B}_0,\ \boldsymbol{v}(0) = \boldsymbol{v}_0, \tag{1.13}$$

here, $\boldsymbol{v}$ is the fluid velocity field, p is the fluid pressure, $\mathcal{B}$ is the magnetic field, and $\boldsymbol{v}_0$ and $\mathcal{B}_0$ are the corresponding initial data. The interested readers are referred to [15, 19] and references cited therein.

This paper has two main correlated objects. The first one is to study the Large Eddy Simulation for the Navier-Stokes equations (LES for NSE) with a general filter $\overline{}^{\theta}$:

$$\operatorname{div}\boldsymbol{w} = 0, \tag{1.14}$$

$$\boldsymbol{w}_{,t} + \operatorname{div}(\overline{\boldsymbol{w}\otimes\boldsymbol{w}}^{\theta}) - \nu\Delta\boldsymbol{w} + \nabla q = \overline{\boldsymbol{f}}, \tag{1.15}$$

$$\alpha^{2\theta}(-\Delta)^{\theta}\overline{\varphi}^{\theta} + \overline{\varphi}^{\theta} = \varphi, \tag{1.16}$$

where the nonlocal operator $(-\Delta)^{\theta}$ is defined through the Fourier transform

$$\widehat{(-\Delta)^{\theta}\varphi}(\boldsymbol{k}) = |\boldsymbol{k}|^{2\theta}\widehat{\varphi}(\boldsymbol{k}). \tag{1.17}$$

Our task is to show that for $\theta \geq \frac{1}{6}$ (see Theorem 2.1), we get global in time existence of a unique weak solution $(\boldsymbol{w}, q)$ to eqs. (1.14)–(1.16) such that:
$(\boldsymbol{w}, q)$ are spatially periodic with period L,

$$\int_{\mathbb{T}_3} \boldsymbol{w}(t,\boldsymbol{x})d\boldsymbol{x} = 0 \quad \text{and} \int_{\mathbb{T}_3} q(t,\boldsymbol{x})d\boldsymbol{x} = 0 \quad \text{for } t\in[0,T), \tag{1.18}$$

and

$$\boldsymbol{w}(0,x) = \boldsymbol{w}_0(x) = \overline{\boldsymbol{v}_0}^{\theta} \quad \text{in } \mathbb{T}_3. \tag{1.19}$$

We note that the value $\theta = \frac{1}{6}$ is optimal and of course the general α family considered here recover the case $\theta = 1$ studied in [18]. The LES for NSE with $\theta = 1$ can be also addressed as the zeroth order Approximate Deconvolution Model referring to the family of models in [1] . The value $\theta = \frac{1}{6}$ is consistent with the critical regularization value needed to get existence and uniqueness to the simplified Bardina model studied in [3]. We note also that fractionnal order Laplace operator has been used in another α models of turbulence in [21, 2, 13].

The second object of this paper is to study the (LES) model for Magneto-hydrohynamics equation (LES for MHD) with a general filter $\overline{}^{\theta}$. Hence, we consider the following LES for MHD problem

$$\partial_t\boldsymbol{w} - \nu_1\Delta\boldsymbol{w} + \operatorname{div}(\overline{\boldsymbol{w}\otimes\boldsymbol{w}}^{\theta}) - \operatorname{div}(\overline{\boldsymbol{W}\otimes\boldsymbol{W}}^{\theta}) + \nabla q = 0, \tag{1.20}$$

$$\partial_t\boldsymbol{W} - \nu_2\Delta\boldsymbol{W} + \operatorname{div}(\overline{\boldsymbol{w}\otimes\boldsymbol{W}}^{\theta}) - \operatorname{div}(\overline{\boldsymbol{W}\otimes\boldsymbol{w}}^{\theta}) = 0, \tag{1.21}$$

$$\int_{\mathbb{T}_3}\boldsymbol{W}\,d\boldsymbol{x} = \int_{\mathbb{T}_3}\boldsymbol{w}\,d\boldsymbol{x} = 0, \quad \operatorname{div}\boldsymbol{w} = \operatorname{div}\boldsymbol{W} = 0, \tag{1.22}$$

$$\boldsymbol{W}(0) = \boldsymbol{W}_0,\ \boldsymbol{w}(0) = \boldsymbol{w}_0, \tag{1.23}$$

where the boundary conditions are taken to be periodic, and we take as before the same filter $\overline{}^{\theta}$ and $(\boldsymbol{w},\boldsymbol{W},q)$ is the approximation of $(\overline{\boldsymbol{v}},\overline{\mathcal{B}},\overline{p})$ solution of the MHD Equations.

The case when $\theta = 1$ is studied in [15] where the authors gived a mathematical description of the problem, performed the numerical analysis of the model and verified their physical fidelity.
In this paper, we show that for $\theta \geq \frac{1}{6}$ (see Theorem 3.1), we get global in time existence of a unique weak solution $(\boldsymbol{w}, \boldsymbol{W}, q)$ to eqs. (1.20)–(1.23). Let us mention that the idea to consider the LES for MHD with critical regularization is a new feature for the present work. The Approximate Deconvolution Model for Navier-Stokes equations with $\theta > \frac{3}{4}$ is studied in [6] and the Approximate Deconvolution Model for Magnetohydrodynamics equations with $\theta = 1$ is studied in [14]. As mentioned in [6] the value $\theta > \frac{3}{4}$ is not optimal in order to prove the existence and the uniqueness of the solutions in the deconvolution case.
"The exponent "3/4" looks like a "critical exponent". We conjecture that we can get an existence and uniqueness result for lower exponents, but concerning the convergence towards the mean Navier-Stokes equations, we think that it is the best exponent, but this question remains an open one."
Notice however that unlike the LES case the value $\theta = \frac{1}{6}$ is not sufficent to get the uniqueness in the deconvolution case. Based on this work, we will study in a forthcoming paper the Approximate Deconvolution Model for both Navier-Stokes equations and Magnetohydrodynamics equations with critical regularizations.
Finally, one may ask questions about the relation between the regularization parameter θ and the model consistency errors. These questions are adressed in [16] where $\theta = 1$. Therefore, the issue is to find the relation between the model consistency errors and the regularization parameter θ.

This paper is organized as follows. In section 2 we prove the global existence and uniqueness of the solution for the LES for NSE with critical regularization. We also prove that the solution $(\boldsymbol{w}, q)$ of the LES for NSE converges in some sense to a solution of the Navier-Stokes equations when α goes to zero. Section 3 treats the questions of global existence, uniqueness and convegence for the LES for MHD with critical regularization.

2 The Critical LES for NSE

Before formulating the main results of this paper, we fix notation of function spaces that we shall employ.
We denote by $L^p(\mathbb{T}_3)$ and $W^{s,p}(\mathbb{T}_3)$, $s \geq -1$, $1 \leq p \leq \infty$ the usual Lebesgue and Sobolev spaces over $\mathbb{T}_3$, and the Bochner spaces $C(0,T;X), L^p(0,T;X)$ are defined in the standard way. In addition we introduce

$$\dot{W}^{s,p}_{\text{div}} = \left\{ \boldsymbol{w} \in W^{s,p}(\mathbb{T}_3)^3;\ \int_{\mathbb{T}_3} \boldsymbol{w} = 0;\ \text{div}\, \boldsymbol{w} = 0 \text{ in } \mathbb{T}_3 \right\}.$$

We present our main results, restricting ourselves to the critical case $\theta = \frac{1}{6}$, and for simplicity we drop some indices of θ so sometimes we will write "$\overline{\varphi}$" instead of "$\overline{\varphi}^{\theta}$", expecting that no confusion will occur.

2.1 Existence and uniqueness results for the LES for NSE

Theorem 2.1. *Assume that* $\theta = \frac{1}{6}$. *Let* $\overline{\boldsymbol{f}} \in L^2(0,T;W^{-\frac{5}{6},2})$ *be a divergence free function and* $\boldsymbol{w}_0 \in W^{\frac{1}{6},2}_{\rm div}$. *Then there exist* $(\boldsymbol{w}, q)$ *a unique "regular" weak solution to* (1.14)–(1.19) *such that*

$$\boldsymbol{w} \in \mathcal{C}(0,T;\dot{W}^{\frac{1}{6},2}_{\rm div}) \cap L^2(0,T;\dot{W}^{1+\frac{1}{6},2}_{\rm div}), \tag{2.1}$$

$$\boldsymbol{w}_{,t} \in L^2(0,T;W^{-\frac{5}{6},2}), \tag{2.2}$$

$$q \in L^2(0,T;W^{\frac{1}{6},2}(\mathbb{T}_3)). \tag{2.3}$$

fulfill

$$\int_0^T \langle \boldsymbol{w}_{,t}, \boldsymbol{\varphi}\rangle - (\overline{\boldsymbol{w}\otimes\boldsymbol{w}}, \nabla\boldsymbol{\varphi}) + \nu(\nabla\boldsymbol{w}, \nabla\boldsymbol{\varphi})\, dt = \int_0^T \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle\, dt \quad \text{for all } \boldsymbol{\varphi} \in L^2(0,T;\dot{W}^{\frac{5}{6},2}_{\rm div}), \tag{2.4}$$

Moreover,

$$\boldsymbol{w}(0) = \boldsymbol{w}_0. \tag{2.5}$$

Remark 2.1. *The notion of "regular weak solution" is introduced in [6]. Here, we use the name "regular" for the weak solution since the weak solution is unique and the velocity part of the solution* $\boldsymbol{w}$ *does not develop a finite time singularity.*

Remark 2.2. *Once existence and uniqueness in the large of a weak solution to the model* (1.14)–(1.19) *with critical regularization is known. Further theoretical properties of the model with critical and subcritical regularizations can then be developed. These are currently under study by the author and will be presented in a subsequent report.*

Proof of Theorem 2.1. The proof of Theorem 2.1 follows the classical scheme. We start by constructing approximated solutions $(\boldsymbol{v}^N, p^N)$ via Galerkin method. Then we seek for a priori estimates that are uniform with respect to N. Next, we passe to the limit in the equations after having used compactness properties. Finaly we show that the solution we constructed is unique thanks to Gronwall's lemma.

Step 1(Galerkin approximation). Consider a sequence $\{\boldsymbol{\varphi}^r\}_{r=1}^{\infty}$ consisting of L^2-orthonormal and $W^{1,2}$-orthogonal eigenvectors of the Stokes problem subjected to the space periodic conditions. We note that this sequence forms a hilbertian basis of L^2.
We set

$$\boldsymbol{w}^N(t,\boldsymbol{x}) = \sum_{r=1}^{N} \boldsymbol{c}_r^N(t)\boldsymbol{\varphi}^r(\boldsymbol{x}), \text{ and } q^N(t,\boldsymbol{x}) = \sum_{|\boldsymbol{k}|=1}^{N} q_{\boldsymbol{k}}^N(t)e^{\boldsymbol{ik}\cdot\boldsymbol{x}}. \tag{2.6}$$

We look for $(\boldsymbol{w}^N(t,\boldsymbol{x}), q^N(t,\boldsymbol{x}))$ that are determined through the system of equations

$$\left(\boldsymbol{w}^N_{,t}, \boldsymbol{\varphi}^r\right) - (\overline{\boldsymbol{w}^N \otimes \boldsymbol{w}^N}, \nabla\boldsymbol{\varphi}^r) + \nu(\nabla\boldsymbol{w}^N, \nabla\boldsymbol{\varphi}^r) = \langle \boldsymbol{f}, \boldsymbol{\varphi}^r\rangle , \qquad r = 1,2,...,N, \tag{2.7}$$

and

$$\Delta q^N = -\operatorname{div}\operatorname{div}\left(\Pi^N(\overline{\boldsymbol{w}^N \otimes \boldsymbol{w}^N})\right). \tag{2.8}$$

Where the projector Π^N assign to any Fourier series $\sum_{\boldsymbol{k}\in\mathbb{Z}^3\setminus\{0\}} \boldsymbol{g}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ its N-dimensional part, i.e. $\sum_{\boldsymbol{k}\in\mathbb{Z}^3\setminus\{0\},|\boldsymbol{k}|\le N} \boldsymbol{g}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$.

Moreover we require that $\boldsymbol{w}^N$ satisfies the following initial condition

$$\boldsymbol{w}^N(0,.) = \boldsymbol{w}^N_0 = \sum_{r=1}^{N} \boldsymbol{c}^N_0 \boldsymbol{\varphi}^r(\boldsymbol{x}), \tag{2.9}$$

and

$$\boldsymbol{w}^N_0 \to \boldsymbol{w}_0 \quad \text{strongly in } W^{\frac{1}{6},2}(\mathbb{T}_3)^3 \quad \text{when } N\to\infty. \tag{2.10}$$

The classical Caratheodory theory [23] then implies the short-time existence of solutions to (2.7)-(2.8). Next we derive estimate on $\boldsymbol{c}^N$ that is uniform w.r.t. N. These estimates then imply that the solution of (2.7)-(2.8) constructed on a short time interval $[0,T^N[$ exists for all $t\in[0,T]$.

Step 2 (A priori estimates) Multilplying the rth equation in (2.7) with $\alpha^{2\theta}|\boldsymbol{k}|^{2\theta}\boldsymbol{c}^N_r(t) + \boldsymbol{c}^N_r(t)$, summing over $r=1,2,...,N$, integrating over time from 0 to t and using the following identities

$$\left(\boldsymbol{w}^N_{,t}, \boldsymbol{w}^N + \alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\boldsymbol{w}^N\right) = \frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}^N\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}^N\|^2_{\frac{1}{6},2}, \tag{2.11}$$

$$\left(-\Delta\boldsymbol{w}^N, \boldsymbol{w}^N + \alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\boldsymbol{w}^N\right) = \|\boldsymbol{w}^N\|^2_{1,2} + \|\boldsymbol{w}^N\|^2_{1+\frac{1}{6},2}, \tag{2.12}$$

$$\langle\overline{\boldsymbol{f}}, \boldsymbol{w}^N + \alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\boldsymbol{w}^N\rangle = \langle\boldsymbol{f}, \boldsymbol{w}^N\rangle, \tag{2.13}$$

and

$$\begin{aligned}\left(\overline{\boldsymbol{w}^N \otimes \boldsymbol{w}^N}, \nabla(\boldsymbol{w}^N + \alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\boldsymbol{w}^N)\right) &= \left(\boldsymbol{w}^N\otimes\boldsymbol{w}^N, \nabla\boldsymbol{w}^N\right)\\ &= -\left(\operatorname{div}\boldsymbol{w}^N, \tfrac{|\boldsymbol{w}^N|^2}{2}\right) = 0\end{aligned} \tag{2.14}$$

leads to the a priori estimates

$$\begin{aligned}\frac{1}{2}\left(\|\boldsymbol{w}^N\|_2^2 + \|\boldsymbol{w}^N\|^2_{\frac{1}{6},2}\right) + \nu\int_0^t \left(\|\boldsymbol{w}^N\|^2_{1,2} + \|\boldsymbol{w}^N\|^2_{1+\frac{1}{6},2}\right)\, ds\\ = \int_0^t \langle\boldsymbol{f}, \boldsymbol{w}^N\rangle\, ds + \frac{1}{2}\left(\|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{w}_0\|^2_{\frac{1}{6},2}\right).\end{aligned} \tag{2.15}$$

Using the duality norm comined with Young inequality we conclude from eqs. (2.15) that

$$\sup_{t\in[0,T^N[} \|\boldsymbol{w}^N\|_2^2 + \sup_{t\in[0,T^N[} \|\boldsymbol{w}^N\|_{\frac{1}{6},2}^2 + \nu \int_0^t \left(\|\boldsymbol{w}^N\|_{1,2}^2 + \|\boldsymbol{w}^N\|_{1+\frac{1}{6},2}^2 \right) \, ds \le C \tag{2.16}$$

that immediately implies that the existence time is independent of N and it is possible to take $T = T^N$.
We deduce from (2.16) that

$$\boldsymbol{w}^N \in L^\infty(0,T; \dot{W}_{\text{div}}^{\frac{1}{6},2}) \cap L^2(0,T; W^{1+\frac{1}{6},2}(\mathbb{T}_3)^3). \tag{2.17}$$

From (2.17) and (3.5) it follows that

$$\overline{\boldsymbol{w}^N \otimes \boldsymbol{w}^N} \in L^2(0,T; W^{\frac{1}{6},2}). \tag{2.18}$$

Consequently from the elliptic theory eqs (2.8) implies that

$$\int_0^T \|p^N\|_{\frac{1}{6},2}^2 dt < K. \tag{2.19}$$

From eqs. (2.7), (2.17), (2.18) and (2.19) we also obtain that

$$\int_0^T \|\boldsymbol{w}_{,t}^N\|_{-\frac{5}{6},2}^2 dt < K. \tag{2.20}$$

Step 3 (Limit $N \to \infty$) It follows from the estimates (2.17)-(2.20) and the Aubin-Lions compactness lemma (see [22] for example) that there are a not relabeled subsequence of $(\boldsymbol{w}^N, q^N)$ and a couple $(\boldsymbol{w}, q)$ such that

$$\begin{aligned}
\boldsymbol{w}^N &\rightharpoonup^* \boldsymbol{w} && \text{weakly}^* \text{ in } L^\infty(0,T; W^{\frac{1}{6},2}), && (2.21)\\
\boldsymbol{w}^N &\rightharpoonup \boldsymbol{w} && \text{weakly in } L^2(0,T; W^{1+\frac{1}{6},2}), && (2.22)\\
\overline{\boldsymbol{w}^N \otimes \boldsymbol{w}^N} &\rightharpoonup \overline{\boldsymbol{w} \otimes \boldsymbol{w}} && \text{weakly in } L^2(0,T; W^{\frac{1}{6},2}), && (2.23)\\
\boldsymbol{w}_{,t}^N &\rightharpoonup \boldsymbol{w}_{,t} && \text{weakly in } L^2(0,T; W^{-\frac{5}{6},2}), && (2.24)\\
q^N &\rightharpoonup q && \text{weakly in } L^2(0,T; W^{\frac{1}{6},2}(\mathbb{T}_3)), && (2.25)\\
\boldsymbol{w}^N &\to \boldsymbol{w} && \text{strongly in } L^2(0,T; W^{s,2}(\mathbb{T}_3)^3), s < 1 + \frac{1}{6} && (2.26)\\
\overline{\boldsymbol{w}^N \otimes \boldsymbol{w}^N} &\to \overline{\boldsymbol{w} \otimes \boldsymbol{w}} && \text{strongly in } L^2(0,T; W^{r,2}(\mathbb{T}_3)^3), r < \frac{1}{6}. && (2.27)
\end{aligned}$$

The above established convergences are clearly sufficient for taking the limit in (2.7) and for concluding that the velocity part $\boldsymbol{w}$ satisfy (2.4). Moreover, from (2.22) and (2.24) one we can deduce by a classical argument (see in [2]) that

$$\boldsymbol{w} \in \mathcal{C}(0,T; W^{\frac{1}{6},2}). \tag{2.28}$$

Furthermore, from the strong continuty of $\boldsymbol{w}$ with respect to the time with value in $W^{\frac{1}{6},2}$ we deduce that $\boldsymbol{w}(0)=\boldsymbol{w}_0$.
Let us mention also that $\boldsymbol{w}+\alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\boldsymbol{w}$ is a possible test in the weak formlation (2.4). Thus $\boldsymbol{w}$ verifies for all $t\in[0,T]$ the follwing equality

$$\begin{aligned}\left(\|\boldsymbol{w}(t)\|_2^2+\|\boldsymbol{w}(t)\|_{\frac{1}{6},2}^2\right)+2\nu\int_0^t\left(\|\boldsymbol{w}\|_{1,2}^2+\|\boldsymbol{w}\|_{1+\frac{1}{6},2}^2\right)ds\\ =2\int_0^t\langle\boldsymbol{f},\boldsymbol{w}\rangle ds+\left(\|\boldsymbol{w}_0\|_2^2+\|\boldsymbol{w}_0\|_{\frac{1}{6},2}^2\right).\end{aligned}\tag{2.29}$$

Step 5 (Uniqueness) Since the pressure part of the solution is uniquely determined by the velocity part it remain to show the uniqueness to the velocity.

Next, we will show the continuous dependence of the solutions on the initial data and in particular the uniqueness.
Let $(\boldsymbol{w}_1,p_1)$ and $(\boldsymbol{w}_2,p_2)$ any two solutions of (1.14)-(3.5) on the interval $[0,T]$, with initial values $\boldsymbol{w}_1(0)$ and $\boldsymbol{w}_2(0)$. Let us denote by $\delta\boldsymbol{w}=\boldsymbol{w}_2-\boldsymbol{w}_1$. We subtract the equation for $\boldsymbol{w}_1$ from the equation for $\boldsymbol{w}_2$ and test it with $\delta\boldsymbol{w}$. We get using successively the relation (3.5), the fact that the averaging operator commutes with differentiation under periodic boundary conditions, the norm duality, Young inequality and Sobolev embedding theorem:

$$\begin{aligned}\|\delta\boldsymbol{w}_{,t}\|_2^2+\alpha^{\frac{1}{3}}\|\delta\boldsymbol{w}_{,t}\|_{\frac{1}{6},2}^2+\nu\|\nabla\delta\boldsymbol{w}\|_2^2+\alpha^{\frac{1}{3}}\|\nabla\delta\boldsymbol{w}\|_{\frac{1}{6},2}^2\\ \le(\overline{\boldsymbol{w}_2\otimes\boldsymbol{w}_2}-\overline{\boldsymbol{w}_1\otimes\boldsymbol{w}_1},\nabla(\delta\boldsymbol{w}+\alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\delta\boldsymbol{w}))\\ \le(\boldsymbol{w}_2\otimes\boldsymbol{w}_2-\boldsymbol{w}_1\otimes\boldsymbol{w}_1,\nabla\delta\boldsymbol{w})\\ \le\frac{4}{\nu}\|\delta\boldsymbol{w}\otimes\boldsymbol{w}_1\|_{-\frac{1}{6},2}^2\\ \le\frac{4}{\nu}\|\delta\boldsymbol{w}\|_{\frac{1}{6},2}^2\|\boldsymbol{w}_1\|_{1+\frac{1}{6}}^2.\end{aligned}\tag{2.30}$$

Using Gronwall's inequality we conclude the continuous dependence of the solutions on the inital data in the $L^\infty([0,T],W^{\frac{1}{6},2})$ norm. In particular, if $\delta\boldsymbol{w}_0=0$ then $\delta\boldsymbol{w}=0$ and the solutions are unique for all $t\in[0,T]$. Since $T>0$ is arbitrary this solution may be uniquely extended for all time.
This finish the proof of Theorem 2.1.

2.2 Limit consistensy for the critical LES for NSE

In this section, we take the limit $\alpha\to 0$ in order to show the following result:

Theorem 2.2. *Let $(\boldsymbol{w}_\alpha,q_\alpha)$ be the solution of* (1.14)–(1.16) *for a fixed α. There is a subsequence α_j such that $(\boldsymbol{w}_{\alpha_j},q_{\alpha_j})\to(\boldsymbol{v},p)$ as $j\to\infty$ where $(\boldsymbol{v},p)\in L^\infty([0,T];L^2(\mathbb{T}_3)^3)\cap L^2([0,T];\dot{W}^{1,2}_{\mathrm{div}})\times L^{\frac{5}{3}}([0,T];L^{\frac{5}{3}}(\mathbb{T}_3))$ is a weak solution of the Navier-Stokes equations with periodic boundary conditions and zero mean value constraint.*
The sequence $\boldsymbol{w}_{\alpha_j}$ converges strongly to $\boldsymbol{v}$ in the space $L^p([0,T];L^p(\mathbb{T}_3)^3)$ for all

$2 \le p < \frac{10}{3}$, *and weakly in* $L^r(0,T;L^{\frac{6r}{3r-4}}(\mathbb{T}_3)^3)$ *for all* $r \ge 2$, *while the sequence* q_{α_j} *converges strongly to* p *in the space* $L^p([0,T];L^p(\mathbb{T}_3))$ *for all* $\frac{4}{3} \le p < \frac{5}{3}$, *and weakly in the space* $L^{\frac{r}{2}}(0,T;L^{\frac{3r}{3r-4}}(\mathbb{T}_3))$ *for all* $r \ge 2$.

Before proving Theorem 2.2, we first record the following three Lemmas.

Lemma 2.1. *Let* $\theta \in \mathbb{R}^+, 0 \le \beta \le 2\theta, s \in \mathbb{R}$ *and assume that* $\varphi \in \dot{W}^{s,2}_{\rm div}$. *Then* $\overline{\varphi} \in \dot{W}^{s+\beta,2}_{\rm div}$ *such that*

$$\|\overline{\varphi}\|_{s+\beta,2} \le \frac{1}{\alpha^\beta}\|\varphi\|_{s,2}, \tag{2.31}$$

and

$$\|\overline{\varphi}\|_{s,2} \le \|\varphi\|_{s,2}. \tag{2.32}$$

Proof. see in [2]

Lemma 2.2. *Assume* $\boldsymbol{w}_\alpha$ *belongs to the energy space of solutions of the Navier-Stokes equations, then*

$$\int_0^T \|\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha\|^{\frac{r}{2}}_{\frac{8-3r}{2r},2} dt < \infty \text{ for any } \frac{8}{3} \le r < \infty. \tag{2.33}$$

Proof. We have by interpolation that

$$\boldsymbol{w}_\alpha \in L^r(0,T;L^{\frac{6r}{3r-4}}(\mathbb{T}_3)^3) \tag{2.34}$$

for any $r \ge 2$, thus we deduce by using Hölder inequality that

$$\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha \in L^{\frac{r}{2}}(0,T;L^{\frac{3r}{3r-4}}(\mathbb{T}_3)^{3\times 3}). \tag{2.35}$$

From Sobolev embedding we deduce that

$$\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha \in L^{\frac{r}{2}}(0,T;W^{\frac{8-3r}{2r},2}(\mathbb{T}_3)^{3\times 3}), \tag{2.36}$$

for any $r \ge \frac{8}{3}$.

Lemma 2.3. *Assume* $\boldsymbol{w}_\alpha$ *belongs to the energy space of solutions of the Navier-Stokes equations, then for all* $p \ge 1$ *and* $q \ge \frac{4}{3}$ *such that*

$$\frac{1}{p} + \frac{2}{3q} < 1, \tag{2.37}$$

we have

$$\int_0^T \|\overline{\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha} - \boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha\|^q_p dt \le C\alpha^{\frac{3q+p-3pq}{p}}. \tag{2.38}$$

Proof. We take $r = 2q$, from the Sobolev injection $W^{\frac{3p-6}{2p},2}(\mathbb{T}_3) \hookrightarrow L^p(\mathbb{T}_3)$, it is sufficent to show that

$$\int_0^T \|\overline{\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha} - \boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha\|^{\frac{r}{2}}_{\frac{3p-6}{2p},2} dt \le C\alpha^{\frac{3r+2p-3pr}{2p}}. \tag{2.39}$$

From the relation between $\overline{\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha}$ and $\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha$ we have

$$\|\overline{\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha} - \boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha\|^{\frac{r}{2}}_{\frac{3p-6}{2p},2} \leq \alpha^{\theta r} \|\overline{\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha}\|^{\frac{r}{2}}_{\frac{3p-6}{2p}+2\theta,2} \tag{2.40}$$

Lemma 2.1 implies that

$$\int_0^T \|\overline{\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha} - \boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha\|^{\frac{r}{2}}_{\frac{3p-6}{2p},2} dt \leq \alpha^{\frac{3r+2p-3pr}{2p}} \int_0^T \|\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha\|^{\frac{r}{2}}_{\frac{8-3r}{2r},2} dt. \tag{2.41}$$

Recall that

$$\int_0^T \|\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha\|^{\frac{r}{2}}_{\frac{8-3r}{2r},2} dt < \infty \text{ for any } \frac{8}{3} \leq r < \infty. \tag{2.42}$$

This yields the desired result for any $p \geq 1$, $q \geq \frac{4}{3}$ such that $\frac{1}{p} + \frac{2}{3q} < 1$.

Proof of Theorem 2.2. The proof of Theorem 2.2 follows the lines of the proof of the Theorem 4 in [18]. The only difference is the strong convergence of the pressure term q_α to the pressure term p of the Navier-Stokes equations. We will use Layton-Lewandowski [18] as a reference and only point out the differences between their proof of convergence to a weak solution of the Navier-Stokes equations and the proof of convergence in our study. First, we need to find estimates that are independent from α. Using the fact that $\boldsymbol{w}_\alpha$ belong to the energy space: $L^\infty([0,T]; L^2(\mathbb{T}_3)^3) \cap L^2([0,T]; \dot{W}^{1,2}_{\text{div}})$ and from the Aubin-Lions compactness Lemma (the same arguments as in section 2.1) we can find a subsequence $(\boldsymbol{w}_{\alpha_j}, q_{\alpha_j})$ and $(\boldsymbol{v}, p)$ such that when α_j tends to zero we have:

$$\boldsymbol{w}_{\alpha_j} \rightharpoonup^* \boldsymbol{v} \quad \text{weakly}^* \text{ in } L^\infty([0,T]; L^2(\mathbb{T}_3)^3), \tag{2.43}$$

$$\boldsymbol{w}_{\alpha_j} \rightharpoonup \boldsymbol{v} \quad \text{weakly in } L^2([0,T]; W^{1,2}(\mathbb{T}_3)^3), \tag{2.44}$$

$$\boldsymbol{w}_{\alpha_j} \rightharpoonup \boldsymbol{v} \quad \text{weakly in } L^r(0,T; L^{\frac{6r}{3r-4}}(\mathbb{T}_3)^3) \text{ for all } r \geq 2, \tag{2.45}$$

$$\boldsymbol{w}_{\alpha_j} \otimes \boldsymbol{w}_{\alpha_j} \rightharpoonup \boldsymbol{v} \otimes \boldsymbol{v} \quad \text{weakly in } L^{\frac{r}{2}}(0,T; L^{\frac{3r}{3r-4}}(\mathbb{T}_3)^{3\times 3}), \text{ for all } r \geq 2, \tag{2.46}$$

$$\boldsymbol{w}_{\alpha_j} \to \boldsymbol{v} \quad \text{strongly in } L^p([0,T]; L^p(\mathbb{T}_3)^3) \text{ for all } 2 \leq p < \frac{10}{3}, \tag{2.47}$$

$$\boldsymbol{w}_{\alpha_j} \otimes \boldsymbol{w}_{\alpha_j} \to \boldsymbol{v} \otimes \boldsymbol{v} \quad \text{strongly in } L^p([0,T]; L^p(\mathbb{T}_3)^{3\times 3}) \text{ for all } \frac{4}{3} \leq p < \frac{5}{3}, \tag{2.48}$$

Having (2.38) and (2.48) at hand we deduce that

$$\overline{\boldsymbol{w}_{\alpha_j} \otimes \boldsymbol{w}_{\alpha_j}} \to \boldsymbol{v} \otimes \boldsymbol{v} \quad \text{strongly in } L^p(0,T; L^p(\mathbb{T}_3)^{3\times 3}) \text{ for all } \frac{4}{3} \leq p < \frac{5}{3}. \tag{2.49}$$

Then, from (2.35) and (2.49) we deduce that

$$\overline{\boldsymbol{w}_{\alpha_j} \otimes \boldsymbol{w}_{\alpha_j}} \rightharpoonup \boldsymbol{v} \otimes \boldsymbol{v} \quad \text{weakly in } L^{\frac{r}{2}}(0,T; L^{\frac{3r}{3r-4}}(\mathbb{T}_3)^{3\times 3}) \text{ for all } r \geq 2. \tag{2.50}$$

Further, we have

$$q(t) = \boldsymbol{R}(\sum_{k,l} \overline{\boldsymbol{w}^k_{\alpha_j} \boldsymbol{w}^l_{\alpha_j}}(t)) \tag{2.51}$$

where the linear map $\boldsymbol{R}$ is defined by

$$\boldsymbol{R} \quad : \quad L^s(\mathbb{T}_3)^9 \longmapsto L^s(\mathbb{T}_3) \tag{2.52}$$

$$(u^{kl})_{k,l=1,2,3} \longmapsto (-\Delta)^{-1}\partial_k\partial_l(u^{kl}) \tag{2.53}$$

By the theory of Riesz transforms, $\boldsymbol{R}$ is a continuous map for any $s \in]1, \infty[$. Consequently, from (2.50) we have

$$\int_0^T \|q_{\alpha_j}\|^{\frac{r}{2}}_{\frac{3r}{3r-4}} dt < \infty, \text{ for all } r \geq 2. \tag{2.54}$$

From (2.49) we deduce that for almost all $t > 0$,

$$\overline{\boldsymbol{w}_{\alpha_j} \otimes \boldsymbol{w}_{\alpha_j}}(t) \to \boldsymbol{v} \otimes \boldsymbol{v}(t) \quad \text{ strongly in } L^p(\mathbb{T}_3)^{3\times 3} \text{ for all } \frac{4}{3} \leq p < \frac{5}{3}. \tag{2.55}$$

Using the dominate convergence theorem and the continuity of the operator $\boldsymbol{R}$, we conclude that

$$q_{\alpha_j} \to p \qquad \text{strongly in } L^p([0,T]; L^p(\mathbb{T}_3)) \text{ for all } \frac{4}{3} \leq p < \frac{5}{3}. \tag{2.56}$$

Finally we deduce from (2.56) and (2.54) that

$$q_{\alpha_j} \rightharpoonup p \qquad \text{weakly in } L^{\frac{r}{2}}(0,T; L^{\frac{3r}{3r-4}}(\mathbb{T}_3)), \text{ for all } r \geq 2. \tag{2.57}$$

These convergence results allow us to prove in the same way as in [18] that $(\boldsymbol{v}, p)$ is a weak solution to the Navier-Stokes equations, so we will not repeat it.

3 Application to the LES for magnetohydrodynamic equations (LES for MHD)

In this section, we consider the critical LES regularization for magnetohydrodynamic (LES for MHD) equations, given by

$$\partial_t \boldsymbol{w} - \nu_1 \Delta \boldsymbol{w} + \operatorname{div}(\overline{\boldsymbol{w} \otimes \boldsymbol{w}}^{\frac{1}{6}}) - \operatorname{div}(\overline{\boldsymbol{W} \otimes \boldsymbol{W}}^{\frac{1}{6}}) + \nabla q = 0, \tag{3.1}$$

$$\partial_t \boldsymbol{W} - \nu_2 \Delta \boldsymbol{W} + \operatorname{div}(\overline{\boldsymbol{w} \otimes \boldsymbol{W}}^{\frac{1}{6}}) - \operatorname{div}(\overline{\boldsymbol{W} \otimes \boldsymbol{w}}^{\frac{1}{6}}) = 0, \tag{3.2}$$

$$\int_{\mathbb{T}_3} \boldsymbol{W}\, d\boldsymbol{x} = \int_{\mathbb{T}_3} \boldsymbol{w}\, d\boldsymbol{x} = 0, \quad \operatorname{div} \boldsymbol{w} = \operatorname{div} \boldsymbol{W} = 0, \tag{3.3}$$

$$\boldsymbol{W}(0) = \boldsymbol{W}_0,\ \boldsymbol{w}(0) = \boldsymbol{w}_0, \tag{3.4}$$

where the boundary conditions are taken to be periodic, and we take as before the same spacing average operator.

$$\alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\overline{\varphi}^{\frac{1}{6}} + \overline{\varphi}^{\frac{1}{6}} = \varphi, \quad \varphi(t, \boldsymbol{x} + L\boldsymbol{e_j}) = \varphi(t, \boldsymbol{x}). \tag{3.5}$$

Here, the unknowns are the averaging fluid velocity field $\boldsymbol{w}(t, \boldsymbol{x})$, the averaging fluid pressure $q(t, \boldsymbol{x})$, and the averaging magnetic field $\boldsymbol{W}(t, \boldsymbol{x})$. Note that when $\alpha = 0$, we formally retrieve the MHD equations. Existence and uniqueness results for MHD equations are established by G. Duvaut and J.L. Lions in [10]. These results are completed by M. Sermange and R. Temam in [20]. They showed that the classical properities of the Navier-Stokes equations can be extended to the MHD system.

The aim in this section is to extend the results of existence uniqueness and convergence established above for the LES for NSE to the LES for MHD. We know, thanks to the work [15], that for $\theta = 1$ these results hold ture. Further, when $\theta = \frac{1}{6}$, we proved in the above section the existence of a unique "regular" weak solution to the LES for NSE. Therefore, it is intersecting to find the critical value of regularization needed to establish global in time existence of a unique "regular" weak solution to LES for MHD.

We divide this section into two subsections. One is devoted to prove the existence of a unique "regular" weak solution to the LES for MHD with $\theta = \frac{1}{6}$. The second one is devoted to prove that this solution converges to a weak solution to the MHD equations when α tends to zero.

3.1 Existence and uniqueness results for the LES for MHD

First, we establish the global existence and uniqueness of solutions for the LES for MHD equations with $\theta = \frac{1}{6}$.
We have the following theorem:

Theorem 3.1. *Assume that $\theta = \frac{1}{6}$. Assume $\boldsymbol{w}_0$ and $\boldsymbol{W}_0$ are both in $W^{\frac{1}{6},2}_{\text{div}}$. Then there exist $(\boldsymbol{w}, \boldsymbol{W}, q)$ a unique "regular" weak solution to* (3.1)–(3.4) *such that*

$$\boldsymbol{w},\ \boldsymbol{W} \in \mathcal{C}(0, T; \dot{W}^{\frac{1}{6},2}_{\text{div}}) \cap L^2(0, T; \dot{W}^{1+\frac{1}{6},2}_{\text{div}}), \tag{3.6}$$

$$\boldsymbol{w}_{,t},\ \boldsymbol{W}_{,t} \in L^2(0, T; W^{-\frac{5}{6},2}), \tag{3.7}$$

$$q \in L^2(0, T; W^{\frac{1}{6},2}(\mathbb{T}_3)). \tag{3.8}$$

fulfill

$$\begin{aligned} &\int_0^T \langle \boldsymbol{w}_{,t}, \boldsymbol{\varphi}\rangle - (\overline{\boldsymbol{w}\otimes\boldsymbol{w}}, \nabla\boldsymbol{\varphi}) + (\overline{\boldsymbol{W}\otimes\boldsymbol{W}}, \nabla\boldsymbol{\varphi}) + \nu_1(\nabla\boldsymbol{w}, \nabla\boldsymbol{\varphi})\, dt = 0 \\ &\int_0^T \langle \boldsymbol{W}_{,t}, \boldsymbol{\varphi}\rangle - (\overline{\boldsymbol{w}\otimes\boldsymbol{W}}, \nabla\boldsymbol{\varphi}) + (\overline{\boldsymbol{W}\otimes\boldsymbol{w}}, \nabla\boldsymbol{\varphi}) + \nu_2(\nabla\boldsymbol{w}, \nabla\boldsymbol{\varphi})\, dt = 0 \\ &\qquad\qquad\qquad\qquad \textit{for all } \boldsymbol{\varphi} \in L^2(0, T; W^{\frac{5}{6},2}_{\text{div}}). \end{aligned} \tag{3.9}$$

Moreover,

$$\boldsymbol{w}(0)=\boldsymbol{w}_0 \quad \textit{and} \quad \boldsymbol{W}(0)=\boldsymbol{W}_0. \tag{3.10}$$

Proof. of Theorem 3.1. We only sketch the proof since is similar to the Navier-Stokes equations case. The proof is obtained by taking the inner product of (1.20) with $\alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\boldsymbol{w}+\boldsymbol{w}$, (1.21) with $\alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\boldsymbol{W}+\boldsymbol{W}$ and then adding them, the existence of a solution to the critical LES for MHD can be derived thanks to the Galerkin method. Notice that $(\boldsymbol{w},\boldsymbol{W})$ satisfy the following estimates

$$\begin{aligned}&\left(\|\boldsymbol{w}(t)\|_2^2+\|\boldsymbol{w}(t)\|_{\frac{1}{6},2}^2\right)+\left(\|\boldsymbol{W}(t)\|_2^2+\|\boldsymbol{W}(t)\|_{\frac{1}{6},2}^2\right)\\&\quad+2\nu_1\int_0^t\left(\|\boldsymbol{w}\|_{1,2}^2+\|\boldsymbol{w}\|_{1+\frac{1}{6},2}^2\right)ds+2\nu_2\int_0^t\left(\|\boldsymbol{W}\|_{1,2}^2+\|\boldsymbol{W}\|_{1+\frac{1}{6},2}^2\right)ds\\&\qquad=\left(\|\boldsymbol{w}_0\|_2^2+\|\boldsymbol{w}_0\|_{\frac{1}{6},2}^2\right)+\left(\|\boldsymbol{W}_0\|_2^2+\|\boldsymbol{W}_0\|_{\frac{1}{6},2}^2\right).\end{aligned} \tag{3.11}$$

The averaging pressure q is reconstructed from $\boldsymbol{w}$ and $\boldsymbol{W}$ (as we work with periodic boundary conditions) and its regularity results from the fact that $\overline{\boldsymbol{w}\otimes\boldsymbol{w}}$ and $\overline{\boldsymbol{W}\otimes\boldsymbol{W}}\in L^2([0,T];W^{\frac{1}{6},2}(\mathbb{T}_3)^{3\times3})$.
It remains to prove the uniqueness. Let $(\boldsymbol{w}_1,\boldsymbol{W}_1,q_1)$ and $(\boldsymbol{w}_2,\boldsymbol{W}_2,q_2)$, be two solutions, $\delta\boldsymbol{w}=\boldsymbol{w}_2-\boldsymbol{w}_1$,$\delta\boldsymbol{W}=\boldsymbol{W}_2-\boldsymbol{W}_1$, $\delta q=q_2-q_1$. Then one has

$$\begin{aligned}\partial_t\delta\boldsymbol{w}-\nu_1\Delta\delta\boldsymbol{w}+\operatorname{div}(\overline{\boldsymbol{w}_2\otimes\boldsymbol{w}_2})-\operatorname{div}(\overline{\boldsymbol{w}_1\otimes\boldsymbol{w}_1})&\\-\operatorname{div}(\overline{\boldsymbol{W}_2\otimes\boldsymbol{W}_2})+\operatorname{div}(\overline{\boldsymbol{W}_1\otimes\boldsymbol{W}_1})+\nabla\delta q=0,&\\\partial_t\delta\boldsymbol{W}-\nu_2\Delta\delta\boldsymbol{W}+\operatorname{div}(\overline{\boldsymbol{w}_2\otimes\boldsymbol{W}_2})-\operatorname{div}(\overline{\boldsymbol{w}_1\otimes\boldsymbol{W}_1})&\\-\operatorname{div}(\overline{\boldsymbol{W}_2\otimes\boldsymbol{w}_2})+\operatorname{div}(\overline{\boldsymbol{W}_1\otimes\boldsymbol{w}_1})=0,&\end{aligned} \tag{3.12}$$

and $\delta\boldsymbol{w}=0$, $\delta\boldsymbol{W}=0$ at initial time. One can take $\alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\delta\boldsymbol{w}+\delta\boldsymbol{w}$ as test in the first equation of (3.12) and $\alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\delta\boldsymbol{W}+\delta\boldsymbol{W}$ as test in the second equations of (3.12). Since $\boldsymbol{w}_1$ is divergence-free we have

$$\int_0^T\int_{\mathbb{T}_3}\boldsymbol{w}_1\otimes\delta\boldsymbol{w}:\nabla\delta\boldsymbol{w}=-\int_0^T\int_{\mathbb{T}_3}(\boldsymbol{w}_1\cdot\nabla)\delta\boldsymbol{w}\cdot\delta\boldsymbol{w}=0, \tag{3.13}$$

Thus we obtain by using the fact that the averaging operator commutes with differentiation under periodic boundary conditions

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}_3}(\operatorname{div}(\overline{\boldsymbol{w}_2\otimes\boldsymbol{w}_2})-\operatorname{div}(\overline{\boldsymbol{w}_1\otimes\boldsymbol{w}_1}))\cdot\left(\alpha^{\frac{1}{3}}(-\Delta)^{\frac{1}{6}}\delta\boldsymbol{w}+\delta\boldsymbol{w}\right)\\&\qquad=\int_0^T\int_{\mathbb{T}_3}(\operatorname{div}(\boldsymbol{w}_2\otimes\boldsymbol{w}_2)-\operatorname{div}(\boldsymbol{w}_1\otimes\boldsymbol{w}_1))\cdot\delta\boldsymbol{w}\\&\qquad=-\int_0^T\int_{\mathbb{T}_3}\delta\boldsymbol{w}\otimes\boldsymbol{w}_2:\nabla\delta\boldsymbol{w}.\end{aligned} \tag{3.14}$$

Similarly, because $(\boldsymbol{w}_1)$ is divergence-free we have

$$\int_0^T\int_{\mathbb{T}_3}\boldsymbol{w}_1\otimes\delta\boldsymbol{W}:\nabla\delta\boldsymbol{W}=-\int_0^T\int_{\mathbb{T}_3}(\boldsymbol{w}_1\cdot\nabla)\delta\boldsymbol{W}\cdot\delta\boldsymbol{W}=0, \tag{3.15}$$

and thus we have the following identity

$$\int_0^T\int_{\mathbb{T}_3}\big(\operatorname{div}(\overline{\boldsymbol{w}_2\otimes\boldsymbol{W}_2})-\operatorname{div}(\overline{\boldsymbol{w}_1\otimes\boldsymbol{W}_1})\big)\cdot\Big(\alpha^{\frac13}(-\Delta)^{\frac16}\delta\boldsymbol{W}+\delta\boldsymbol{W}\Big)$$
$$=\int_0^T\int_{\mathbb{T}_3}(\operatorname{div}(\boldsymbol{w}_2\otimes\boldsymbol{W}_2)-\operatorname{div}(\boldsymbol{w}_1\otimes\boldsymbol{W}_1))\cdot\delta\boldsymbol{W} \tag{3.16}$$
$$=-\int_0^T\int_{\mathbb{T}_3}\delta\boldsymbol{w}\otimes\boldsymbol{W}_2:\nabla\delta\boldsymbol{W}.$$

Concerning the remaining terms we get by integrations by parts and by using the using the fact that the averaging operator commutes with differentiation under periodic boundary conditions

$$\int_0^T\int_{\mathbb{T}_3}\big(-\operatorname{div}(\overline{\boldsymbol{W}_2\otimes\boldsymbol{W}_2})+\operatorname{div}(\overline{\boldsymbol{W}_1\otimes\boldsymbol{W}_1})\big)\cdot\Big(\alpha^{\frac13}(-\Delta)^{\frac16}\delta\boldsymbol{w}+\delta\boldsymbol{w}\Big)$$
$$=\int_0^T\int_{\mathbb{T}_3}(-\operatorname{div}(\boldsymbol{W}_2\otimes\boldsymbol{W}_2)+\operatorname{div}(\boldsymbol{W}_1\otimes\boldsymbol{W}_1))\cdot\delta\boldsymbol{w} \tag{3.17}$$
$$=\int_0^T\int_{\mathbb{T}_3}\boldsymbol{W}_1\otimes\delta\boldsymbol{W}:\nabla\delta\boldsymbol{w}+\int_0^T\int_{\mathbb{T}_3}\delta\boldsymbol{W}\otimes\boldsymbol{W}_2:\nabla\delta\boldsymbol{w}.$$

and similarly

$$\int_0^T\int_{\mathbb{T}_3}\big(-\operatorname{div}(\overline{\boldsymbol{W}_2\otimes\boldsymbol{w}_2})+\operatorname{div}(\overline{\boldsymbol{W}_1\otimes\boldsymbol{w}_1})\big)\cdot\Big(\alpha^{\frac13}(-\Delta)^{\frac16}\delta\boldsymbol{W}+\delta\boldsymbol{W}\Big)$$
$$=\int_0^T\int_{\mathbb{T}_3}(\operatorname{div}(\boldsymbol{W}_2\otimes\boldsymbol{w}_2)-\operatorname{div}(\boldsymbol{W}_1\otimes\boldsymbol{w}_1))\cdot\delta\boldsymbol{W} \tag{3.18}$$
$$=-\int_0^T\int_{\mathbb{T}_3}(\boldsymbol{W}_1\cdot\nabla)\delta\boldsymbol{w}\cdot\delta\boldsymbol{W}+\int_0^T\int_{\mathbb{T}_3}\delta\boldsymbol{W}\otimes\boldsymbol{w}_2:\nabla\delta\boldsymbol{W}.$$

Therefore by adding (3.14)-(3.18) and using the fact that the averaging operator commutes with differentiation under periodic boundary conditions we obtain

$$\begin{aligned}&\frac{d}{2dt}\int_{\mathbb{T}_3}\Big(|\delta\boldsymbol{w}|^2+\alpha^{\frac16}|\nabla^{\frac16}\delta\boldsymbol{w}|^2\Big)+\frac{d}{2dt}\Big(\int_{\mathbb{T}_3}|\delta\boldsymbol{W}|^2+\alpha^{\frac16}|\nabla^{\frac16}\delta\boldsymbol{W}|^2\Big)\\&+\nu_1\Big(\int_{\mathbb{T}_3}|\nabla\delta\boldsymbol{u}|^2+|\nabla^{1+\frac16}\delta\boldsymbol{u}|^2\Big)+\nu_2\Big(\int_{\mathbb{T}_3}|\nabla\delta\mathcal{B}|^2+|\nabla^{1+\frac16}\delta\mathcal{B}|^2\Big)\\&\qquad=\int_{\mathbb{T}_3}\delta\boldsymbol{w}\otimes\boldsymbol{w}_2:\nabla\delta\boldsymbol{w}+\int_{\mathbb{T}_3}\delta\boldsymbol{w}\otimes\boldsymbol{W}_2:\nabla\delta\boldsymbol{W}\\&\qquad\qquad-\int_{\mathbb{T}_3}\delta\boldsymbol{W}\otimes\boldsymbol{W}_2:\nabla\delta\boldsymbol{w}-\int_{\mathbb{T}_3}\delta\boldsymbol{W}\otimes\boldsymbol{w}_2:\nabla\delta\boldsymbol{W}.\end{aligned} \tag{3.19}$$

By the norm duality

$$|\int_{\mathbb{T}_3} \delta\boldsymbol{w}\otimes\boldsymbol{w}_2 : \nabla\delta\boldsymbol{w}| \le \|\delta\boldsymbol{w}\otimes\boldsymbol{w}_2\|_{-\frac{1}{6},2}\|\nabla\delta\boldsymbol{w}\|_{\frac{1}{6},2}, \tag{3.20}$$

$$|\int_{\mathbb{T}_3} \delta\boldsymbol{w}\otimes\boldsymbol{W}_2 : \nabla\delta\boldsymbol{W}| \le \|\delta\boldsymbol{w}\otimes\boldsymbol{W}_2\|_{-\frac{1}{6},2}\|\nabla\delta\boldsymbol{W}\|_{\frac{1}{6},2}, \tag{3.21}$$

$$|\int_{\mathbb{T}_3} \delta\boldsymbol{W}\otimes\boldsymbol{W}_2 : \nabla\delta\boldsymbol{w}| \le \|\delta\boldsymbol{W}\otimes\boldsymbol{W}_2\|_{-\frac{1}{6},2}\|\nabla\delta\boldsymbol{w}\|_{\frac{1}{6},2}, \tag{3.22}$$

$$|\int_{\mathbb{T}_3} \delta\boldsymbol{W}\otimes\boldsymbol{w}_2 : \nabla\delta\boldsymbol{W}| \le \|\delta\boldsymbol{W}\otimes\boldsymbol{w}_2\|_{-\frac{1}{6},2}\|\nabla\delta\boldsymbol{W}\|_{\frac{1}{6},2}. \tag{3.23}$$

By Young's inequality,

$$|\int_{\mathbb{T}_3} \delta\boldsymbol{w}\otimes\boldsymbol{w}_2 : \nabla\delta\boldsymbol{w}| \le \frac{1}{\nu_1}\|\delta\boldsymbol{w}\otimes\boldsymbol{w}_2\|^2_{-\frac{1}{6},2} + \frac{\nu_1}{4}\|\nabla\delta\boldsymbol{w}\|^2_{\frac{1}{6},2}, \tag{3.24}$$

$$|\int_{\mathbb{T}_3} \delta\boldsymbol{w}\otimes\boldsymbol{W}_2 : \nabla\delta\boldsymbol{W}| \le \frac{1}{\nu_2}\|\delta\boldsymbol{w}\otimes\boldsymbol{W}_2\|^2_{-\frac{1}{6},2} + \frac{\nu_2}{4}\|\nabla\delta\boldsymbol{W}\|^2_{\frac{1}{6},2}, \tag{3.25}$$

$$|\int_{\mathbb{T}_3} \delta\boldsymbol{W}\otimes\boldsymbol{W}_2 : \nabla\delta\boldsymbol{w}| \le \frac{1}{\nu_1}\|\delta\boldsymbol{W}\otimes\boldsymbol{W}_2\|^2_{-\frac{1}{6},2} + \frac{\nu_1}{4}\|\nabla\delta\boldsymbol{w}\|^2_{\frac{1}{6},2}, \tag{3.26}$$

$$|\int_{\mathbb{T}_3} \delta\boldsymbol{W}\otimes\boldsymbol{w}_2 : \nabla\delta\boldsymbol{W}| \le \frac{1}{\nu_2}\|\delta\boldsymbol{W}\otimes\boldsymbol{w}_2\|^2_{-\frac{1}{6},2} + \frac{\nu_2}{4}\|\nabla\delta\boldsymbol{W}\|^2_{\frac{1}{6},2}. \tag{3.27}$$

By Hölder inequality combined with Sobolev injection

$$\frac{1}{\nu_1}\|\delta\boldsymbol{w}\otimes\boldsymbol{w}_2\|^2_{-\frac{1}{6},2} \le \frac{1}{\nu_1}\|\delta\boldsymbol{w}\|^2_{\frac{1}{6},2}\|\boldsymbol{w}_2\|^2_{1+\frac{1}{6},2} \tag{3.28}$$

$$\tfrac{1}{\nu_2}\|\delta\boldsymbol{w}\otimes\boldsymbol{W}_2\|^2_{-\frac{1}{6},2} \le \frac{1}{\nu_2}\|\delta\boldsymbol{w}\|^2_{\frac{1}{6},2}\|\boldsymbol{W}_2\|^2_{1+\frac{1}{6},2} \tag{3.29}$$

$$\frac{1}{\nu_1}\|\delta\boldsymbol{W}\otimes\boldsymbol{W}_2\|^2_{-\frac{1}{6},2} \le \frac{1}{\nu_1}\|\delta\boldsymbol{W}\|^2_{\frac{1}{6},2}\|\boldsymbol{W}_2\|^2_{1+\frac{1}{6},2} \tag{3.30}$$

$$\frac{1}{\nu_2}\|\delta\boldsymbol{W}\otimes\boldsymbol{w}_2\|^2_{-\frac{1}{6},2} \le \frac{1}{\nu_2}\|\delta\boldsymbol{W}\|^2_{\frac{1}{6},2}\|\boldsymbol{w}_2\|^2_{1+\frac{1}{6},2} \tag{3.31}$$

Hence,

$$\begin{aligned}
&\frac{d}{2dt}\int_{\mathbb{T}_3}\left(\|\delta\boldsymbol{w}\|_2^2 + \|\delta\boldsymbol{W}\|_2^2\right) + \alpha^{\frac{1}{6}}\frac{d}{2dt}\left(\|\delta\boldsymbol{w}\|^2_{\frac{1}{6},2} + \|\delta\boldsymbol{W}\|^2_{\frac{1}{6},2}\right)\\
&+\min(\nu_1,\nu_2)\left(\|\delta\boldsymbol{w}\|^2_{1,2} + \|\delta\boldsymbol{W}\|^2_{1,2}\right) + \alpha^{\frac{1}{6}}\min(\nu_1,\nu_2)\left(\|\delta\boldsymbol{w}\|^2_{1+\frac{1}{6},2} + \|\delta\boldsymbol{W}\|^2_{1+\frac{1}{6},2}\right)\\
&\qquad\le \frac{1}{\min(\nu_1,\nu_2)}\left(\|\delta\boldsymbol{w}\|^2_{\frac{1}{6},2} + \|\delta\boldsymbol{W}\|^2_{\frac{1}{6},2}\right)\left(\|\boldsymbol{w}_2\|^2_{1+\frac{1}{6},2} + \|\boldsymbol{W}_2\|^2_{1+\frac{1}{6},2}\right)
\end{aligned} \tag{3.32}$$

We conclude that $\delta\boldsymbol{u} = \delta\mathcal{B} = 0$ thanks to Grönwall's Lemma.

3.2 Limit consistensy for the critical LES for MHD

Next, we will deduce that the LES for MHD with critical regularization gives rise to a weak solution to the MHD equations.

Theorem 3.2. *Let* $(\boldsymbol{w}_\alpha, \boldsymbol{W}_\alpha, q_\alpha)$ *be the solution of* (1.14)–(1.16) *for a fixed* α. *There is a subsequence* α_j *such that* $(\boldsymbol{w}_{\alpha_j}, \boldsymbol{W}_{\alpha_j}, q_{\alpha_j}) \to (\boldsymbol{v}, \mathcal{B}, p)$ *as* $j \to \infty$ *where* $(\boldsymbol{v}, \boldsymbol{W}, p) \in [L^\infty([0,T]; L^2(\mathbb{T}_3)^3) \cap L^2([0,T]; \dot{W}^{1,2}_{\text{div}})]^2 \times L^{\frac{5}{3}}([0,T]; L^{\frac{5}{3}}(\mathbb{T}_3))$ *is a weak solution of the Navier-Stokes equations with periodic boundary conditions and zero mean value constraint.*
The sequence $\boldsymbol{w}_{\alpha_j}$ *converges strongly to* $\boldsymbol{v}$ *in the space* $L^p([0,T]; L^p(\mathbb{T}_3)^3)$ *for all* $2 \le p < \frac{10}{3}$, *and weakly in* $L^r(0,T; L^{\frac{6r}{3r-4}}(\mathbb{T}_3)^3)$ *for all* $r \ge 2$.
The sequence $\boldsymbol{W}_{\alpha_j}$ *converges strongly to* $\mathcal{B}$ *in the space* $L^p([0,T]; L^p(\mathbb{T}_3)^3)$ *for all* $2 \le p < \frac{10}{3}$, *and weakly in* $L^r(0,T; L^{\frac{6r}{3r-4}}(\mathbb{T}_3)^3)$ *for all* $r \ge 2$, *while the sequence* q_{α_j} *converges strongly to* p *in the space* $L^p([0,T]; L^p(\mathbb{T}_3))$ *for all* $\frac{4}{3} \le p < \frac{5}{3}$, *and weakly in the space* $L^{\frac{r}{2}}(0,T; L^{\frac{3r}{3r-4}}(\mathbb{T}_3))$ *for all* $r \ge 2$.

Proof of Theorem 3.2. As in the proof of Theorem 2.2 we can show that for all $\frac{4}{3} \le p < \frac{5}{3}$ we have

$$\overline{\boldsymbol{w}_\alpha \otimes \boldsymbol{w}_\alpha} \to \boldsymbol{w} \otimes \boldsymbol{w} \qquad \text{strongly in } L^p(0,T; L^p(\mathbb{T}_3)^{3\times 3}), \tag{3.33}$$

$$\overline{\boldsymbol{w}_\alpha \otimes \boldsymbol{W}_\alpha} \to \boldsymbol{w} \otimes \mathcal{B} \qquad \text{strongly in } L^p(0,T; L^p(\mathbb{T}_3)^{3\times 3}), \tag{3.34}$$

$$\overline{\boldsymbol{W}_\alpha \otimes \boldsymbol{w}_\alpha} \to \mathcal{B} \otimes \boldsymbol{w} \qquad \text{strongly in } L^p(0,T; L^p(\mathbb{T}_3)^{3\times 3}), \tag{3.35}$$

$$\overline{\boldsymbol{W}_\alpha \otimes \boldsymbol{W}_\alpha} \to \mathcal{B} \otimes \mathcal{B} \qquad \text{strongly in } L^p(0,T; L^p(\mathbb{T}_3)^{3\times 3}). \tag{3.36}$$

The above L^pL^p convergences combined with the fact that $\boldsymbol{w}_\alpha$ and $\boldsymbol{W}_\alpha$ belong to the energy space of the solutions of the Navier-Stokes equations and the Aubin-Lions compactness Lemma allow us to take the limit $\alpha \to 0$ in order to deduce that $(\boldsymbol{w}_\alpha, \boldsymbol{W}_\alpha, q_\alpha)$ converge to $(\boldsymbol{v}, \mathcal{B}, p)$ a weak solution to the MHD equations. The rest can be done in exactly way as in [18], so we omit the details.

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