THE COMPUTATIONAL COMPLEXITY OF RECOGNISING EMBEDDINGS IN FINITELY PRESENTED GROUPS

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Abstract. We extend a result by Lempp that recognising torsion-freeness for finitely presented groups is Π_2^0-complete; we show that the problem of recognising embeddings of finitely presented groups is at least Π_2^0-hard, Σ_2^0-hard, and lies in Σ_3^0. We conjecture that this problem is indeed Σ_3^0-complete. We use our constructions to form a universal finitely presented torsion-free group.

1. Introduction

It is well known ([10]) that recognising the following group properties amongst finitely presented groups are Σ_1^0-complete: trivial, abelian, finite, free, nilpotent, polycyclic, automatic. More recently ([11]), being word hyperbolic is Σ_1^0-complete. In addition, having solvable word problem is known to be Σ_3^0-complete ([2]). In [8] it was shown by Lempp that the problem of recognising torsion-freeness for finitely presented groups is Π_2^0-complete. In [3] it was shown that there exists a finitely presented group whose finitely presented subgroups are not recursively enumerable. In this paper we use the techniques established in [3] to extend the result by Lempp, to show the following:

Theorem A (Theorem 3.6)**.** *Take an enumeration $P_1, P_2, \dots$ of all finite presentations of groups. Then the set $K = \{\langle i,j\rangle \in \mathbb{N} \mid P_i \text{ embeds in } P_j \text{ as groups}\}$ is Σ_2^0-hard, Π_2^0-hard, has a Σ_3^0 description, and is conjectured to be Σ_3^0-complete.*

The main technical result of this paper is that the Higman embedding theorem can strictly preserve the orders of torsion elements:

Theorem B (Theorem 2.5)**.** *There is a uniform procedure than, on input of a countably generated recursive presentation P, constructs a finite presentation $T(P)$ such that for each $k \in \mathbb{N}$, the group P has an element of order k if and only if the group $T(P)$ has an element of order k.*

As part of the proof of this, we make the following remarkable observation, which also strengthens the result from [3] on non-enumerablilty of subgroups.

Theorem C (Theorem 4.4)**.** *There is a universal finitely presented torsion-free group G. That is, for any finitely presented group H, we have that $H \hookrightarrow G$ if and only if H is torsion-free. Hence the set of finite presentations defining subgroups of G is Π_2^0-complete.*

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In actual fact, G can be made to have an embedded copy of every infinitely generated recursively presentable torsion-free group (yet still be finitely presented).

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2. Preliminaries

2.1. **Notation.** We take φ_m to be the m^{th} *partial recursive function* $\varphi_m : \mathbb{N} \to \mathbb{N}$, and the m^{th} *partial recursive set* (r.e. set) W_m as the domain of φ_m. If $P = \langle X|R \rangle$ is a group presentation with generating set X and relators R, then we denote by $\overline{P}$ the group presented by P. A presentation $P = \langle X|R \rangle$ is said to be a *finitely generated recursive presentation* if X is a finite set and R is a recursive enumeration of relators; P is said to be a *countably generated recursive presentation* if instead X is a recursive enumeration of generators. A group G is said to be *finitely presentable* if $G \cong \overline{P}$ for some finite presentation P. If P, Q are group presentations then we denote their free product presentation by $P*Q$, which is given by taking the disjoint union of their generators and relators; this extends to the free product of arbitrary collections of presentations. If X is a set, then we denote by X^{-1} a set of the same cardinality as X (considered an 'inverse' set to X), and X^* the finite words on $X \cup X^{-1}$, including the empty word. If $\phi : X \to Y^*$ is a set map, then we write $\overline{\phi} : X^* \to Y^*$ for the extension of ϕ. We let $|g|$ denote the order of a group element g, and say g is *torsion* if $1 < |g| < \infty$. *Cantor's pairing function* is given by $\langle .,. \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $\langle x, y \rangle := \frac{1}{2}(x+y)(x+y+1)+y$ which is a computable bijection. An introduction to the relevant concepts in recursion theory, including Π^0_n and Σ^0_n sets, can be found in [12] or the introduction in [8].

2.2. **Embedding theorems.**

Definition 2.1. Let G be a group. We define the set $\text{TorOrd}(G) := \{n \in \mathbb{N} \mid \exists g \in G \text{ with } |g| = n \geq 2\}$, the set of orders of non-trivial torsion elements of G. Note that $\text{TorOrd}(G)$ never contains 0 or 1.

We begin with the following three results found in [13] as theorem 11.69, corollary 11.72, and theorem 12.18 respectively.

Theorem 2.2 (Torsion theorem for amalgamated products and HNN extensions)**.**
Let $g \in G$ have finite order in G. Then:
1. If $G = K_1 *_H K_2$ is an amalgamated product, then g is conjugate to an element of K_1 or K_2. Hence $\text{TorOrd}(K_1 *_H K_2) = \text{TorOrd}(K_1) \cup \text{TorOrd}(K_2)$.
2. If $G = K*_H$ is an HNN extension, then g lies in the base group K. Hence $\text{TorOrd}(K*_H) = \text{TorOrd}(K)$.

Lemma 2.3. *Let G be a countable group with generator and relator sets that are recursively enumerable. Then G can be uniformly embedded into some 2-generator recursively presented group E such that* $\mathrm{TorOrd}(G) = \mathrm{TorOrd}(E)$.

Theorem 2.4 (Higman)**.** *Let G be a finitely generated recursively presented group. Then G can be uniformly embedded into some finitely presented group H such that* $\mathrm{TorOrd}(G) = \mathrm{TorOrd}(H)$.

In both cases, the group so constructed is built up from amalgamated products and HNN extensions, beginning with G and some finitely generated free groups. Hence, by theorem 2.2, $\mathrm{TorOrd}(G) = \mathrm{TorOrd}(E) = \mathrm{TorOrd}(H)$.

The above machinery is used to prove the following theorem, which will become the cornerstone of the main results contained herein.

Theorem 2.5. *There is a uniform procedure than, on input of a countably generated recursive presentation $P = \langle X|R \rangle$, constructs a finite presentation $T(P)$ such that* $\mathrm{TorOrd}(\overline{T(P)}) = \mathrm{TorOrd}(\overline{P})$.

Proof. Given a countably generated recursive presentation P, we use lemma 2.3 to uniformly construct a 2-generator recursive presentation H with $\overline{P} \hookrightarrow \overline{H}$ and $\mathrm{TorOrd}(\overline{H}) = \mathrm{TorOrd}(\overline{P})$, and theorem 2.4 to uniformly construct a finite presentation $T(P)$ with $\overline{H} \hookrightarrow \overline{T(P)}$ and $\mathrm{TorOrd}(\overline{P}) = \mathrm{TorOrd}(\overline{H}) = \mathrm{TorOrd}(\overline{T(P)})$. And as all stages in this construction have been uniform, we conclude that such a presentation $T(P)$ can be uniformly constructed from P. □

We note that Collins ([6]), extending the work of Clapham ([4], [5]), has shown that such an embedding can be made to simultaneously preserve the Turing Degree of the word problem, order problem, and power problem (see [12] for an introduction to Turing Degrees). However, this does not immediately imply that TorOrd is preserved under the embedding, or even many-one equivalent.

Theorem 2.5 was used in [3] to show the following:

Theorem 2.6 (Chiodo)**.** *There is a uniform procedure than, on input of any $n \in \mathbb{N}$, constructs a finite presentation Q_n such that $\mathrm{TorOrd}(\overline{Q_n})$ is one-one equivalent to $\mathbb{N} \setminus W_n$. Taking n' with $W_{n'}$ non-recursive thus gives that $\mathrm{TorOrd}(\overline{Q_{n'}})$ is not recursively enumerable; thus the finitely presented subgroups of $\overline{Q_{n'}}$ are not recursively enumerable.*

3. Complexity Results

Our initial motivation was to investigate if the finitely presented subgroups of a finitely presented group always form a recursively enumerable set. This was done in [3], and follows from theorem 2.6.

Theorem 3.1 (Chiodo)**.** *There exists a finitely presented group G such that the set of all finite presentations that define groups which embed into G is not recursively enumerable.*

Using the machinery described in section 2, we can encode the following recursion theory facts (found in [12]) into groups.

Lemma 3.2. *The set* $\{n \in \mathbb{N} |\ W_n = \mathbb{N}\}$ *is* Π_2^0*-complete;*
the set $\{n \in \mathbb{N} |\ |W_n| < \infty\}$ *is* Σ_2^0*-complete.*

Our existing observations immediately lead to the following result, first proved in [8] by Lempp.

Theorem 3.3 (Lempp)**.** *The set of finite presentations of torsion-free groups is* Π_2^0*-complete.*

Proof. Given $n \in \mathbb{N}$, we use theorem 2.6 to construct a finite presentation Q_n such that $\mathrm{TorOrd}(\overline{Q}_n)$ is one-one equivalent to $\mathbb{N} \setminus W_n$. Thus $\overline{Q}_n$ is torsion-free if and only if $W_n = \mathbb{N}$. From lemma 3.2, $\{n \in \mathbb{N} |\ W_n = \mathbb{N}\}$ is Π_2^0-complete, so the set of torsion-free finite presentations is at least Π_2^0-hard. But this set has the following Π_2^0 description (taken from [8]):

$$G \text{ is torsion-free if and only if } \forall w \forall n > 0 (w^n \neq_G e \text{ or } w =_G e)$$

and hence is Π_2^0-complete. □

A similar construction gives us the following, which we can prove after making a preliminary observation in recursion theory.

Theorem 3.4. *For any fixed prime* p*, the set of finite presentations into which* C_p *embeds is* Σ_2^0*-complete.*

We start by showing that given an index n with W_n finite, we can recursively compress it to $\{1, \dots, |W_n|\}$.

Lemma 3.5. *There is a recursive function* $h : \mathbb{N} \to \mathbb{N}$ *satisfying the following:*
1. If $|W_n| = \emptyset$ *then* $W_{h(n)} = \emptyset$*.*
2. If $1 \leq |W_n| < \infty$ *then* $W_{h(n)} = \{1, \dots, |W_n|\}$*.*
3. If $|W_n| = \infty$ *then* $W_{h(n)} = \mathbb{N}$*.*

Proof. Given n, we begin an enumeration of W_n. For each element enumerated into W_n we increase the size of $W_{h(n)}$ by 1, by adding the next smallest number not already in $W_{h(n)}$. If $|W_n| = \emptyset$ then $W_{h(n)} = \emptyset$. If $1 \leq |W_n| < \infty$, then $W_{h(n)} = \{1, \dots, |W_n|\}$. If $|W_n| = \infty$ then we will continue to enumerate elements of $\mathbb{N}$ into $W_{h(n)}$, so $W_{h(n)} = \mathbb{N}$. As this is an effective description of $W_{h(n)}$, we have that h is recursive. □

Proof of theorem 3.4. With h as in lemma 3.5, given n we form the infinitely generated recursive presentation P_n as follows:
$P_n := \langle x_0, x_1, \dots |\ x_i^p = e\ \forall\ i,\ x_j = e\ \forall\ j \in W_{h(n)} \rangle$.
If $|W_n| < \infty$ then $W_{h(n)} < \infty$ and hence $\overline{P}_n \cong C_p * C_p * C_p * \dots$ On the other hand, if $|W_n| = \infty$ then $W_{h(n)} = \mathbb{N}$ and hence $\overline{P}_n \cong \{1\}$. So

$$\mathrm{TorOrd}(\overline{P}_n) = \begin{cases} \{p\} \text{ if } |W_n| < \infty \\ \quad\emptyset \text{ if } |W_n| = \infty \end{cases}$$

That is, $C_p \hookrightarrow \overline{P}_n$ if and only if $|W_n| < \infty$. Now use theorem 2.5 to construct a finite presentation $T(P_n)$ such that $\overline{P}_n \hookrightarrow \overline{T(P_n)}$ with $\mathrm{TorOrd}(\overline{P}_n) = \mathrm{TorOrd}(\overline{T(P_n)})$. Hence $C_p \hookrightarrow \overline{T(P_n)}$ if and only if $|W_n| < \infty$, so by lemma 3.2

the set of finite presentations in to which C_p embeds is Σ^0_2-hard. But this set has the following straightforward Σ^0_2 description:

$$C_p \hookrightarrow G \text{ if and only if } \exists w(w \neq_G e \text{ and } w^p =_G e)$$

and hence is Σ^0_2-complete. □

We will later see in theorem 4.4 that there is a group whose finitely presented subgroups form a Π^0_2-complete set; combining this with the above theorem gives the following:

Theorem 3.6. *Take an enumeration $P_1, P_2, \dots$ of all finite presentations of groups; $P_i = \langle X_i | R_i \rangle$. Then the set $K = \{\langle i,j\rangle \in \mathbb{N} \,|\overline{P}_i \hookrightarrow \overline{P}_j\}$ is Σ^0_2-hard, Π^0_2-hard, and has a Σ^0_3 description.*

Proof. Theorem 4.4 shows that K is Π^0_2-hard, theorem 3.4 gives that K is Σ^0_2-hard, and the following is a Σ^0_3 description for K:

$$K = \{\langle i,j\rangle \in \mathbb{N} | (\exists \phi : X_i \to X_j^*)(\forall w \in X_i^*)(\overline{\phi}(w) =_{\overline{P}_j} e \text{ if and only if } w =_{\overline{P}_i} e)\}$$

□

Based on the above, we conjecture the following:

Conjecture 1. The set K defined above is Σ^0_3-complete. That is, the problem of deciding for finite presentations P_i, P_j if $\overline{P}_i \hookrightarrow \overline{P}_j$ is Σ^0_3-complete.

Further work. The positions of the following properties in the arithmetic hierarchy have not been fully determined (see [10] which refers to the first three, and [7] lemmata 2.2.1 and 2.2.2 for the final two). However, it may be possible that some are neither Π^0_n-complete nor Σ^0_2-complete for any n:
Soluble: Known to have a Σ^0_3 description.
Residually finite: Known to have a Π^0_2 description.
Simple: Known to have a Π^0_2 description.
Orderable: Known to have a Π^0_3 description (the Ohnishi condition).
Right orderable: Known to have a Π^0_3 description (the Ohnishi condition).

4. Applications: Universality

Definition 4.1. For ρ an algebraic property of groups, we say a finitely presented group G is a *universal ρ group* if both of the following occur:
1. G has property ρ.
2. Every finitely presented group H with property ρ embeds in G.

The motivation for such a definition comes from the following famous result by Higman:

Theorem 4.2 (Higman). *There is a universal finitely presented group. That is, a finitely presented group into which every finitely presented group embeds.*

Proof. Take an enumeration $P_1, P_2, \dots$ of all finite presentations of groups, and form the infinitely generated recursive presentation $P := P_1 * P_2 * \dots$ Now use theorem 2.5 to embed $\overline{P}$ into a finitely presented group $\overline{T(P)}$. By construction, $\overline{T(P)}$ has an embedded copy of every finitely presented group, since $\overline{P}$ did. □

It seems reasonable to ask for which (if any) algebraic properties of groups ρ there exists a universal ρ group. An easy example is free groups.

Lemma 4.3. *There is a universal free group (namely, F_2).*

Proof. Take the standard presentation $P := \langle a, b | - \rangle$ of F_2. Then, for any $n \in \mathbb{N}$, the set $\{b^{-i}ab^i | 1 \leq i \leq n\}$ freely generates F_n in $\overline{P}$. □

A much deeper result, which comes as a consequence of theorem 2.5, is the following:

Theorem 4.4. *There is a universal torsion-free group G (finitely presentable). That is, for any finite presentation P, we have that $\overline{P} \hookrightarrow G$ if and only if $\overline{P}$ is torsion-free. Hence the set of finite presentations defining subgroups of G is Π_2^0-complete.*

To show this, we need the following lemma:

Lemma 4.5. *There is a uniform procedure that, on input of a countably generated recursive presentation $P = \langle X | R \rangle$ of a group, outputs a countably generated recursive presentation $\mathrm{TK}(P) = \langle X | R' \rangle$ (on the same generating set X, and with $R \subseteq R'$) such that:*
1. $\overline{\mathrm{TK}(P)}$ is torsion-free.
2. If $\overline{P}$ is torsion-free, then $\overline{\mathrm{TK}(P)} \cong \overline{P}$ by extending the identity map ϕ on X.
3. Any homomorphism $f : \overline{P} \to G$ to a torsion-free group G factors through $\overline{\phi}$.
We call the process of forming $\mathrm{TK}(P)$ the Torsion-Killing *of P.*

Proof. Set $P_0 := P$ for convenience. Enumerate all words $w_1, w_2, \ldots$ in X^*. For P_0, enumerate all trivial words $v_1, v_2, \ldots$ in P_0. Begin checking if any non-zero finite power of some word w_{i_1} is equal to some trivial word v_j in P_0; if so form a new presentation P_1 by adding the word w_{i_1} to the relating set of P_0. Now repeat this process for P_1, while still running the process for P_0 in parallel. If either process yields another word w_{i_2}, add the word w_{i_2} to the relating set of P_1, and call this P_2. Continue in this manner; whenever any of the parallel processes on the $P_0, \ldots, P_n$ yield a word $w_{i_{n+1}}$, form a new presentation $P_{n+1} := \langle X | R, w_{i_1}, \ldots, w_{i_{n+1}} \rangle$ and begin the process on P_{n+1} as well. Finally, define $\mathrm{TK}(P) := \langle X | R, w_{i_1}, w_{i_2}, \ldots \rangle$ (think of this as P_∞). It is clear that $\mathrm{TK}(P)$ is finitely generated, recursively presented, and has the desired properties. □

It should be noted that the torsion killing of a group is unique, and universal in the sense that any map from a group G to a torsion-free group factors through the torsion killing quotient. But what other group properties admit such a (recursive) killing procedure? It is well known that the abelianisation of a countably generated recursive presentation can be recursively constructed (think of this as the non-abelian killing). Moreover, the residual $G^{\mathrm{res}} := G / \cap_{i \in I} N_i$ of a group G (where $\{N_i\}_{i \in I}$ is the collection of all finite index normal subgroups of G) is universal in the sense that it is residually finite, and any map from G to a residually finite group factors through G^{res}. However, G^{res} need not be recursively presentable, so can not always be effectively constructed from a presentation for G. At the moment the author is unaware of any other (non-trivial) properties which admit an effective killing procedure.

Proof of theorem 4.4. Take an enumeration $P_1, P_2, \ldots$ of all finite presentations of groups, and form the infinitely generated recursive presentation $Q := \mathrm{TK}(P_1) * \mathrm{TK}(P_2) * \ldots$. Thus by lemma 4.5, $\overline{Q}$ describes a torsion-free group (as we have successfully annihilated all the torsion), and contains an embedded copy of every torsion-free group (as we have left the torsion-free factors untouched). Now use theorem 2.5 to embed $\overline{Q}$ into a finitely presented group $\overline{T(Q)}$. By construction, $\emptyset = \mathrm{TorOrd}(\overline{Q}) = \mathrm{TorOrd}(\overline{T(Q)})$, so $\overline{T(Q)}$ is torsion-free. Finally, $\overline{T(Q)}$ has an embedded copy of every finitely presented torsion-free finitely presented group, since $\overline{Q}$ did. Taking G to be $\overline{T(Q)}$ completes the proof of the first part. The second part follows immediately from theorem 3.3. □

Remark. By taking $P_1, P_2, \ldots$ to be an enumeration of all countably generated recursive presentations of groups in the above proof, we can construct a finitely presented G that has an embedded copy of every countably generated recursively presented torsion-free group.

But which other properties admit such a group? Clearly, there is no universal abelian group (the rank of such a group would be bounded, but the rank of its subgroups would be unbounded). Similarly, there is no universal nilpotent group or universal soluble group. This is because the nilpotency class (resp. derived length) of such a group bounds the nilpotency class (resp. derived length) of all its subgroups. A very interesting question is whether there exists a universal simple group. Existence of such a group, in conjunction with a result by Miller ([9]), would imply that the Boone-Higman theorem ([1]) cannot be strengthened any further than the following form ([14]):

Theorem 4.6 (Thompson)**.** *A finitely presented group G has solvable word problem if and only if G embeds into a finitely generated simple group S which in turn embeds into a finitely presented group H.*

Question 1. Is it true that a finitely presented group G has solvable word problem if and only if G embeds into some finitely presented simple group S?

It is worthwhile noting that a finitely presented group G is simple if and only if all of its homomorphic images are either isomorphic to itself or trivial. Miller ([9]), building on a result of Boone and Rogers ([2]), showed the following:

Theorem 4.7 (Boone-Rogers)**.** *There is no uniform partial algorithm which solves the word problem in all finitely presented groups with solvable word problem.*

Theorem 4.8 (Miller)**.** *There is no universal solvable word problem group.*

This immediately leads to the following observation:

Corollary 4.9. *The existence of a universal simple group implies that the answer to question 1 is no.*

Proof. Use theorem 4.8, together with the fact that every finitely presented simple group has solvable word problem. For suppose we had a universal simple group S, and assume that every finitely presented group with solvable word problem embeds into a finitely presented simple. Then every finitely presented group with solvable word problem would embed into S. But S has solvable word problem as it is finitely presented and simple, so this contradicts theorem 4.8. □

References

[1] W. W. Boone, G. Higman, *An algebraic characterization of the solvability of the word problem*, J. Austral. Math. Soc. **18**, 41–53 (1974).

[2] W. W. Boone, H. Rogers Jr., *On a problem of J.H.C. Whitehead and a problem of Alonzo Church*, Math. Scand. **19**, 185–192 (1966).

[3] M. Chiodo, *Finding non-trivial elements and splittings in groups*, J. Algebra. **331**, 271–284 (2011).

[4] C. R. J. Clapham, *Finitely presented groups with word problems of arbitrary degrees of insolubility*, Proc. London Math. Soc. (3) **14**, 633–676 (1964).

[5] C. R. J. Clapham, *An embedding theorem for finitely generated groups*, Proc. London Math. Soc. (3) **14**, 419–430 (1967).

[6] D. J. Collins *The word, power and order problems in finitely presented groups*, in Word Problems, eds. Boone, Cannonito, and Lyndon, Amsterdam, North-Holland, 401–420 (1973)

[7] A. M. W. Glass, *Partially ordered groups*, Series in algebra Vol. 7, World Sci., Singapore, (1999).

[8] S. Lempp, *The computational complexity of recognising torsion-freeness of finitely presented groups*, Bull. Austral. Math. Soc. **56**, 273–277 (1997).

[9] C. F. Miller III, *On group theoretic decision problems and their classification*, Annals of Math. Study **68**, Princeton University Press, Princeton, NJ, (1971).

[10] C. F. Miller III, *Decision problems for groups-survey and reflections*. Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., **23**, Springer, New York, 1–59 (1992).
161-164 (1950).

[11] P. Papasoglu, *An algorithm detecting hyperbolicity*. In *Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994)*, volume 25 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 193-200. Amer. Math. Soc., Providence, RI, 1996.

[12] H. Rogers Jr, *Theory of recursive functions and effective computability*, MIT Press, (1987).

[13] J. Rotman, *An introduction to the theory of groups*, Springer-Verlag, New York, (1995).

[14] R. J. Thompson, *Embeddings into Finitely Generated Simple Groups Which Preserve the Word Problem*, Word Problems II: The Oxford Book, Studies in Logic and the Foundations of Mathematics, Volume 95, (1980).

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