# SPACES OF MEASURABLE FUNCTIONS

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Abstract. For a metrizable space $X$ and a finite measure space $(\Omega, \mathfrak{M}, \mu)$ let $M_\mu(X)$ and $M_\mu^f(X)$ be the spaces of all equivalence classes (under the relation of equality almost everywhere mod $\mu$) of $\mathfrak{M}$-measurable functions from $\Omega$ to $X$ whose images are separable and finite, respectively, equipped with the topology of convergence in measure. The main aim of the paper is to prove the following result: if $\mu$ is (nonzero and) nonatomic and $X$ has more than one point, then the space $M_\mu(X)$ is a noncompact absolute retract and $M_\mu^f(A)$ is homotopy dense in $M_\mu(X)$ for each dense subset $A$ of $X$. In particular, if $X$ is completely metrizable, then $M_\mu(X)$ is homeomorphic to an infinite-dimensional Hilbert space.
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In [8] Bessaga and Pełczyński have proved that whenever $X$ is a separable completely metrizable topological space having more than one point, then the space $M_X$ of Borel functions from $[0, 1]$ to $X$ with the topology of convergence in measure is homeomorphic to $l^2$. Later it turned out that the topology of $l^2$ can be well characterized. This was done by Toruńczyk[20, 21]. After publication of the latter papers the number of results on spaces homeomorphic to the separable infinite-dimensional Hilbert space has highly rised. For example, Dobrowolski and Toruńczyk[11] have shown that every separable completely metrizable non-locally compact topological group which is an AR is homeomorphic to a Hilbert space. However, the problem whether the assumption of separability in the latter may be omitted is still open (see [7]). In this paper we shall introduce a class of nonseparable completely metrizable topological groups which are homeomorphic to Hilbert spaces. Namely, if $G$ is any (nonzero) completely metrizable topological group and $\mu$ is a (nonzero) finite nonatomic measure, then the space $M_\mu(G)$ (defined in Abstract) has a natural structure (induced by the one of $G$) of a topological group and is homeomorphic to a Hilbert space. In fact we shall prove the following, quite more general, result: if $X$ is a nonempty metrizable space, $\mu$ is a finite nonatomic measure and $Y = M_\mu^r(X)$ is the subspace of $M_\mu(X)$ consisting of all (equivalence classes of) functions whose images are contained in $\sigma$-compact subsets of $X$, then $Y$ is an absolute retract such that $Y^\omega \cong Y$. Since infinite-dimensional Hilbert spaces are the only completely metrizable noncompact AR's homeomorphic to their own countable infinite Cartesian powers ([20]), the latter mentioned result may be seen as a generalization of earlier results of Bessaga and Pełczyński[8] as well as of Toruńczyk[19].

Other purpose of the paper is to present the idea of extending maps between metrizable spaces to maps between AR's via functors. Namely, whenever $\mu$ is a finite (nonzero) nonatomic measure, every map $f\colon X \to Y$ has a natural extension $M_\mu(f)\colon M_\mu(X) \to M_\mu(Y)$. What is more, the correspondence $f \leftrightarrow M_\mu(f)$ preserves many properties (such as: being an injection, an embedding, a map with dense image). We shall show that if $m$ is the Lebesgue measure on $[0,1]$, the space $W = M_m(X)$ is always an AR satisfying the following conditions: $X$ is a $Z$-set in $W$ (provided $X$ has more than one point), $W^\omega \cong W$, $W$ has RIP and is an S-space (in the sense of Schori[16]). As an immediate consequence of this, we shall obtain that if $U$ is a metrizable manifold modelled on $W$, then $U$ is $W$-stable, i.e. $U \times W \cong U$.

Another issue we shall discuss here concerns the question of whether $M_m(M_m(X))$ is homeomorphic to $M_m(X)$. We shall see that the answer is affirmative for a huge class of metrizable spaces (namely, for spaces in which every closed separable subset is absolutely measurable), which contains locally absolutely Borel spaces and (separable) Souslin ones. However, in general we leave this question as an open problem.

The article is organized as follows. In the first section we establish notation and terminology, define general spaces of measurable functions and collect several results on them. Section 2 deals with spaces $M_\mu^r(X)$, defined in this introduction. We show there that if $\mu$ and $\nu$ are two homogeneous (nonatomic) measures of the same weight, then the spaces $M_\mu^r(X)$ and $M_\nu^r(X)$ are naturally homeomorphic, whatever $X$ is. The third part is devoted to spaces of measurable functions over metrizable AM-spaces (i.e. in which every closed separable subset is absolutely measurable). We prove there that if $X$ is an AM-space, then $M_\mu(X) = M_\mu^r(X)$ for each finite measure $\mu$. In Section 4 we state and prove the main result of the paper, which includes the claim that spaces of measurable functions are absolute retracts. We conclude from this that such spaces over completely metrizable ones are homeomorphic to Hilbert spaces. In the last part we generalize our results of [15] to nonseparable case. Also the idea of extending maps to AR's via the functors $M_\mu$ is presented.

## 1. Preliminaries

In this paper $\mathbb{R}_+$ and $\mathbb{N}$ denote the sets of nonnegative reals and integers, respectively, $I = [0,1]$ and $m$ stands for the Lebesgue measure on $I$. If $g$ is any function, $\operatorname{im} g$ stands for the image of $g$. If, in addition, $g$ takes values in a topological space, $\overline{\operatorname{im}} g$ denotes the closure of $\operatorname{im} g$ in the whole space. The weight of a topological space $X$ is denoted by $w(X)$ and is understood as an **infinite** cardinal number (i.e. $w(X) = \aleph_0$ for finite $X$). All topological spaces which appear in the paper are metrizable and all measures are nonnegative, finite and nonzero. For topological spaces $Y$ and $Z$ we shall write $Y \cong Z$ iff $Y$ and $Z$ are homeomorphic. By a *map* we mean a continuous function. If $X$ is a metrizable space, $X^\omega$ stands for the countable infinite Cartesian power of $X$, equipped with the Tichonov topology, and $\mathrm{Metr}(X)$ denotes the family of all bounded metrics on $X$ which induce the given topology of $X$. $\mathfrak{B}(X)$ stands for the $\sigma$-algebra of all Borel subsets of $X$, that is, $\mathfrak{B}(X)$ is the smallest $\sigma$-algebra containing all open subsets of $X$. If $(\Omega_1 \times \Omega_2, \mathfrak{M}, \mu)$ is the product space of measure spaces $(\Omega_1, \mathfrak{M}_1, \mu_1)$ and $(\Omega_2, \mathfrak{M}_2, \mu_2)$, then we shall write $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ and $\mu_1 \otimes \mu_2$ for $\mathfrak{M}$ and $\mu$, respectively.

Whenever $(\Omega, \mathfrak{M})$ is a measurable space and $X$ is a metrizable space, a function $f\colon \Omega \to X$ is $\mathfrak{M}$-*measurable*, if $f^{-1}(U) \in \mathfrak{M}$ for each open subset $U$ of $X$. Sets which are members of $\mathfrak{M}$ are said to be *measurable.* By a $\mu$-*partition* of $B \in \mathfrak{M}$ we mean any family $\{B_j\}_{j\in J}$ (with $J \subset \mathbb{N}$) of measurable pairwise disjoint sets such that $\mu(B_j) > 0$ for each $j \in J$ and $B = \bigcup_{j\in J} B_j$. If the images of $\mathfrak{M}$-measurable functions $f_j\colon \Omega \to X_j$, where $j \in J \subset \mathbb{N}$, are separable, then also the function $\Omega \ni \omega \mapsto (f_j(\omega))_{j\in J} \in \prod_{j\in J} X_j$ is $\mathfrak{M}$-measurable. Therefore, if $J = \{1, 2\}$ and $X_2 = X_1 = X$, the set $\{\omega \in \Omega\colon\ f_1(\omega) \neq f_2(\omega)\}$ is measurable.

We use standard terminology and ideas of measure theory. For details the Reader is referred e.g. to [12]. For example, every measurable function $f\colon \Omega \to X$ with separable image defined on a measure space $(\Omega, \mathfrak{M}, \mu)$ will be identified with its equivalence class (in the set of all measurable functions $\Omega \to X$ with separable images) with respect to the relation of almost everywhere equality mod $\mu$. The set of all such (equivalence classes of) functions is denoted by $M_\mu(X)$. The subfamilies of $M_\mu(X)$ consisting of all those functions whose images are, respectively, finite, (at most) countable and contained in $\sigma$-compact subsets of $X$ are denoted by $M_\mu^f(X)$, $M_\mu^c(X)$ and $M_\mu^r(X)$. We clearly have $M_\mu^f(X) \subset M_\mu^c(X) \subset M_\mu^r(X) \subset M_\mu(X)$. Each of the latter inclusions may be proper (the example for the last one is given in Section 3, see Example 3.5). If $A$ is a subset of $X$, we may and shall naturally identify the members of $M_\mu(A)$ with elements of $M_\mu(X)$. Thus, if $N$ stands for $M^f$, $M^c$, $M^r$ or $M$, then $N_\mu(A) \subset N_\mu(X)$. Analogously, if $\mathfrak{N}$ is a $\sigma$-subalgebra of $\mathfrak{M}$ and $\nu = \mu\big|_{\mathfrak{N}}$, then for $N = M, M^f, M^c, M^r$ the function $N_\nu(X) \ni f \mapsto f \in N_\mu(X)$ is well defined (and is isometric with respect to the metrics $M_\nu(d)$ and $M_\mu(d)$, defined in the sequel, for every $d \in \mathrm{Metr}(X)$). The Boolean $\sigma$-algebra (equipped with the metric induced by the measure) associated with a measure space $(\Omega, \mathfrak{M}, \mu)$ will be denoted by $\mathfrak{A}(\mu)$. The weight of $\mathfrak{A}(\mu)$ is called by us the *weight of* $\mu$ and is denoted by $w(\mu)$. We call the measure $\mu$ *simple* if $\mu(B) \in \{0, \mu(\Omega)\}$ for each $B \in \mathfrak{M}$ and $\mu$ is *nonatomic* if for every $B \in \mathfrak{M}$ of positive $\mu$-measure there is a subset $A \in \mathfrak{M}$ of $B$ with $0 < \mu(A) < \mu(B)$. Finally, $\mu$ is *homogeneous* if it is nonatomic and for each $B \in \mathfrak{M}$ of positive $\mu$-measure, $w(\mu) = w(\mu\big|_B)$, where $\mu\big|_B = \mu\big|_{\mathfrak{M}_B}$ is a measure on $B$ and $\mathfrak{M}_B = \{A \in \mathfrak{M}\colon\ A \subset B\}$.

From now on, we assume that $(\Omega, \mathfrak{M}, \mu)$ is a measure space with (nonzero) finite measure $\mu$ and that $X$ is a (nonempty) metrizable space. The space $M_\mu(X)$ and all its subsets will always be equipped with the topology of convergence in measure. In other words, a sequence $(f_n)_n$ of elements of $M_\mu(X)$ converges to $f \in M_\mu(X)$ iff every its subsequence contains a subsequence $(f_{\nu_n})_n$ such that $f_{\nu_n}(\omega) \to f(\omega)$ $(n \to \infty)$ for $\mu$-almost all $\omega \in \Omega$. It is well known that if $\varrho \in \operatorname{Metr}(X)$, then $M_\mu(\varrho) \in \operatorname{Metr}(M_\mu(X))$, where $M_\mu(\varrho)(f,g) = \int_\Omega \varrho(f(\omega), g(\omega))\, \mathrm{d}\mu(\omega)$.

It is clear that if $(X, \cdot)$ is a metrizable group, then $M_\mu(X)$ has a natural topological group structure (that is, with the pointwise multiplication) induced by the one of $X$.

For each $x \in X$ denote by $\delta_{\mu,x} \in M_\mu(X)$ the constant function with the only value equal to $x$ and let $\Delta_\mu(X) = \{\delta_{\mu,x}\colon\ x \in X\}$ and $\delta_{\mu,X}\colon X \ni x \mapsto \delta_{\mu,x} \in \Delta_\mu(X) \subset M_\mu(X)$.

The following are a kind of folklore. Most of them can easily be proved.

(M1) $\Delta_\mu(X)$ is closed in $M_\mu(X)$ and $\delta_{\mu,X}\colon (X,d) \to (\Delta_\mu(X), M_\mu(d))$ is an isometry for each $d \in \operatorname{Metr}(X)$. In particular, $\Delta_\mu(X) \cong X$. If $X$ is a group, $\delta_{\mu,X}$ is a homomorphism.

(M2) If $\mathfrak{N} \subset \mathfrak{M}$ is an algebra of subsets of $X$ which is dense in $\mathfrak{A}(\mu)$ and $D$ is a dense subset of $X$, then the set $M^f(\mathfrak{N}, D)$ consisting of such functions $f \in M^f_\mu(D)$ that $f^{-1}(\{x\}) \in \mathfrak{N}$ for each $x \in D$ is dense in $M_\mu(X)$. In particular, $w(M_\mu(X)) = \max(w(\mu), w(X))$.

(M3) If $d \in \operatorname{Metr}(X)$, then $M_\mu(d)$ is complete (in the whole space $M_\mu(X)$) iff $d$ is complete. The space $M_\mu(X)$ is completely metrizable iff $X$ is so. Moreover, if $\operatorname{card} X > 1$, then $M_\mu(X)$ is noncompact.

(M4) For each $A \subset X$, $\overline{M_\mu(A)} = M_\mu(\bar{A})$ (the first closure is in $M_\mu(X)$).

(M5) The measure $\mu$ is nonatomic iff there is a family $\{A_t\}_{t\in I}$ of measurable sets such that $A_s \subset A_t$ for $s \leqslant t$ and $\mu(A_t) = t\mu(\Omega)$.

(M6) If $\mu$ is nonatomic and $\{A_t\}_{t\in I}$ is a family as in (M5), then the map $\lambda\colon M_\mu(X) \times M_\mu(X) \times I \ni (f,g,t) \mapsto f\big|_{\Omega\setminus A_t} \cup g\big|_{A_t} \in M_\mu(X)$ is continuous. Moreover, $\lambda(f,g,0) = f$, $\lambda(f,g,1) = g$ and $\lambda(N_\mu(X) \times N_\mu(X) \times I) = N_\mu(X)$ for $N = M^f, M^c, M^r$. In particular, each of the spaces $N_\mu(X)$ with $N = M, M^f, M^c, M^r$ is contractible, provided $X$ is nonempty (in fact they are equiconnected).

(M7) If $\{A_j\}_{j\in J}$ ($J \subset \mathbb{N}$) is a $\mu$-partition of $\Omega$; $\lambda = \frac{\mu}{\mu(\Omega)}$ and $\lambda_j = \frac{\mu\big|_{A_j}}{\mu(A_j)}$, then the map $\Phi\colon (M_\lambda(X), M_\lambda(d)) \ni f \mapsto (f\big|_{A_j})_{j\in J} \in \left(\prod_{j\in J} M_{\lambda_j}(X), \tilde{d}\right)$ is an isometry, where $\tilde{d}((f_j)_{j\in J}, (g_j)_{j\in J}) = \sum_{j\in J} \mu(A_j) M_{\lambda_j}(d)(f_j, g_j)$ for $d \in \mathrm{Metr}(X)$. Moreovoer,

$$\Phi(N_\lambda(X)) = \prod_{j\in J} N_{\lambda_j}(X)$$

for $N = M^c, M^r$. In particular, $N_\lambda(X)$ is homeomorphic to $\prod_{j\in J} N_{\lambda_j}(X)$ for $N = M, M^c, M^r$.

(M8) Let $\{(X_j, d_j)\}_{j\in J}$ ($J \subset \mathbb{N}$) be a collection of metric spaces with metrics upper bounded by 1 and let $\{a_j\}_{j\in J}$ be a family of positive numbers such that $\sum_{j\in J} a_j < +\infty$. Let $X = \prod_{j\in J} X_j$ be a metric space with metric $d((x_j)_{j\in J}, (y_j)_{j\in J}) = \sum_{j\in J} a_j d_j(x_j, y_j)$. Analogously, let $D$ be the metric on $\prod_{j\in J} M_\mu(X_j)$ given by

$$D((f_j)_{j\in J}, (g_j)_{j\in J}) = \sum_{j\in J} a_j M_\mu(d_j)(f_j, g_j).$$

Then the map

$$\Psi\colon (M_\mu(X), M_\mu(d)) \ni F \mapsto (p_j \circ F)_{j\in J} \in (\prod_{j\in J} M_\mu(X_j), D),$$

where $p_j\colon X \to X_j$ is the natural projection, is an isometry. In particular, $M_\mu(\prod_{j\in J} X_j)$ is homeomorphic to $\prod_{j\in J} M_\mu(X_j)$. If $J$ is finite, then

$$\Psi(M_\mu^r(X)) = \prod_{j\in J} M_\mu^r(X_j) \tag{1-1}$$

and $\Psi(M_\mu^c(X)) = \prod_{j\in J} M_\mu^c(X_j)$.

(M9) There is a finite or countable collection $\{A_j\}_{j\in J} \cup \{B_k\}_{k\in K}$ (each of $J$ and $K$ may be empty) of measurable sets of positive $\mu$-measure such that $\mu\big|_{A_j}$ is simple for each $j \in J$, while the measures $\mu\big|_{B_k}$ with $k \in K$ are homogeneous and of different weights.

(M10) If $\mu$ is an atom, then $M_\mu(X) = M_\mu^f(X) = \Delta_\mu(X)$ and thus $M_\mu(X) \cong X$.

(M11) (Maharam[14]) If $(\Omega_j, \mathfrak{M}_j, \mu_j)$ ($j = 1, 2$) are probabilistic spaces such that both $\mu_1$ and $\mu_2$ are homogeneous and $w(\mu_1) = w(\mu_2)$, then the Boolean $\sigma$-algebras $\mathfrak{A}(\mu_1)$ and $\mathfrak{A}(\mu_2)$ are isometrically isomorphic.

The property (M1) says that $X$ may naturally be identified (via the map $\delta_{\mu,X}$) with $\Delta_\mu(X)$. The points (M7) and (M9)–(M11) imply that if $N = M, M^c$ or $M^r$, then $N_\mu(X) \cong X^p \times \prod_{j\in J} N_{\mu_j}(X)$, where $p = n \in \mathbb{N}$ if $\mu$ has exactly $n$ atoms and $p = \omega$ if $\mu$ has infinitely many atoms (if $p = 0$, we omit the factor $X^p$); and $J \subset \mathbb{N}$ (if $J$ is empty, we omit the factor

$\prod_{j\in J} N_{\mu_j}(X)$) and the measures $\mu_j$ are probabilistic homogeneous and of different weights. We shall prove in Section 2 that $M_\lambda^r(X)$ is *naturally* homeomorphic to $M_\nu^r(X)$ if $\lambda$ and $\nu$ are homogeneous and of the same weight. We shall also show that the connection (1-1) is fulfilled without assumption of finiteness of $J$.

Our next aim is to prove that if $(\Omega_j, \mathfrak{N}_j, \nu_j)$ for $j = 1, 2$ are two measure spaces, then there is a measure space $(\Omega, \mathfrak{N}, \nu)$ such that $M_{\nu_1}(M_{\nu_2}(X))$ is *naturally* homeomorphic to $M_\nu(X)$ for each metrizable space $X$. To do this, let $\Omega = \Omega_1 \times \Omega_2$ and $\pi\colon \Omega \to \Omega_2$ be the natural projection. Let $\mathfrak{N}$ be the $\sigma$-algebra of all subset $A$ of $\Omega$ such that $\pi(A \cap (\{\omega_1\} \times \Omega_2)) \in \mathfrak{N}_2$ for each $\omega_1 \in \Omega_1$ and the function $\Omega_1 \ni \omega_1 \mapsto \pi(A \cap (\{\omega_1\} \times \Omega_2)) \in \mathfrak{A}(\nu_2)$ is $\mathfrak{N}_1$-measurable and its image is separable. Finally, let $\nu\colon \mathfrak{N} \to \mathbb{R}_+$ be given by $\nu(A) = \int_{\Omega_1} \nu_2(\pi(A \cap (\{\omega_1\} \times \Omega_2)))\,\mathrm{d}\nu_1(\omega_1)$. It is easy to see that $\mathfrak{N}$ is indeed a $\sigma$-algebra and that $\nu$ is a finite measure on $\Omega$. Note also that $\mathfrak{N}_1 \otimes \mathfrak{N}_2 \subset \mathfrak{N}$ and $\nu$ extends $\nu_1 \otimes \nu_2$. We call $\nu$ the *directed product* of $\nu_1$ and $\nu_2$. It would be quite more reasonable to define $(\Omega, \mathfrak{N}, \nu)$ as the product space of $(\Omega_1, \mathfrak{N}_1, \nu_1)$ and $(\Omega_2, \mathfrak{N}_2, \nu_2)$. However, as we will see in Section 3 (Example 3.5), the product space (as $(\Omega, \mathfrak{N}, \nu)$ below) does not satisfy the following claim:

(M12) For every bounded metric space $(X, d)$ the map

$$\Lambda\colon (M_\nu(X), M_\nu(d)) \to (M_{\nu_1}(M_{\nu_2}(X)), M_{\nu_1}(M_{\nu_2}(d)))$$

given by the formula $(\Lambda f(\omega_1))(\omega_2) = f(\omega_1, \omega_2)$ is a well defined (bijective) isometry.

To show that $\operatorname{im}\Lambda \subset M_{\nu_1}(M_{\nu_2}(X))$, use the fact that if $f\colon \Omega \to X$ is $\mathfrak{N}$-measurable and $\operatorname{im} f$ is separable, then there is a sequence of $\mathfrak{N}$-measurable functions $f_n\colon \Omega \to X$ with finite images such that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ for each $\omega \in \Omega$. Further, direct calculation shows that $\Lambda$ is isometric. To see the surjectivity, fix an $\mathfrak{N}_1$-measurable function $g\colon \Omega_1 \to M_{\nu_2}(X)$ with separable image. Let $\bar{X}$ be the completion of $X$ with respect to $d$. Since $M_{\nu_2}^f(X)$ is dense in $M_{\nu_2}(X)$, there is a sequence of $\mathfrak{N}_1$-measurable functions $g_n\colon \Omega_1 \to M_{\nu_2}^f(X)$ with finite images such that $\lim_{n\to\infty} g_n(\omega_1) = g(\omega_1)$ for every $\omega_1 \in \Omega_1$. It is easy to check that for each $n$ there is an $\mathfrak{N}$-measurable function $f_n\colon \Omega \to X$ whose image is finite and such that $f_n(\omega_1, \cdot)$ and $g_n(\omega_1)$ concide in $M_{\nu_2}(X)$ for every $\omega_1 \in \Omega_1$ (in fact, each $f_n$ is $\mathfrak{N}_1 \otimes \mathfrak{N}_2$-measurable). Thus (since $\Lambda$ is isometric), $(f_n)_n$ is a fundamental sequence in $M_\nu(\bar{X})$. This means that there is an $\mathfrak{N}$-measurable function $f\colon \Omega \to \bar{X}$ with separable image which is the limit of $(f_n)_n$ in $M_\nu(\bar{X})$. We conclude from this that $\bar{\Lambda} f = g$, where $\bar{\Lambda}$ is the suitable map '$\Lambda$' for $\bar{X}$. So, the set $A^1 = \{\omega_1 \in \Omega_1\colon\ f(\omega_1, \cdot) \neq g(\omega_1) \text{ in } M_{\nu_2}(X)\}$ belongs to $\mathfrak{N}_1$ and $\nu_1(A^1) = 0$. Now fix $\omega_1 \in \Omega_1 \setminus A^1$. Let $h\colon \Omega_2 \to X$ be an $\mathfrak{N}_2$-measurable function with separable image which coincides with $g(\omega_1)$ in $M_{\nu_2}(\Omega_2)$. Then the set $A_{\omega_1} = \{\omega_2 \in \Omega_2\colon\ f(\omega_1, \omega_2) \neq h(\omega_2)\}$ belongs to $\mathfrak{N}_2$ and

$\nu_2(A_{\omega_1}) = 0$. Finally, put $A = (A^1 \times \Omega_2) \cup \bigcup_{\omega_1 \in \Omega_1 \setminus A^1} (\{\omega_1\} \times A_{\omega_1}) \subset \Omega$ and let $f_*\colon \Omega \to X$ be such that $f_*\big|_A = f\big|_A$ and $f_*\big|_{\Omega \setminus A} \equiv b$, where $b$ is a fixed element of $X$. By the construction, $A \in \mathfrak{N}$, $f_* \in M_\nu(X)$ and $\Lambda(f_*) = g$.

The above defined $\sigma$-algebra $\mathfrak{N}$ and measure $\nu$ will be denoted by us by $\mathfrak{N}_1 \overset{\to}{\otimes} \mathfrak{N}_2$ and $\nu_1 \overset{\to}{\otimes} \nu_2$, respectively. Since $\mathfrak{A}(\nu_1 \overset{\to}{\otimes} \nu_2)$ is naturally isometric to $M_{\nu_1 \overset{\to}{\otimes} \nu_2}(\{0,1\})$, the presented proof of (M12) (especially $\mathfrak{N}_1 \otimes \mathfrak{N}_2$-measurability of the functions $f_n$) yields that

(M13) For each $A \in \mathfrak{N}_1 \overset{\to}{\otimes} \mathfrak{N}_2$ there is $A_0 \in \mathfrak{N}_1 \otimes \mathfrak{N}_2$ such that $(\nu_1 \overset{\to}{\otimes} \nu_2)(A \setminus A_0) = (\nu_1 \overset{\to}{\otimes} \nu_2)(A_0 \setminus A) = 0$. In particular, $\mathfrak{A}(\nu_1 \overset{\to}{\otimes} \nu_2) = \mathfrak{A}(\nu_1 \otimes \nu_2)$ and if $\nu_1$ and $\nu_2$ are homogeneous, so is $\nu_1 \overset{\to}{\otimes} \nu_2$.

Now we shall give a sufficient condition (on a measure $\mu$) under which the space $Y = M_\mu(X)$ is homeomorphic to $Y^\omega$ (for each $X$). To formulate it, we need an additional notion. We say that two measure spaces $(\Omega_1, \mathfrak{M}_1, \mu_1)$ and $(\Omega_2, \mathfrak{M}_2, \mu_2)$ are *pointwisely isomorphic* if there is a bijection $\psi\colon \Omega_1 \to \Omega_2$ such that for any $A \subset \Omega_1$, $\psi(A) \in \mathfrak{M}_2$ iff $A \in \mathfrak{M}_1$ and $\mu_2(\psi(A)) = \mu_1(A)$ for every $A \in \mathfrak{M}_1$. In such a situation $\psi$ is called an *isomorphism.* These spaces are said to be *almost pointwisely isomorphic* if there are sets $A_1 \in \mathfrak{M}_1$ and $A_2 \in \mathfrak{M}_2$ such that $\mu_j(\Omega_j \setminus A_j) = 0$ $(j = 1, 2)$ and the spaces $(A_1, \mathfrak{M}_1\big|_{A_1}, \mu_1\big|_{A_1})$ and $(A_2, \mathfrak{M}_2\big|_{A_2}, \mu_2\big|_{A_2})$ are *pointwisely isomorphic.* Basicly, every isomorphism $\varphi\colon \Omega_1 \to \Omega_2$ induces isometries $(M_{\mu_1}(X), M_{\mu_1}(d)) \ni f \mapsto f \circ \varphi^{-1} \in (M_{\mu_2}(X), M_{\mu_2}(d))$ for any $X$ and $d \in \operatorname{Metr}(X)$ (the same for $M^f$,$M^c$ and $M^r$-spaces). We also have:

(M14) If there is a measurable set $A$ such that $0 < \mu(A) < \mu(\Omega)$ and the spaces $(\Omega, \mathfrak{M}, \frac{\mu}{\mu(\Omega)})$ and $(A, \mathfrak{M}\big|_A, \frac{\mu|_A}{\mu(A)})$ are almost pointwisely isomorphic, then $M_\mu(X) \cong M_\mu(X)^\omega$ for each metrizable space $X$.

To see this, first of all observe that there are measurable sets $\Omega_0$ and $A_0$ such that $A_0 \subset A \cap \Omega_0$, $\mu(\Omega \setminus \Omega_0) = \mu(A \setminus A_0) = 0$ and the spaces $(\Omega_0, \mathfrak{M}\big|_{\Omega_0}, \mu\big|_{\Omega_0})$ and $(A_0, \mathfrak{M}\big|_{A_0}, \mu\big|_{A_0})$ are pointwisely isomorphic. (Indeed, if $\tau\colon \Omega_1 \to A_1$ is an isomorphism, where $\Omega_1 \subset \Omega$ and $A_1 \subset A$ are measurable and $\mu(\Omega \setminus \Omega_1) = \mu(A \setminus A_1) = 0$, then for $n \geqslant 2$ put $A_n = A_{n-1} \cap \Omega_{n-1}$ and $\Omega_n = \tau^{-1}(A_n)$ and finally $A_0 = \bigcap_{n=1}^\infty A_n$ and $\Omega_0 = \bigcap_{n=1}^\infty \Omega_n$.) Since the maps $(M_\mu(X), M_\mu(d)) \ni f \mapsto f\big|_{\Omega_0} \in (M_{\mu|_{\Omega_0}}(X), M_{\mu|_{\Omega_0}}(d))$ and $(M_{\mu|_A}(X), M_{\mu|_A}(d)) \ni f \mapsto f\big|_{A_0} \in (M_{\mu|_{A_0}}(X), M_{\mu|_{A_0}}(d))$ are (bijective) isometries for every bounded metric space $(X, d)$, we may assume that $\Omega_0 = \Omega$ and $A_0 = A$. Let $\varphi\colon \Omega \to A$ be an isomorphism. For a moment we will think of $\varphi$ as of a function from $\Omega$ to $\Omega$. Let $B_0 = \Omega \setminus A$ and $B_n = \varphi^n(B_0)$ $(n \geqslant 1)$, where $\varphi^n$ denotes the $n$-th iterate of $\varphi$. Note that $\{B_n\}_{n=0}^\infty$ is a $\mu$-partition of $B = \bigcup_{n=0}^\infty B_n$. What is more, $\varphi(\Omega \setminus B) = \Omega \setminus B$. But $\frac{\mu(\Omega \setminus B)}{\mu(\Omega)} = \frac{\mu(\varphi(\Omega \setminus B))}{\mu(A)}$ and thus $\mu(\Omega \setminus B) = 0$. Therefore, as before, we may assume that

$B = \Omega$. Since $\varphi(B_n) = B_{n+1}$, all the spaces $(B_n, \mathfrak{M}\big|_{B_n}, \mu\big|_{B_n})$ are pointwisely isomorphic. Take a bijection $\kappa\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ and for each $n, l \in \mathbb{N}$ let $\psi_{l,n}\colon B_n \to B_{\kappa(l,n)}$ be an isomorphism. Finally, for a metrizable space $X$ put $h\colon M_\mu(X) \ni f \mapsto (\bigcup_{n=0}^{\infty}[f\big|_{B_{\kappa(l,n)}} \circ \psi_{l,n}])_{l=0}^{\infty} \in M_\mu(X)^\omega$. We leave this as a simple exercise that $h$ is a homeomorphism.

The point (M14) will be applied in Section 2. We shall end the section with the two more properties of spaces of measurable functions. Recall that a metrizable space $X$ has the *reflective isotopy property* (in short: RIP) if there is an ambient invertible isotopy $H\colon X \times X \times I \to X \times X$ such that $H(x, y, 0) = (x, y)$ and $H(x, y, 1) = (y, x)$ (that is, $H$ needs to be such a homotopy that for each $t \in I$, the map $h_t(x, y) = H(x, y, t)$ is a homeomorphism of $X \times X$ and the function $(x, y, t) \mapsto h_t^{-1}(x, y)$ is continuous) (compare [9, Definition IX.2.1]). There are other definitions of RIP (see [23],[22]), all 'invertible' versions of it are however equivalent for spaces $X$ such that $X \cong X^\omega$. (M6) implies that:

(M15) If $\mu$ is nonatomic, then the space $N_\mu(X)$ has RIP for each metrizable $X$ and $N = M, M^f, M^c, M^r$.

Indeed, if $\{A_t\}_{t\in I}$ is as in (M5), then the map

$$H(f, g, t) = (f\big|_{\Omega\setminus A_t} \cup g\big|_{A_t}, g\big|_{\Omega\setminus A_t} \cup f\big|_{A_t})$$

is an isotopy we searched for.

Following Toruńczyk[17], we say that a closed subset $K$ of a metrizable space $X$ is a *$Z$-set* if the set $\mathcal{C}(Q, X \setminus K)$, where $Q$ is the Hilbert cube, is dense in $\mathcal{C}(Q, X)$ in the topology of uniform convergence. (This definition differs from the original one by Anderson[1], but both these definitions are equivalent in ANR's.) Countable unions of $Z$-sets are called *$\sigma$-$Z$-sets*. The last property established in this section, which shall be used in Section 5, is

(M16) Let $\mu$ be nonatomic. If $X$ has more than one point, then $\Delta_\mu(X)$ is a $Z$-set in $M_\mu(X)$. If $X$ is infinite, the set $M_\mu^f(X)$ is a $\sigma$-$Z$-set in $M_\mu(X)$.

We shall prove only the second claim (the first one has similar proof). Let $\{A_t\}_{t\in I}$ be as in (M5). It is easy to see that for each $n$ the set of all measurable functions whose images have at most $n$ elements is closed in $M_\mu(X)$ and thus $M_\mu^f(X)$ is of type $\mathcal{F}_\sigma$. What is more, there is $u \in M_\mu(X)$ such that $u\big|_{A_t} \notin M^f_{\mu|_{A_t}}(X)$ for each $t \in I$. Now if $F\colon Q \to M_\mu(X)$ is continuous, then the maps $F_n\colon Q \ni x \mapsto u\big|_{A_{1/n}} \cup F(x)\big|_{\Omega\setminus A_{1/n}} \in M_\mu(X)$ converge uniformly to $F$ and have images disjoint from $M_\mu^f(X)$.

## 2. $M^r$-SPACES

At the beginning we shall study certain spaces of measurable functions.

Fix an infinite cardinal number $\alpha$. Each of the sets $I^J$, where $J$ is countable (infinite), will be equipped with the Tichonov topology. Let $T$ be a set of cardinality $\alpha$. Let $\Omega_\alpha = I^T$ $(= I^\alpha)$ and $\mathfrak{M}_\alpha$ be the $\sigma$-algebra of all subsets $B$ of $\Omega_\alpha$ for which there are a countable infinite set $J \subset T$ and $B_0 \in \mathfrak{B}(I^J)$ such that $B = \{(x_t)_{t\in T}\colon\ (x_j)_{j\in J} \in B_0\}$. In other words, $\mathfrak{M}_\alpha$ is the product of $\alpha$ copies of $\mathfrak{B}(I)$. (Note also that, when consider $\Omega_\alpha$ with the Tichonov topology, not every open subset of $\Omega_\alpha$ is a member of $\mathfrak{M}_\alpha$. Open sets which are measurable are exactly those which are $\mathcal{F}_\sigma$.) Finally, let $m_\alpha\colon \mathfrak{M}_\alpha \to I$ be the product measure of $\alpha$ copies of the Lebesgue measure $m$ on $I$. The following is well known:

(M17) The measure $m_\alpha$ is homogeneous and $w(m_\alpha) = \alpha$. The measure spaces $(I, \mathfrak{B}(I), m)$ and $(\Omega_{\aleph_0}, \mathfrak{M}_{\aleph_0}, m_{\aleph_0})$ are pointwisely isomorphic.

We need to know a little bit more about the space $(\Omega_\alpha, \mathfrak{M}_\alpha, m_\alpha)$. But first a few necessary definitions.

A *Polish* space is a separable completely metrizable one. A subset $B$ of a Polish space $Y$ is said to be *absolutely measurable in* $Y$ if for every probabilistic Borel measure $\mu$ on $Y$ there are two Borel subsets $A$ and $C$ of $Y$ such that $A \subset B \subset C$ and $\mu(C \setminus A) = 0$. A separable metrizable space $X$ is *absolutely measurable*, if for every embedding $\varphi$ of $X$ into the Hilbert cube $Q$, $\varphi(X)$ is absolutely measurable in $Q$. Equivalently, $X$ is absolutely metrizable if there is $d \in \mathrm{Metr}(X)$ such that $X$ is absolutely measurable in the completion of $(X, d)$.

A (separable) *Souslin space* is the empty space or a continuous image of the space of all irrational numbers; or, equivalently, it is a continuous image of some Polish space. The following are important for us properties of Souslin spaces:

(So1) the image of a Borel function defined on a Borel subset of a Polish space is a Souslin space,

(So2) every Souslin space is absolutely measurable (compare with [13, Theorem XIII.4.1]).

It is a kind of folklore that every finite Borel measure on a Polish space is regular, i.e. it is supported on a $\sigma$-compact subset of the whole space. This implies that every finite Borel measure on a (separable) absolutely measurable space is also supported on a $\sigma$-compact set.

All the above facts yield the following result.

2.1. **Lemma.** *Let $(\Omega, \mathfrak{M}, \mu)$ be a finite measure space and let $X$ be a metrizable space.*

(A) *If the image of an $\mathfrak{M}$-measurable function $f\colon \Omega \to X$ is contained in a separable absolutely measurable subset of $X$, then $f \in M^r_\mu(X)$.*

(B) *If $\mathfrak{N}$ is a $\sigma$-subalgebra of $\mathfrak{M}$, $\nu = \mu\big|_{\mathfrak{N}}$ and a function $f \in M^r_\mu(X)$ belongs to the closure of $M_\nu(X)$, then $f \in M^r_\nu(X)$, i.e. there is an $\mathfrak{N}$-measurable function $g\colon \Omega \to X$ whose image is separable and which is $\mu$-almost everywhere equal to $f$. In particular, $M^r_\nu(X)$ is closed in $M^r_\mu(X)$ and $M^r_\nu(X) = M^r_{\bar\nu}(X)$, where $\bar\nu = \mu\big|_{\bar{\mathfrak{N}}}$ and $\bar{\mathfrak{N}}$ consists of those $A \in \mathfrak{M}$ for which there is $B \in \mathfrak{N}$ with $\mu(A \setminus B) = \mu(B \setminus A) = 0$.*

*Proof.* (A): Let $A \subset X$ be a separable absolutely measurable superset of $\operatorname{im} f$. Let $\lambda\colon \mathfrak{B}(A) \ni B \mapsto \mu(f^{-1}(B)) \in \mathbb{R}_+$. Since $\lambda$ is a finite measure, there is a $\sigma$-compact subset $K$ of $A$ such that $\lambda(K) = \lambda(A)$. So, $\mu(U) = \mu(\Omega)$ for $U = f^{-1}(K)$. Then $f$ coincides with $\hat f \in M^r_\mu(X)$ in $M_\mu(X)$, where $\hat f\big|_U = f\big|_U$ and $\hat f\big|_{\Omega\setminus U} \equiv b$ with $b$ taken from $K$.

(B): We only need to prove the first claim. We may assume that the image of $f$ is contained in a $\sigma$-compact subset of $X$, say $K_0$. By the assumption, there is a sequence of $\mathfrak{N}$-measurable functions $f_n\colon \Omega \to X$ with finite images which is pointwisely convergent $\mu$-almost everywhere to $f$. Let $K = K_0 \cup \bigcup_n \operatorname{im} f_n$. Fix $d \in \operatorname{Metr}(K)$ and let $\bar K$ be the completion of $(K, d)$. Note that $K$ is $\sigma$-compact and therefore it is a Borel subset of $\bar K$. Let $B$ be the set of all those $\omega \in \Omega$ such that the sequence $(f_n(\omega))_n$ is convergent in $\bar K$. Since $f_n$'s are $\mathfrak{N}$-measurable, $B \in \mathfrak{N}$. What is more, $\mu(\Omega \setminus B) = 0$. Thus, after changing each $f_n$ so that $f_n\big|_{\Omega\setminus B} \equiv b$, where $b \in K$, there is an $\mathfrak{N}$-measurable function $\bar g\colon \Omega \to \bar K$ such that $\lim_{n\to\infty} f_n(\omega) = \bar g(\omega)$ for each $\omega$ and $\bar g$ is equal to $f$ in $M_\mu(\bar K)$. This yields that the set $C = \bar g^{-1}(K)$ belongs to $\mathfrak{N}$ and $\mu(\Omega \setminus C) = 0$. Therefore, to end the proof, it suffices to put $g = \bar g\big|_C$ and $g\big|_{\Omega\setminus C} \equiv b$. □

As we shall see in the next section (Example 3.5), all claims of the point (B) of the above lemma fail when we replace each $M^r$ by $M$.

For a metrizable space $X$ let $M(X) = M_m(X)$ and for an infinite cardinal $\alpha$, let $M_\alpha(X) = M_{m_\alpha}(X)$ (analogous notation for metrics). The second claim of (M17) yields that $M_{\aleph_0}(X) \cong M(X)$.

2.2. **Theorem.** *For every infinite cardinal number $\alpha$ and each metrizable space $X$, $M_\alpha(X) = M^r_{m_\alpha}(X)$.*

*Proof.* We assume that $\Omega_\alpha = I^T$. Let $u\colon \Omega_\alpha \to X$ be $\mathfrak{M}_\alpha$-measurable with separable image. Since $u$ is the pointwise limit of a sequence of $\mathfrak{M}_\alpha$-measurable functions with finite images, we conclude from this that there is a countable infinite set $J \subset T$ such that $u(x) = u(y)$ whenever $x$ and $y$ are elements of $\Omega_\alpha$ such that $p_J(x) = p_J(y)$, where $p_J\colon I^T \to I^J$ is the natural projection. This means that there is a Borel function $v\colon I^J \to X$ such that $u = v \circ p_J$. Let $S = \operatorname{im} v = \operatorname{im} u$. By

(So1) and (So2), $S$ is absolutely measurable and thus Lemma 2.1–(A) finishes the proof. $\square$

The argument used in the proof of the above theorem shows also that $M(X) = M_m^r(X)$.

Following Schori[16], we say that a space $Y$ is an *S-space* if there are an element $\theta \in Y$ and a map $f\colon Y \times I \to Y$ such that:

(S1) $f(x, 0) = \theta$, $f(x, 1) = x$, $f(\theta, t) = \theta$ for each $x \in Y$ and $t \in I$,
(S2) for every neighbourhood $U$ of $\theta$ in $Y$ there is $t \in (0, 1]$ such that $f(Y \times [0, t]) \subset U$,
(S3) the map $Y \times (0, 1] \ni (x, t) \mapsto (f(x, t), t) \in Y \times (0, 1]$ is an embedding,
(S4) $f(f(x, t), s) = f(x, ts)$ for each $t, s \in I$ and $x \in Y$.

2.3. **Theorem.** *For every infinite cardinal number $\alpha$ and each non-empty metrizable space $X$, $M_\alpha(X)$ is an $S$-space.*

*Proof.* As usual, we assume that $\Omega_\alpha = I^T$. Fix $\xi \in T$ and $a \in X$ and put $\Omega = I^{T\setminus\{\xi\}}$, $\theta = \delta_{m_\alpha,a}$ and $Y = M_{m_\alpha}(X)$. We shall identify $\Omega_\alpha$ with $\Omega \times I$. For every $t \in (0, 1]$ let $\kappa_t\colon \Omega \times [0, t] \ni (x, s) \mapsto (x, s/t) \in \Omega \times I$. Finally, let $f\colon Y \times I \ni (u, t) \mapsto (u \circ \kappa_t) \cup \theta\big|_{\Omega\times(t,1]} \in Y$. It is not too difficult to show that $f$ is continuous. What is more, $f$ satisfies the axioms (S1)–(S4), which finishes the proof. $\square$

It is easily seen that $\mu = m_\alpha$ satisfies the assumption of (M14) (for example, look at $\kappa_{1/2}$ defined in the foregoing proof). So, Theorem 2.2, Theorem 2.3, (M14), (M15) and the results of Schori[16] imply

2.4. **Corollary.** *Let $Y = M_\alpha(X)$. Then $Y \cong Y^\omega$ and for every metrizable manifold $U$ modelled on $Y$, $U \times Y$ is homeomorphic to $U$.*

Before we prove the main result of this section, let us show the following

2.5. **Proposition.** *Let $\mu$ be any measure (defined on a $\sigma$-algebra of subsets of $\Omega$) and $X_0, X_1, X_2, \dots$ be an infinite sequence of metrizable spaces. Let $J = \mathbb{N}$ and $X$ and $\Psi$ be as in* (M8). *Then $\Psi(M_\mu^r(X)) = \prod_{j\in J} M_\mu^r(X_j)$. In other words, for any sequence $(f_n)_{n=0}^\infty$ such that $f_n \in M_\mu^r(X_n)$ there is $g \in M_\mu^r(\prod_{n\in\mathbb{N}} X_n)$ such that $(f_n(\omega))_{n=0}^\infty = g(\omega)$ for $\mu$-almost all $\omega \in \Omega$.*

*Proof.* Let $f\colon \Omega \ni \omega \mapsto (f_n(\omega))_{n=0}^\infty \in \prod_{n\in\mathbb{N}} X_n$. Let $\bar{X}_n$ be the completion of $(X_n, d_n)$, where $d_n$ is a fixed metric on $X_n$. Let $K_n$ be a $\sigma$-compact subset of $X_n$ such that $\operatorname{im} f_n \subset K_n$. Then $K_n \in \mathfrak{B}(\bar{X}_n)$ and thus $K = \prod_{n=0}^\infty K_n \in \mathfrak{B}(\prod_{n=0}^\infty \bar{X}_n)$. So, $K$ is absolutely measurable and $\operatorname{im} f \subset K$. Now it remains to apply Lemma 2.1–(A). $\square$

And now the main result of the section.

2.6. **Theorem.** *Let $(\Omega_1, \mathfrak{M}_1, \mu_1)$ and $(\Omega_2, \mathfrak{M}_2, \mu_2)$ be two nonatomic measure spaces such that $\mathfrak{A}(\mu_1)$ and $\mathfrak{A}(\mu_2)$ are isometrically isomorphic. Let $\Phi\colon \mathfrak{A}(\mu_1) \to \mathfrak{A}(\mu_2)$ be an isometric isomorphism of Boolean algebras. Then for every metrizable $X$ there is a unique homeomorphism $H\colon M^r_{\mu_1}(X) \to M^r_{\mu_2}(X)$ such that for each function $f \in M^c_{\mu_1}(X)$ there is a function $g \in M^c_{\mu_2}(X)$ such that $g = H(f)$, $\operatorname{im} g = \operatorname{im} f$ and $g^{-1}(\{x\}) = \Phi(f^{-1}(\{x\}))$ in $\mathfrak{A}(\mu_2)$ for every $x \in X$. What is more, $H(\delta_{\mu_1,x}) = H(\delta_{\mu_2,x})$ for each $x \in X$; and for any $d \in \operatorname{Metr}(X)$, $H$ is an isometry with respect to the metrics $M_{\mu_1}(d)$ and $M_{\mu_2}(d)$.*

*Proof.* It is clear that the connections between $f \in M^c_{\mu_1}(X)$ and $g \in M^c_{\mu_2}(X)$ described in the statement of the theorem well (and uniquely) define $H$ on $M^c_{\mu_1}(X)$. Moreover, in this step $H$ is a bijection between $M^c$-spaces. It is also clear that $H$ is isometric with respect to the suitable metrics (described in the statement). Fix $d \in \operatorname{Metr}(X)$ and let $(\bar{X}, \bar{d})$ be the completion of $(X, d)$. Since the spaces $(M_{\mu_j}(\bar{X}), M_{\mu_j}(\bar{d}))$ $(j = 1, 2)$ are complete, there is a unique continuous extension

$$\bar{H}\colon M_{\mu_1}(\bar{X}) \to M_{\mu_2}(\bar{X}),$$

which is simultaneously a (bijective) isometry. It is enough to check that $\bar{H}(M^r_{\mu_1}(X)) \subset M^r_{\mu_2}(X)$ (because then we infer analogous inclusion for $\bar{H}^{-1}$). Take an $\mathfrak{M}_1$-measurable function $f\colon \Omega_1 \to X$ whose image is contained in a $\sigma$-compact subset of $X$. This implies that there is a $\mu_1$-partition $\{A_n\}_{n=1}^{\infty}$ of $\Omega_1$ such that $K_n = \overline{f(A_n)}$ (the closure taken in $X$) is compact for each $n \geqslant 1$. There is a $\mu_2$-partition $\{B_n\}_{n=1}^{\infty}$ of $\Omega_2$ such that $B_n = \Phi(A_n)$ in $\mathfrak{A}(\mu_2)$ for any $n$. For each $l \geqslant 1$ take a sequence $(f_n^{(l)}\colon A_l \to K_l)_{n=1}^{\infty}$ of $\mathfrak{M}_1$-measurable functions with finite images which converges pointwisely to $f\big|_{A_l}$. For every $n$ and $l$ let $\operatorname{im} f_n^{(l)} = \{x_1^{(l,n)}, \ldots, x_{p_{l,n}}^{(l,n)}\}$ and let $B_1^{(l,n)}, \ldots, B_{p_{l,n}}^{(l,n)}$ be a $\mu_2$-partition of $B_l$ such that $\Phi((f_n^{(l)})^{-1}(\{x_j^{(l,n)}\})) = B_j^{(l,n)}$ in $\mathfrak{A}(\mu_2)$. Define $g_n^{(l)}\colon B_l \to K_l$ in the following way: $g_n^{(l)}\big|_{B_j^{(l,n)}} \equiv x_j^{(l,n)}$. Of course $H(\bigcup_{l=1}^{\infty} f_n^{(l)}) = \bigcup_{l=1}^{\infty} g_n^{(l)}$ $(n \geqslant 1)$. So — since $f_n = \bigcup_{l=1}^{\infty} f_n^{(l)}$ tends to $f$ in $M_{\mu_1}(\bar{X})$ and $\bar{H}$ is isometric — $g_n = \bigcup_{l=1}^{\infty} g_n^{(l)}$ is a fundamental sequence in $M_{\mu_2}(\bar{X})$ and thus also the sequence $(g_n^{(l)})_n$ is fundamental in $M_{\mu_2|_{B_l}}(\bar{X})$. But $g_n^{(l)}$ is a member of $M_{\mu_2|_{B_l}}(K_l)$, which is closed in $M_{\mu_2|_{B_l}}(\bar{X})$. This implies that there is $g^{(l)} \in M_{\mu_2|_{B_l}}(K_l)$ which is the limit of $(g_n^{(l)})_n$. Then the function $g = \bigcup_{l=1}^{\infty} g^{(l)}$ is the limit of $(g_n)_n$ in $M_{\mu_2}(\bar{X})$. Finally we conclude that $g \in M^r_{\mu_2}(X)$ and $H(f) = g$. □

We shall denote the unique homeomorphism $H$ corresponding to an isometric isomorphism $\Phi$ between Boolean measure algebras, described in Theorem 2.6, by $\widehat{\Phi}$.

The above result and (M11) give

2.7. **Corollary.** *If $\mu$ is homogeneous and of weight $\alpha$, then $M_\mu^r(X) \cong M_\alpha(X)$.*

The note following (M11) combined with the results of this section leads us to

2.8. **Theorem.** *Let $\mu$ be nonatomic and let $Y = M_\mu^r(X)$. Then there is a finite or countable collection $\{\alpha_j\}_{j\in J}$ of different infinite cardinals such that $Y \cong \prod_{j\in J} M_\alpha(X)$. In particular, $Y^\omega \cong Y$, $Y$ has RIP and is an S-space and therefore every metrizable manifold $U$ modelled on $Y$ is $Y$-stable.*

## 3. AM-class

3.1. **Definition.** A metrizable space is said to be an *AM-space* [a *So-space*] if every its closed separable subset is absolutely measurable [a Souslin space].

Every So-space is an AM-space and all locally absolutely Borel spaces (in particular, completely metrizable spaces) are So-spaces. It is also well known that finite or countable Cartesian products of AM-spaces [So-spaces] are AM-spaces [So-spaces] as well.

AM-spaces may be characterized as follows:

3.2. **Proposition.** *For a metrizable space $X$ the following conditions are equivalent:*

(i) *$X$ is an AM-space,*
(ii) *$M_\mu^r(X) = M_\mu(X)$ for every finite measure space $(\Omega, \mathfrak{M}, \mu)$,*
(iii) *$M_\nu^r(X) = M_\nu(X)$ for any separable metric space $Y$ and each probabilistic nonatomic measure $\nu$ defined on $\mathfrak{B}(Y)$.*

*Proof.* Thanks to Lemma 2.1–(A), we only need to prove the implication (iii) $\Longrightarrow$ (i). Let $X$ satisfies the claim of (iii) and let $A$ be a separable closed subset of $X$. Fix $d \in \mathrm{Metr}(A)$ and denote by $\hat{A}$ the completion of $(A, d)$. Let $\mu$ be a finite Borel measure on $\hat{A}$. We may assume that $\mu$ is nonatomic. Put $\nu\colon \mathfrak{B}(A) \ni B \mapsto \inf\{\mu(C)\colon\ C \in \mathfrak{B}(\hat{A}), B \subset C\} \in \mathbb{R}_+$. It is well known that $\nu$ is a measure. By (iii), there is a Borel function $f\colon A \to X$ whose image is contained in a $\sigma$-compact subset of $X$ and such that $f(a) = a$ for $\nu$-almost all $a \in A$. Since $A$ is closed in $X$, we may assume that $\operatorname{im} f$ is contained in a $\sigma$-compact subset of $A$, say $K$. Then $K \in \mathfrak{B}(\hat{A})$ and $\nu(A \setminus K) = 0$. Clearly, there is $B \in \mathfrak{B}(\hat{A})$ such that $A \subset B$ and $\nu(A) = \mu(B)$. Then $K \subset A \subset B$ and $\mu(B \setminus K) = 0$, which finishes the proof. □

As an application of the above characterization, thanks to (M12), (M13) and the results of Section 2, we obtain

3.3. **Theorem.** *Let $X$ be an AM-space and $d \in \mathrm{Metr}(X)$.*

(A) *For any finite measure spaces* $(\Omega_1,\mathfrak{M}_1,\mu_1)$ *and* $(\Omega_2,\mathfrak{M}_2,\mu_2)$ *the map*

$$\Lambda\colon (M_{\mu_1\otimes\mu_2}(X), M_{\mu_1\otimes\mu_2}(d)) \to (M_{\mu_1}(M_{\mu_2}(X)), M_{\mu_1}(M_{\mu_2}(d)))$$

*given by* $(\Lambda f(\omega_1))(\omega_2) = f(\omega_1,\omega_2)$ *is a (bijective) isometry.*

(B) *For every two infinite cardinal numbers* $\alpha$ *and* $\beta$, $M_\alpha(M_\beta(X)) \cong M_\gamma(X)$, *where* $\gamma = \max(\alpha,\beta)$. *In particular,* $M(M(X)) \cong M(X)$.

(C) *If* $(\Omega,\mathfrak{M},\mu)$ *is a finite measure space,* $\mathfrak{N}$ *is a* $\sigma$*-subalgebra of* $\mathfrak{M}$ *and* $\nu = \mu\big|_{\mathfrak{N}}$, *then* $M_\nu(X)$ *is closed in* $M_\mu(X)$.

(D) *If* $\mu$ *is a finite nonatomic measure and* $Y = M_\mu(X)$, *then* $Y \cong Y^\omega$, $Y$ *has RIP and is an S-space.*

It turns out that the classes of AM-spaces and of So-spaces are invariant under the operators $M_\mu$, as it is shown in the following

3.4. **Theorem.** *If* $X$ *is an AM-space [a So-space], then* $M_\mu(X)$ *is an AM-space [a So-space] as well for every finite measure space* $(\Omega,\mathfrak{M},\mu)$.

*Proof.* Take a separable and closed subset $Y$ of $M_\mu(X)$. Let $\{f_n\}_{n=1}^\infty$ be a dense subset of $Y$. Put $A = \overline{\bigcup_{n=1}^\infty \operatorname{im} f_n}$ (the closure taken in $X$) and let $\mathfrak{N}$ be the smallest $\sigma$-subalgebra of $\mathfrak{M}$ such that each of the functions $f_n$ is $\mathfrak{N}$-measurable. Then $A$ is separable and $\mathfrak{N}$ is a countably generated $\sigma$-algebra. This means that $\mathfrak{A}(\nu)$ is separable, where $\nu = \mu\big|_{\mathfrak{N}}$. Therefore $M_\nu(A)$ is separable as well. What is more, by Theorem 3.3–(C), the space $M_\nu(A)$ is closed in $M_\mu(X)$ and thus $Y \subset M_\nu(X)$. Since the classes of AM-spaces and So-spaces are closed hereditary, it suffices to show that $M_\nu(X)$ is an AM-space [a So-space] if so is $X$. Further, thanks to the note following (M11), we may assume that $\nu$ is nonatomic. But then (see Proposition 3.2 and Corollary 2.7) $M_\nu(X) \cong M(X)$. So, we have reduced the proof to showing that $M(X)$ is an AM-space [a So-space], provided $X$ is so and $X$ is separable. First we shall show this for the So-class.

Suppose $X$ is a separable nonempty Souslin space. Then there is a continuous surjection $g\colon \mathbb{R}\setminus\mathbb{Q} \to X$. Put $M(g)\colon M(\mathbb{R}\setminus\mathbb{Q}) \ni f \mapsto g\circ f \in M(X)$ (see the last section). By [15, Theorem 3.3], $M(g)$ is a continuous surjection. So, by the complete metrizability and the separability of $M(\mathbb{R}\setminus\mathbb{Q})$, $M(X)$ is indeed a Souslin space.

Now assume that $X$ is a separable absolutely measurable space. Let $S$ be a separable metrizable space and let $\lambda$ be a probabilistic Borel nonatomic measure on $S$. It is enough to prove that $M_\lambda(M(X)) = M_\lambda^r(M(X))$. Let $u \in M_\lambda(M(X))$. By Theorem 3.3–(A), there is a Borel function $v\colon S\times I \to X$ such that $u(s)$ and $v(s,\cdot)$ coincide in $M(X)$ for $\lambda$-almost all $s \in S$. Since $X$ is absolutely measurable, there is a Borel function $w\colon S\times I \to X$ whose image is contained in a $\sigma$-compact subset of $X$ (say $K$) and such that $v$ and $w$ coincide in $M_{\lambda\otimes m}(X)$. Put $\widetilde{u}\colon S \ni s \mapsto w(s,\cdot) \in M(K) \subset M(X)$. Then $\widetilde{u}$ and $u$ represent the same element of $M_\lambda(M(X))$. What is more, $\widetilde{u} \in M_\lambda(M(K))$ and $M(K)$

is a Souslin space, which yields that $\widetilde{u} \in M^r_\lambda(M(K)) \subset M^r_\lambda(M(X))$. This finishes the proof. $\square$

We end the section with the following

3.5. **Example.** It is well known that there exists a subset $X$ of the square $I^2$ which is not Lebesgue measurable, but for each $t \in I$ the set $X_t = \{s \in I\colon\ (t,s) \in X\}$ is a Borel subset of $I$ and $m(X_t) = 1$. This implies that $X \in \mathfrak{B}(I) \overset{\rightarrow}{\otimes} \mathfrak{B}(I)$. So, the map $f\colon I^2 \to X$ which is the identity on $X$ and constant on its complement is $\mathfrak{B}(I) \overset{\rightarrow}{\otimes} \mathfrak{B}(I)$-measurable. However, since $X$ is nonmeasurable, there is no $g \in M_{m\otimes m}(X)$ which coincides with $f$ in $M_{m\overset{\rightarrow}{\otimes}m}(X)$; and $f \notin M^r_{m\overset{\rightarrow}{\otimes}m}(X)$. Thus we have obtained that $M^r_{m\overset{\rightarrow}{\otimes}m}(X) \subsetneq M_{m\overset{\rightarrow}{\otimes}m}(X)$ and $M_{m\otimes m}(X) \subsetneq M_{m\overset{\rightarrow}{\otimes}m}(X)$ as well. The example shows that (M12) is in general not true if we put there $\nu = \nu_1 \otimes \nu_2$. It also shows that if $\nu$ is the restriction of $\mu$ to a dense (in $\mathfrak{A}(\mu)$) $\sigma$-subalgebra, then $M_\nu(X)$, in spite of its density in $M_\mu(X)$, may differ from $M_\mu(X)$.

We do not know whether $M(M(X)) \cong M(X)$ if $X$ is as in Example 3.5.

## 4. Main results

In this section we shall show that all considered by us spaces of measurable functions with respect to nonatomic measures are absolute retracts. In our proof we shall use the following three results:

4.1. **Lemma** ([8, Theorem 3.1])**.** *Every metrizable space admits an embedding into the unit sphere of a Hilbert space whose image is linearly independent.*

4.2. **Lemma** ([8, the proof of Lemma 4.3])**.** *Let $T$ be a finite linearly independent subset of the unit sphere of a Hilbert space $(\mathcal{H}, \langle\cdot,-\rangle)$. Let $K$ be the convex hull of $T$ (in $\mathcal{H}$) and let $D = M(K)$ be equipped with the topology $\tau_w$ induced by the weak one of $L^2[\operatorname{lin} T]$ $(= L^2(m, \operatorname{lin} T))$, i.e. a sequence $(f_n)_n$ of members of $D$ converges to $f \in D$ iff*

$$\int_0^1 \langle f_n(t), g(t)\rangle \,\mathrm{d}t \to \int_0^1 \langle f(t), g(t)\rangle \,\mathrm{d}t \ (n \to \infty)$$

*for each $g \in D$. Then:*

(BP1) *$(D, \tau_w)$ is metrizable compact and convex,*
(BP2) *the inclusion map of $M(T)$ into $D$ is an embedding, when $M(T)$ is equipped with the topology of convergence in measure,*
(BP3) *there is a sequence of maps from $D$ into $D$ whose images are contained in $M(T)$ which is uniformly convergent to the identity map on $D$.*

4.3. **Theorem** ([18, Theorem 1.1])**.** *If a metrizable space $X$ has a basis (consisting of open sets) such that all finite intersections of its members are homotopically trivial, then $X$ is an ANR.*

Recall that a topological space $X$ is *homotopically trivial* iff for every $n \geqslant 1$ each map of $\partial I^n$ into $X$ is extendable to a map of $I^n$ into $X$. Note also that the empty space is homotopically trivial.

Following [6],[4], a subset $A$ of a space $X$ is said to be *homotopy dense* (in $X$) if there is a homotopy $H\colon X \times [0,1] \to X$ such that $H(x,0) = x$ for each $x \in X$ and $H(X \times (0,1]) \subset A$. If $X$ is an ANR, then $A$ is homotopy dense in $X$ iff $X \setminus A$ is locally homotopy negligible in $X$ ([18]). The main result of the paper has the following form:

4.4. **Theorem.** *Let $(\Omega, \mathfrak{M}, \mu)$ be a finite nonatomic measure space, $X$ a nonempty metrizable space and $A$ its dense subset. Then the space $M_\mu(X)$ is an AR and $M_\mu^f(A)$ is homotopy dense in $M_\mu(X)$.*

The proof of the above theorem is divided into a few lemmas. Let us fix a finite nonatomic measure space $(\Omega, \mathfrak{M}, \mu)$, a nonempty metrizable space $X$ and its dense subset $A$. By Lemma 4.1, we may assume that $X$ is a linearly independent subset of the unit sphere $\mathcal{S}$ of a Hilbert space $(\mathcal{H}, \langle \cdot, - \rangle)$. For each bounded subset $E$ of $\mathcal{H}$, the topology of convergence in measure in $M_\mu(E)$ coincides with the topology induced by the metric $\varrho_E(u,v) = (\int_\Omega \|u(\omega) - v(\omega)\|^2 \,\mathrm{d}\mu(\omega))^{1/2}$ $(u, v \in M_\mu(E))$. For each $Y \subset X$, we shall denote by $B_{\varrho_Y}(u,r)$ the open ball in $(M_\mu(Y), \varrho_Y)$ with center at $u \in M_\mu(Y)$ and of radius $r > 0$. Our purpose is to prove that if $u_1, \ldots, u_p \in M_\mu(X)$ and $r_1, \ldots, r_p > 0$, then $\bigcap_{j=1}^p B_{\varrho_X}(u_j, r_j)$ is homotopically trivial. First we shall show a special case of this.

4.5. **Lemma.** *Let $T$ be a finite subset of $X$. Then for every $u_1, \ldots, u_p \in M_\mu(T)$ and each $r_1, \ldots, r_p > 0$, the set $G = \bigcap_{j=1}^p B_{\varrho_T}(u_j, r_j)$ is homotopically trivial.*

*Proof.* Fix $k \geqslant 1$ and take a map $f\colon \partial(I^k) \to G$. Let $\{A_t\}_{t \in I}$ be as in (M5) and let $E$ be an at most countable dense subset of $\operatorname{im} f$. There is a countably generated $\sigma$-subalgebra $\mathfrak{N}$ of $\mathfrak{M}$ such that each member of $E$ is $\mathfrak{N}$-measurable and $A_q \in \mathfrak{N}$ for $q \in \mathbb{Q} \cap I$. Put $\nu = \mu\big|_{\mathfrak{N}}$. Then $\nu$ is nonatomic and $\mathfrak{A}(\nu)$ is separable. What is more, since $T$ is clearly an AM-space, $M_\nu(T)$ is closed in $M_\mu(T)$ (Theorem 3.3–(C)). This implies that $\operatorname{im} f \subset M_\nu(T)$. Now by Corollary 2.7, $M_\nu(T) \cong M(T)$. Note also that the homeomorphism $H$ (between $M_\nu(T)$ and $M(T)$) appearing in the statement of Theorem 2.6 is an isometry with respect to the metrics $\varrho_T\big|_{M_\nu(T) \times M_\nu(T)}$ and $d_T\colon M(T) \times M(T) \ni (u,v) \mapsto (\int_0^1 \|u(t) - v(t)\|^2 \,\mathrm{d}t)^{1/2} \in \mathbb{R}_+$. So, the inverse image of $G$ under $H$ coincides with the finite intersection of open $d_T$-balls in $M(T)$. This reduces the problem to the case when $(\Omega, \mathfrak{M}, \mu) = (I, \mathfrak{B}(I), m)$, which we now assume. Let $V = \operatorname{lin} T \subset \mathcal{H}$. Following Bessaga and Pełczyński[8], consider the Hilbert space $L^2[V]$ of all (equivalence classes of) Borel functions $w\colon I \to V$ such that $\int_0^1 \|w(t)\|^2 \,\mathrm{d}t < +\infty$ with the scalar product $\langle u, v\rangle_V = \int_0^1 \langle u(t), v(t)\rangle \,\mathrm{d}t$. Put $\delta_j = 1 - \frac{r_j^2}{2}$ and $U_j = \{g \in L^2[V]\colon\ \langle g, u_j\rangle_V > \delta_j\}$ $(j = 1, \dots, p)$. It is easily seen that each $U_j$ is convex and open in the weak topology of $L^2[V]$. Let $K$ and $(D, \tau_w)$ be as in the statement of Lemma 4.2. By (BP2), the topology of $M(T)$ coincides with the one induced by $\tau_w$ and therefore to the end of the proof we shall deal only with the topology $\tau_w$. Put $U = D \cap \bigcap_{j=1}^p U_j$. Note that $U$ is open in $(D, \tau_w)$ and $U$ is convex. What is more, since $T \subset \mathcal{S}$, $M(T)$ is contained in the unit sphere of $L^2[V]$ and therefore

$$U \cap M(T) = G. \tag{4-1}$$

This implies that $f\colon \partial(I^k) \to U$. Since $U$ is convex, there exists a continuous extension $\hat{f}\colon I^k \to U$ of $f$. Further, applying Lemma 4.2, take a sequence of maps $\varphi_n\colon D \to D$ which is uniformly convergent to the identity map on $D$ and such that $\operatorname{im}\varphi_n \subset M(T)$. Then the sequence $f_n = \varphi_n \circ \hat{f}\colon I^k \to D$ is uniformly convergent to $\hat{f}$. This yields that for infinitely many $n$ we have $\operatorname{im} f_n \subset U$ and thus we may assume that the latter inclusion is satisfied for each $n$. But $\operatorname{im} f_n \subset \operatorname{im}\varphi_n \subset M(T)$, which combined with (4-1) gives $\operatorname{im} f_n \subset G$. Again by (BP2), the sequence $(f_n\big|_{\partial(I^k)}\colon \partial(I^k) \to G)_n$ tends uniformly to $f$ (with respect to the topology of $M(T)$). Finally, since $G$ is open in $M(T)$ and thanks to (M6), $G$ is locally equiconnected, which implies that for some $n$, $f_n\big|_{\partial(I^k)}$ and $f$ are homotopic in $G$. So, the homotopy extension property finishes the proof. □

The main result (Theorem 4.4) is an easy consequence of Theorem 4.3 and the following

4.6. **Lemma.** *If $u_1, \dots, u_p \in M_\mu(X)$ and $r_1, \dots, r_p > 0$, then the set $W = \bigcap_{j=1}^p B_{\varrho_X}(u_j, r_j)$ is homotopically trivial.*

*Proof.* We may assume that $\mu$ is probabilistic. For each $k \geqslant 1$ let $\Delta_k = \{(t_0, \dots, t_k) \in I^{k+1}\colon\ \sum_{j=0}^k t_j = 1\}$ be the $k$-dimensional simplex and let $\partial(\Delta_k) = \{x \in \Delta_k\colon\ x_j = 0 \text{ for some } j\}$ be its combinatorial boundary. It is enough to prove that each map of $\partial(\Delta_k)$ into $W$ is extendable to a map of $\Delta_k$ into $W$. Fix a map $f\colon \partial(\Delta_k) \to W$. Since $W$ is open, $\operatorname{im} f$ is compact and $M_\mu^f(A)$ is dense in $M_\mu(X)$, there are functions $u_1^*, \dots, u_p^* \in M_\mu^f(A)$ and numbers $r_1^*, \dots, r_p^* > 0$ such that $\operatorname{im} f \subset W^* \subset W$, where $W^* = \bigcap_{j=1}^p B_{\varrho_X}(u_j^*, r_j^*)$. Thus we may and shall assume that

$$u_1, \dots, u_p \in M_\mu^f(A). \tag{4-2}$$

As in the proof of Lemma 4.5, take a family $\{A_t\}_{t\in I}$ satisfying the claim of (M5). One may show that for each $n \geqslant 1$ the function

$\lambda_n\colon M_\mu^f(A)^{n+1}\times\Delta_n\to M_\mu^f(A)$ given by

$$\lambda_n(v_0,\dots,v_n;t_0,\dots,t_n)=\bigcup_{j=0}^n v_j\big|_{A_{t_j}\setminus A_{t_{j-1}}}$$

(with $t_{-1}=0$) is continuous. Further, take a positive number $\varepsilon$ such that

$$\bigcup_{x\in\partial(\Delta_k)} B_{\varrho_X}(f(x),\varepsilon)\subset W \tag{4-3}$$

and put $\delta=\frac{\varepsilon}{3\sqrt{k}}$. Fix $l\geqslant 1$. Let $\mathcal{K}_0$ be the collection of all faces of $\Delta_k$ and for each $n\geqslant 1$ let $\mathcal{K}_n$ be the collection of all geometric simplices obtained by the barycentric divisions of all members of $\mathcal{K}_{n-1}$. There is $N\geqslant 1$ such that $\operatorname{diam}_{\varrho_X} f(\sigma)\leqslant\frac{\delta}{l}$ for every $\sigma\in\mathcal{K}_N$. Now for each vertex $x$ of any member of $\mathcal{K}=\mathcal{K}_N$ take $v_x\in M_\mu^f(A)$ such that $\varrho_X(f(x),v_x)\leqslant\delta/l$. Let '$\preccurlyeq$' be a total order on the set of all vertices of all members of $\mathcal{K}$. Take any $\sigma\in\mathcal{K}$ and assume that $x_0\prec\ldots\prec x_k$ are vertices of $\sigma$. We define $g_\sigma\colon\sigma\to M_\mu^f(A)$ by $g_\sigma(\sum_{j=0}^k t_jx_j)=\lambda_k(v_{x_0},\dots,v_{x_k};t_0,\dots,t_k)$ $((t_0,\dots,t_k)\in\Delta_k)$. Since $x_0,\dots,x_k$ are linearly independent, $g_\sigma$ is continuous. What is more, if also $\sigma'\in\mathcal{K}$, then $g_\sigma$ and $g_{\sigma'}$ coincide on $\sigma\cap\sigma'$. This yields that the union $g_l$ of all $g_\sigma$'s is a well defined continuous function from $\partial(\Delta_k)$ into $M_\mu^f(A)$. And, what is important, there is a finite subset $T_l$ of $A$ such that $\operatorname{im} g_l\subset M_\mu(T_l)$. Moreover, if $x\in\sigma$, where $\sigma\in\mathcal{K}$ has vertices $x_0\prec\ldots\prec x_k$, then

$$\begin{aligned}\varrho_X(g_l(x),f(x))^2&\leqslant\sum_{j=0}^k\varrho_X(v_{x_j},f(x))^2\\&\leqslant 2\sum_{j=0}^k\Big(\varrho_X(v_{x_j},f(x_j))^2+\varrho_X(f(x_j),f(x))^2\Big)\leqslant 4k\frac{\delta^2}{l^2}<\frac{\varepsilon^2}{l^2}\end{aligned}\tag{4-4}$$

and thus, by (4-3), $\operatorname{im} g_l\subset W$.

Now for $t\in(l,l+1)$ put $g_t\colon\partial(\Delta_k)\ni x\mapsto\lambda_1(g_l(x),g_{l+1}(x);l+1-t,t-l)\in M_\mu^f(A)$. It is clear that $(g_t)_{t\geqslant 1}$ is a homotopy. Furthermore, one checks, using (4-4), that for each $t\in[l,l+1]$ with $l\geqslant 2$ and every $x\in\partial(\Delta_k)$, $\varrho_X(g_t(x),f(x))^2<\frac{2\varepsilon^2}{l^2}\leqslant\varepsilon^2$. So, $\operatorname{im} g_t\subset W$ for $t\geqslant 2$ and the function $h\colon\partial(\Delta_k)\times[2,\infty]\to W$ given by $h(x,t)=g_t(x)$ for $t<+\infty$ and $h(x,\infty)=f(x)$ is a homotopy connecting $g_2$ and $f$. By the homotopy extension property, it suffices to show that $g_2$ is extendable to a map of $\Delta_k$ into $W$. To this end, put $T=T_2\cup\bigcup_{j=1}^p\operatorname{im} u_j\subset A$. By (4-2), $T$ is finite. What is more, $u_1,\dots,u_p\in M_\mu(T)$. Finally, $W\cap M_\mu(T)=\bigcap_{j=1}^p B_{\varrho_T}(u_j,r_j)$ and $g_2\colon\partial(\Delta_k)\to W\cap M_\mu(T)$. So, by Lemma 4.5, $g_2$ admits a continuous extension of $\Delta_k$ into $W\cap M_\mu(T)$, which finishes the proof. □

*Proof of Theorem* 4.4. Let $X$ be embedded as a linearly independent subset of the unit sphere of a Hilbert space. By Theorem 4.3 and Lemma 4.6, $M_\mu(X)$ is a homotopically trivial ANR. This yields that it is an AR. Finally, the last paragraph of the proof of Lemma 4.6 shows that for every open ball $B$ in $M_\mu(X)$ (with respect to the metric $\varrho_X$) the inclusion map $B \cap M_\mu^f(A) \to B$ is a (weak) homotopy equivalence and hence $M_\mu(X) \setminus M_\mu^f(A)$ is locally homotopy negligible in $M_\mu(X)$ ([18]). □

4.7. **Corollary.** *If $\mu$ is a finite nonatomic measure and $X$ is a non-empty metrizable space, then the spaces $M_\mu^f(X)$, $M_\mu^c(X)$, $M_\mu^r(X)$ and $M_\mu(X)$ are AR's.*

4.8. *Remark.* If $\mathfrak{N}$, $A$ and $M^f(\mathfrak{N}, A)$ are as in (M2) and additionally $\mathfrak{N}$ contains a subfamily $\{A_t\}_{t\in I}$ as in (M5), the proof of Lemma 4.6 shows that $M^f(\mathfrak{N}, A)$ is homotopy dense in $M_\mu(X)$ and thus it is an AR. This implies that the space $M^s(X) \subset M(X)$ of all piecewise constant functions is an AR.

As a first consequence of Theorem 4.4 we obtain a generalization of theorems of Bessaga and Pełczyński[8] and of Toruńczyk[19]:

4.9. **Theorem.** *If $\mu$ is a finite nonatomic (nonzero) measure and $X$ is a completely metrizable space which has more than one point, then $M_\mu(X)$ is homeomorphic to an infinite-dimensional Hilbert space of dimension $\alpha = \max(w(\mu), w(X))$.*

*Proof.* Put $Y = M_\mu(X)$. By Theorem 3.3–(D), $Y^\omega \cong Y$. But $Y$ is a noncompact AR and thus, by [20, Theorem 5.1], $Y$ is homeomorphic to a Hilbert space of dimension $w(Y)$. So, the observation that $w(Y) = \alpha$ finishes the proof. □

Now repeating the proofs (with $M_G$ replaced by $M_\alpha(G)$) of Theorem 5.1 and Corollary 5.2 of [8] we get

4.10. **Corollary.** *Let $\mathcal{H}$ be a Hilbert space of dimension $\alpha \geqslant \aleph_0$ and let $G$ be a completely metrizable topological group of weight no greater than $\alpha$. Then $G$ is (algebraically and topologically) isomorphic to a closed subgroup of a group homeomorphic to $\mathcal{H}$ and $G$ admits a free action in $\mathcal{H}$.*

## 5. Extending maps

We begin this section with

5.1. **Definition.** Let $\mu$ be a finite measure and let $f\colon X \to Y$ be a map. Let

$$M_\mu(f)\colon M_\mu(X) \ni u \mapsto f \circ u \in M_\mu(Y).$$

$M_\mu(f)$ is said to be the *$\mu$-extension of $f$*. Additionally, let $M(f) = M_m(f)$ and $M_\alpha(f) = M_{m_\alpha}(f)$ for every infinite cardinal $\alpha$.

Note that $M_\mu(f)$ is continuous and that $M_\mu(f)(N_\mu(X)) \subset N_\mu(Y)$ for $N = M^f, M^c, M^r$. The connection

$$M_\mu(f)(\delta_{\mu,X}(x)) = \delta_{\mu,Y}(f(x)) \qquad (x \in X) \tag{5-1}$$

says that $M_\mu(f)$ extends $f$, when we identify the elements of $Z$ with the ones of $\Delta_{\mu,Z}$ via $\delta_{\mu,Z}$ with $Z = X, Y$, which justifies the undertaken terminology. If, in addition, $X$ and $Y$ are topological groups and $f$ is a group homomorphism, so is $M_\mu(f)$.

The Reader will easily check that whenever $\mu$ is a fixed finite measure, the operations $X \mapsto M_\mu(X)$ and $f \mapsto M_\mu(f)$ define a functor. This functor has interesting properties, whose proofs are left as exercises (below we assume that $g_n, g\colon X \to Y$ are maps):

(F1) $M_\mu(g)$ is an injection [embedding] iff $g$ is so,
(F2) $\overline{\operatorname{im}} M_\mu(g) = M_\mu(\overline{\operatorname{im}}\, g)$,
(F3) the sequence $(M_\mu(g_n))_n$ is pointwisely [uniformly on compact subsets of $M_\mu(X)$] convergent to $M_\mu(g)$ iff the sequence $(g_n)_n$ pointwisely [uniformly on compact subsets of $X$] converges to $g$,
(F4) for each $\varrho \in \operatorname{Metr}(Y)$ the map

$$(\mathcal{C}(X,Y), \varrho_{\sup}) \ni h \mapsto M_\mu(h) \in (\mathcal{C}(M_\mu(X), M_\mu(Y)), (M_\mu(\varrho))_{\sup})$$

is isometric ('$\mathcal{C}(A,B)$' denotes the collection of all maps from $A$ to $B$ and '$d_{\sup}$' stands for the supremum metric induced by a bounded metric $d$).

It is clear that for each $f \in \mathcal{C}(X,Y)$, $\operatorname{im} M_\mu(f) \subset \bigcup_A M_\mu(f(A))$ where $A$ runs over all separable closed subsets of $X$. We do not know whether the latter inclusion can always be replaced by the equality. We are only able to show the following result, the proof of which is similar to the proof of [15, Theorem 3.3].

5.2. **Proposition.** *Whenever $\mu$ is a finite measure and $f\colon X \to Y$ is a map,* $\operatorname{im} M^r_\mu(f) = \bigcup_K M^r_\mu(f(K))$ *where $K$ runs over all $\sigma$-compact subsets of $X$ and $M^r_\mu(f) = M_\mu(f)\big|_{M^r_\mu(X)}$. What is more, if $v \in M^r_\mu(f(A))$, where $A$ is a (separable) Souslin subset of $X$, then $v \in \operatorname{im} M^r_\mu(f)$.*

*Proof.* We only need to prove the second claim. Put $C = f(A)$ and let $L$ be a $\sigma$-compact subset of $C$ such that $\operatorname{im} v \subset L$. Let $\nu\colon \mathfrak{B}(L) \ni B \mapsto \mu(v^{-1}(B)) \in \mathbb{R}_+$. Put $K = A \cap f^{-1}(L)$. Then $K \in \mathfrak{B}(A)$ and thus $K$ is a Souslin space. Now it suffices to apply [13, Theorem XIV.3.1] to obtain a function $h\colon L \to K$ such that $f \circ h$ is the identity map on $L$ and for every open in $K$ set $U \subset K$, $h^{-1}(U)$ is a member of the $\sigma$-algebra generated by the family of all Souslin subsets of $L$. This implies that for every Borel subset $B$ of $K$, $h^{-1}(B)$ is absolutely measurable and therefore there is a Borel function $w\colon L \to K$ and a set $B_0 \in \mathfrak{B}(L)$ such that $\nu(B_0) = 0$ and $w = h$ on $L \setminus B_0$. Now put $u = w \circ v$. By Lemma 2.1–(A), $u \in M^r_\mu(X)$. What is more, $f \circ u$ is $\mu$-almost everywhere equal to $f \circ h \circ v = v$, which finishes the proof. $\square$

Under the notation of Proposition 5.2 we get

5.3. **Corollary.** (i) *If $X$ is a So-space, then*

$$\operatorname{im} M^r_\mu(f) = \bigcup_A M^r_\mu(f(A))$$

*where $A$ runs over all separable closed subsets of $X$.*

(ii) *If for every compact subset $L$ of* $\operatorname{im} f$ *there is a (separable) Souslin subset $K$ of $X$ such that $L \subset f(K)$, then* $\operatorname{im} M^r_\mu(f) = M^r_\mu(\operatorname{im} f)$.

The above result leads to the following

5.4. **Definition.** A map $f\colon X \to Y$ is said to be an *s-map* if $f$ satisfies the assumption of the point (ii) of Corollary 5.3.

Basic examples of s-maps are closed maps whose domains are So-spaces and proper maps.

Now applying the General Scheme and main ideas of Section 3 of [15] (with the same functor $M$), thanks to the homeomorphism extension theorem proved in [10], we easily obtain

5.5. **Theorem.** *Let $\Omega$ be a topological space homeomorphic to a non-separable Hilbert space. Let $\mathcal{Z}$ be the family (category) of maps between $Z$-sets of $\Omega$ consisting of all pairs $(\varphi, L)$, where* $\operatorname{dom}\varphi$, *i.e. the domain of $\varphi$, and $L$ are $Z$-sets of $\Omega$ and $\varphi$ is an $L$-valued continuous function. There is a functor $\mathcal{Z} \ni (\varphi, L) \mapsto \widehat{\varphi}_L \in \mathcal{C}(\Omega, \Omega)$ of extension which satisfies all the claims of the points* (a), (b), (h), (i) *stated on pages 1–2 of* [15] *and the claims of the points* (d), (f) *and* (g) *(of* [15]*) concerning closures of images. The functor preserves the properties of being an injection, an embedding or a map with dense image; and satisfies all the claims of the points* (c)–(g) *of* [15] *for any s-map $\varphi$.*

5.6. *Remark.* Analogous functor as in Theorem 5.5 can be built using the functor $\widehat{P}$ studied by Banakh[2, 3] and Banakh and Radul[5]. (For a metrizable space $X$, $\widehat{P}(X)$ is the space of all Borel probabilistic measures supported on $\sigma$-compact subsets of $X$ and for a map $f\colon X \to Y$ and $\mu \in \widehat{P}(X)$, $\widehat{P}(f)(\mu)$ is the transport of $\mu$ under $f$.) Theorem 2.11 of [5] says that $\widehat{P}(X)$ is homeomorphic to an infinite-dimensional Hilbert space, provided $X$ is completely metrizable and noncompact. Thus it is enough to apply General Scheme of [15] and results of Banakh[2, 3] on extending maps and bounded metrics via the functor $\widehat{P}$.

We end the paper with the following two questions.

**Question 1.** Is $M(M(X))$ homeomorphic to $M(X)$ for an arbitrary metrizable space $X$ ?

**Question 2.** Is $M_\mu(X)$ homeomorphic to $M^r_\mu(X)$ for an arbitrary metrizable space $X$ and any finite measure space $(\Omega, \mathfrak{M}, \mu)$ ?

Note that the affirmative answer for Question 2 implies the affirmative one for Question 1.

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