NORM CLOSURES OF ORBITS OF BOUNDED OPERATORS

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Abstract. To every bounded linear operator A between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ three cardinals $\iota_r(A)$, $\iota_i(A)$ and $\iota_f(A)$ and a binary number $\iota_b(A)$ are assigned in terms of which the descriptions of the norm closures of the orbits $\{GAL^{-1}\colon\ L \in \mathcal{G}_1,\ G \in \mathcal{G}_2\}$ are given for $\mathcal{G}_1$ and $\mathcal{G}_2$ (chosen independently) being the trivial group, the unitary group or the group of all invertible operators on $\mathcal{H}$ and $\mathcal{K}$, respectively.
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1. Introduction

For a Hilbert space $\mathcal{H}$, denote by $\mathcal{G}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ the group of all isomorphisms (i.e. linear homeomorphisms) of $\mathcal{H}$ and the unitary group of $\mathcal{H}$, respectively. Additionally, let $I_{\mathcal{H}}$ be the identity operator on $\mathcal{H}$. For study of the geometry of the Banach space $\mathcal{B}(\mathcal{H},\mathcal{K})$ of all bounded linear operators from the Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ the natural action $(\mathcal{G}(\mathcal{H}) \times \mathcal{G}(\mathcal{K})) \times \mathcal{B}(\mathcal{H},\mathcal{K}) \ni ((L,G),X) \mapsto GXL^{-1} \in \mathcal{B}(\mathcal{H},\mathcal{K})$ of the group $\mathcal{G}(\mathcal{H}) \times \mathcal{G}(\mathcal{K})$ plays an important role. Especially the literature concerning the orbits (and their closures) under this action of closed range operators is still growing up (see e.g. [1] and references there). This includes the theory of Fredholm and semi-Fredholm operators, for which the index 'ind' is naturally defined and well behaves. However, most of results on closed range operators is settled in a separable (infinite-dimensional) Hilbert space. Also hardly ever operators with nonclosed ranges are considered when speaking about indices or orbits under the group action. The aim of the paper is to fill this lack and give a full answer (see Theorem 3.9) to the following problem:

Given an operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, describe the closure of the orbit of A under the natural action of $\mathcal{G}(\mathcal{H}) \times \mathcal{G}(\mathcal{K})$.

We shall also solve analogous problems for group orbits with respect to the following subgroups of $\mathcal{G}(\mathcal{H}) \times \mathcal{G}(\mathcal{K})$ (under the same action): $\mathcal{G}(\mathcal{H}) \times \{I_{\mathcal{K}}\}$ and $\{I_{\mathcal{H}}\} \times \mathcal{G}(\mathcal{K})$ (Theorem 6.6), $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{K})$ (Proposition 3.14), $\mathcal{U}(\mathcal{H}) \times \{I_{\mathcal{H}}\}$ and $\{I_{\mathcal{H}}\} \times \mathcal{U}(\mathcal{K})$ (Theorem 6.1; note that the orbit of A with respect to the groups $\mathcal{G}(\mathcal{H}) \times \mathcal{U}(\mathcal{K})$, $\mathcal{U}(\mathcal{H}) \times \mathcal{G}(\mathcal{K})$ and

$\mathcal{G}(\mathcal{H}) \times \mathcal{G}(\mathcal{K})$ coincide, which follows e.g. from Theorem 3.1 of [2]—see Proposition 3.2). The main results on these are settled in any Hilbert spaces (without restriction on dimensions) and deal with any bounded operator.

In comparison to the characterization of the members of the orbit of an operator under the action of $\mathcal{G}(\mathcal{H}) \times \mathcal{G}(\mathcal{K})$ which highly depends on the geometry of the range of the operator (see Theorem 3.4 of [2] or Proposition 3.2 in Section 3 below), the description of the closure of this orbit is given only in terms of four indices (that is, cardinal numbers), which seems to be surprising. Among many applications one may find: generalizations of results of Izumino and Kato [3] on the closure of $\mathcal{G}(\mathcal{H})$ and of Mbekhta [4] on the boundaries of sets of semi-Fredholm operators of arbitrarily fixed index, an extension of the notion of the index 'ind' and characterization of all closed two-sided ideals of $\mathcal{B}(\mathcal{H})$ for nonseparable $\mathcal{H}$.

Notation. In this paper $\mathcal{H}$ and $\mathcal{K}$ denote (complex) Hilbert spaces. $\mathcal{B}(\mathcal{H},\mathcal{K})$ is the Banach space of all bounded operators from $\mathcal{H}$ into $\mathcal{K}$; $\mathcal{G}(\mathcal{H},\mathcal{K})$ and $\mathcal{U}(\mathcal{H},\mathcal{K})$ are, respectively, the set of all isomorphisms and unitary operators from $\mathcal{H}$ onto $\mathcal{K}$. When $\mathcal{K} = \mathcal{H}$, we write $\mathcal{B}(\mathcal{H})$, $\mathcal{G}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ instead of $\mathcal{B}(\mathcal{H},\mathcal{H})$, $\mathcal{G}(\mathcal{H},\mathcal{H})$ and $\mathcal{U}(\mathcal{H},\mathcal{H})$. Additionally, $\mathcal{B}_+(\mathcal{H})$ stands for the set of all nonnegative (bounded) operators on $\mathcal{H}$. Whenever V is a closed subspace of $\mathcal{H}$, $\mathcal{H} \ominus V$ and $P_V \in \mathcal{B}(\mathcal{H})$ denote the orthogonal complement of V in $\mathcal{H}$ and the orthogonal projection onto V. $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$ and $\dim \mathcal{H}$ is the dimension of $\mathcal{H}$ as a Hilbert space (i.e. $\dim \mathcal{H}$ is the power of any orthonormal basis of $\mathcal{H}$). For $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $\overline{\mathcal{R}}(A)$ denote the kernel, the range and the closure of the range of A. The *polar decomposition of* A has the form $A = Q|A|$ where $|A| := \sqrt{A^*A}$ and Q is a partial isometry such that $\mathcal{N}(Q) = \mathcal{N}(A)$. Whenever we speak about convergence, closures, open sets, etc., in $\mathcal{B}(\mathcal{H},\mathcal{K})$, all they are understood in the norm topology. By $\mathfrak{B}(\mathbb{R}_+)$ we denote the σ-algebra of all Borel subsets of $\mathbb{R}_+ := [0,+\infty)$.

2. Operator ranges

In this part we recall the characterization of operator ranges and we define two auxiliary indices of such spaces which will find an application in the sequel.

Whenever $\mathcal{E}$ is a pre-Hilbert space, $\bar{\mathcal{E}}$ stands for its completion. A pre-Hilbert space $\mathcal{E}$ is said to be an *operator range* iff there is a Hilbert space $\mathcal{H}$ and a bounded operator $T\colon \mathcal{H} \to \bar{\mathcal{E}}$ such that $\mathcal{R}(T) = \mathcal{E}$.

Whenever $\mathcal{H}_1, \mathcal{H}_2, \ldots$ is a sequence of mutually orthogonal closed subspaces of a Hilbert space $\mathcal{H}$, let us denote by $\mathcal{S}(\mathcal{H}_1,\mathcal{H}_2,\ldots)$ the linear subspace of $\mathcal{H}$ consisting of all vectors $x \in \mathcal{H}$ of the form $x = \sum_{n=1}^{\infty} x_n$ where $x_n \in \mathcal{H}_n$ and $\sum_{n=1}^{\infty} 4^n \|x_n\|^2 < +\infty$.

A fundamental result on operator ranges is the following

2.1. **Theorem.** *For a pre-Hilbert space $\mathcal{E}$ the following conditions are equivalent:*
(a) *$\mathcal{E}$ is an operator range,*
(b) *there is a sequence $\mathcal{H}_1, \mathcal{H}_2, \ldots$ of mutually orthogonal closed subspaces of $\bar{\mathcal{E}}$ such that $\mathcal{E} = \mathcal{S}(\mathcal{H}_1, \mathcal{H}_2, \ldots)$.*

For proof, see e.g. [2].

2.2. **Definition.** Let $\mathcal{E}$ be a pre-Hilbert space. The cardinal

$$\mathrm{IC}(\mathcal{E}) = \min\{\dim(\bar{\mathcal{E}} \ominus V)\colon\ V \text{ is a complete subspace of } \mathcal{E}\}$$

is called the *index of incompleteness of* $\mathcal{E}$. The *binary index of* $\mathcal{E}$, in symbol $\mathfrak{b}(\mathcal{E})$, is defined as follows: $\mathfrak{b}(\mathcal{E}) = 1$ iff $\mathcal{E}$ contains a (necessarily complete) subspace isomorphic to $\bar{\mathcal{E}}$; otherwise $\mathfrak{b}(\mathcal{E}) = 0$.

The following result shows how useful are just defined indices. Its proof is left as an exercise (the points (d) and (e) of it follow from Theorem 2.1).

2.3. **Proposition.** *Let $\mathcal{E}$ be a pre-Hilbert space.*
(a) *$\mathcal{E}$ is complete iff $\mathrm{IC}(\mathcal{E}) = 0$. If $\mathcal{E}$ is incomplete, $\mathrm{IC}(\mathcal{E})$ is infinite.*
(b) *If $\mathrm{IC}(\mathcal{E}) < \dim \bar{\mathcal{E}}$, then $\mathfrak{b}(\mathcal{E}) = 1$.*
(c) *If $\mathfrak{b}(\mathcal{E}) = 1$, there is a complete subspace V of $\mathcal{E}$ such that $\dim V = \dim \bar{\mathcal{E}}$ and $\dim(\bar{\mathcal{E}} \ominus V) = \mathrm{IC}(\mathcal{E})$.*
(d) *If $\mathcal{E}$ is a range space, then for each $\beta < \dim \bar{\mathcal{E}}$ there is a complete subspace of $\mathcal{E}$ of dimension β.*
(e) *If $\mathcal{E}$ is a range space and $\mathfrak{b}(\mathcal{E}) = 0$, then $\dim \bar{\mathcal{E}}$ is an (infinite) limit cardinal of countable cofinality.*

It is clear that if $\mathcal{E}$ and $\mathcal{E}'$ are two linearly isometric pre-Hilbert spaces, then $\mathrm{IC}(\mathcal{E}) = \mathrm{IC}(\mathcal{E}')$ and $\mathfrak{b}(\mathcal{E}) = \mathfrak{b}(\mathcal{E}')$. This property combined with the well known fact that the ranges of a bounded operator and its adjoint operator are linearly isometric yields

2.4. **Proposition.** *If A is a bounded operator between two Hilbert spaces, then $\mathrm{IC}(\mathcal{R}(A)) = \mathrm{IC}(\mathcal{R}(A^*))$ and $\mathfrak{b}(\mathcal{R}(A)) = \mathfrak{b}(\mathcal{R}(A^*))$.*

With use of Theorem 3.3 of [2] on linearly isometric operator ranges we now give formulas for both the indices $\mathrm{IC}(\mathcal{E})$ and $\mathfrak{b}(\mathcal{E})$ in case $\mathcal{E}$ is an operator range.

2.5. **Proposition.** *Let $\mathcal{E}$ be an infinite-dimensional operator range and $\mathcal{H}_1, \mathcal{H}_2, \ldots$ be a sequence as in the point* (b) *of* Theorem 2.1*; that is, $\mathcal{E} = \mathcal{S}(\mathcal{H}_1, \mathcal{H}_2, \ldots)$.*
(a) *There is $N \geqslant 1$ such that $IC(\mathcal{E}) = \sum_{n=N}^{\infty} \dim \mathcal{H}_n$. More precisely, $\mathrm{IC}(\mathcal{E}) = \min\{\sum_{n=m}^{\infty} \dim \mathcal{H}_n\colon\ m \geqslant 1\}$.*
(b) *$\mathfrak{b}(\mathcal{E}) = 1$ iff $\dim \mathcal{H}_j = \dim \bar{\mathcal{E}}$ for some $j \geqslant 1$.*

Proof. Let V be a complete subspace of $\mathcal{E}$. Note that the assertions of both the points (a) and (b) follow from the following property:

$$(\star)\qquad \begin{array}{l}\text{there is } U \in \mathcal{U}(\bar{\mathcal{E}}\,) \text{ and } N \geqslant 1 \text{ such that}\\ U(\mathcal{E}) = \mathcal{E} \text{ and } U(V) \subset \bigoplus_{n=1}^{N} \mathcal{H}_n,\end{array}$$

which we now prove. Let $\mathcal{E}' = \mathcal{E} \cap (\bar{\mathcal{E}} \ominus V)$. Observe that $\mathcal{E}'$ is an operator range (one does not need Corollary 2 of Theorem 2.2 of [2] to see this) and therefore $\mathcal{E}' = \mathcal{S}(\mathcal{H}'_1, \mathcal{H}'_2, \ldots)$ for suitable spaces $\mathcal{H}'_1, \mathcal{H}'_2, \ldots$ But then $\mathcal{E} = \mathcal{S}(V, \mathcal{H}'_1, \mathcal{H}'_2, \ldots) = \mathcal{S}(\mathcal{H}_1, \mathcal{H}_2, \ldots)$ and it follows from the proof of Theorem 3.3 of [2] that there are U and N satisfying $(\star)$. $\square$

3. Two-sided actions

3.1. **Definition.** Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\mathcal{G}_1$ and $\mathcal{G}_2$ be subgroups of $\mathcal{G}(\mathcal{H})$ and $\mathcal{G}(\mathcal{K})$ respectively. For $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ let $\mathcal{O}\,_{\mathcal{G}_1}^{\mathcal{G}_2}(A)$ be the orbit of A with respect to the left action of $\mathcal{G}_1 \times \mathcal{G}_2$ on $\mathcal{B}(\mathcal{H}, \mathcal{K})$ given by $(\mathcal{G}_1 \times \mathcal{G}_2) \times \mathcal{B}(\mathcal{H}, \mathcal{K}) \ni ((G_1, G_2), X) \mapsto G_2 X G_1^{-1} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$; that is, $\mathcal{O}\,_{\mathcal{G}_1}^{\mathcal{G}_2}(A) = \{G_2 A G_1^{-1}\colon\ G_j \in \mathcal{G}_j\}$. The closure in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ of this orbit is denoted by $\overline{\mathcal{O}}\,_{\mathcal{G}_1}^{\mathcal{G}_2}(A)$.

When $\mathcal{G}_j$ coincides with the whole group of invertible operators or with the unitary group, then $\mathcal{G}_j$ in the notation $\mathcal{O}\,_{\mathcal{G}_1}^{\mathcal{G}_2}$ will be replaced by the letter G or U, respectively. When $\mathcal{G}_j$ is the trivial group (consisting only of the identity operator), $\mathcal{G}_j$ is omitted in the latter notation.

Notation. For $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ let $\Upsilon(A)$ constist of all closed linear subspaces V of $\mathcal{H}$ for which there is a positive constant c such that $\|Ax\| \geqslant c\|x\|$ for any $x \in V$. Observe that $A(V)$ is closed in $\mathcal{K}$ for every $V \in \Upsilon(A)$. Additionally, to simplify further arguments, for each $V \in \Upsilon(A)$ we use the following notation: $\xi_A(V) = (\dim(\mathcal{H} \ominus V), \dim V, \dim(\mathcal{K} \ominus A(V))$. Put $\Lambda(A) = \{\xi_A(V)\colon\ V \in \Upsilon(A)\}$. Notice that $\Upsilon(A) = \Upsilon(|A|)$.

For completeness of the lecture, we begin with recalling the characterization of members of suitable orbits.

3.2. **Proposition.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Let $\mathcal{N}_A$ be the set of all $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that* $\dim \mathcal{N}(B) = \dim \mathcal{N}(A)$ *and* $\dim \mathcal{N}(B^*) = \dim \mathcal{N}(A^*)$.

(a) $\mathcal{O}\,_G^G(A)$ *consists of all $B \in \mathcal{N}_A$ such that the ranges of A and B are isomorphic.*
(b) $\mathcal{O}\,_U^U(A)$ *is the set of all $B \in \mathcal{N}_A$ such that AA^* and BB^* are unitarily equivalent.*
(c) $\mathcal{O}\,_G(A)$ *consists of all $B \in \mathcal{N}_A$ such that $\mathcal{R}(B) = \mathcal{R}(A)$.*
(d) $\mathcal{O}\,_U(A)$ *is the set of all $B \in \mathcal{N}_A$ such that $BB^* = AA^*$.*
(e) $\mathcal{O}\,_U^G(A) = \mathcal{O}\,_G^U(A) = \mathcal{O}\,_G^G(A)$, $\mathcal{O}\,^G(A) = \{B^*\colon\ B \in \mathcal{O}\,_G(A^*)\}$ *and* $\mathcal{O}\,^U(A) = \{B^*\colon\ B \in \mathcal{O}\,_U(A^*)\}$.

Proof. The point (c) is Corollary 1 of Theorem 2.1 of [2]; (a) and the first assertion of (e) is proved also in [2], see Theorem 3.4 and its proof. To show the sufficiency of the conditions of (d), observe that if Q_A and Q_B are the partial isometries appearing in the polar decompositions of A and $B \in \mathcal{N}_A$, respectively, then $Q_A^* Q_B\big|_{\overline{\mathcal{R}}(B^*)}$ is extendable to a unitary $U \in \mathcal{U}(\mathcal{H})$ which automatically satisfies $B = AU$. Finally, notice that (b) basicly follows from (d) and the remainder of (e) is immediate. $\square$

In Lemma 3.5–(c) we shall show, independently of the foregoing result, that $\overline{\mathcal{O}}{}_G^G(A) = \overline{\mathcal{O}}{}_G^U(A) = \overline{\mathcal{O}}{}_U^G(A)$ for each A.

The proof of the following easy result is omitted.

3.3. **Lemma.** *If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, W is a closed subspace of $\mathcal{H}$ such that $W \cap \mathcal{N}(A) = \{0\}$, then $\dim(W \ominus V) = \dim(\overline{A(W)} \ominus A(V))$ for any space $V \in \Upsilon(A)$ contained in W.*

The next two results are our main tools.

3.4. **Proposition.** *For every $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and each $\varepsilon > 0$ there is $V \in \Upsilon(A)$ such that $|A|(V) = V$ and $\|A - AP_V\| \leqslant \varepsilon$.*

Proof. Let $E\colon \mathfrak{B}(\mathbb{R}_+) \to \mathcal{B}(\mathcal{H})$ be the spectral measure of $|A|$. It suffices to put $P = E([\varepsilon,+\infty))$ and $V = \mathcal{R}(P)$. $\square$

A part of the point (a) of the following result is certainly known in perturbation theory. However, its short proof is used to establish the remainder of (a) and therefore below we give full details.

3.5. **Lemma.** *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$.*

(a) *If $A_1, A_2, \ldots \in \mathcal{B}(\mathcal{H},\mathcal{K})$ converge to A, then*
$$\Upsilon(A) \subset \bigcup_{n=1}^{\infty}\bigcap_{k=1}^{\infty}\Upsilon(A_k) \quad \text{and} \quad \Lambda(A) \subset \bigcup_{n=1}^{\infty}\bigcap_{k=1}^{\infty}\Lambda(A_k).$$
What is more, for each $V \in \Upsilon(A)$ there is $N \geqslant 1$ and a sequence $(Z_n)_{n=N}^{\infty} \in \mathcal{U}(\mathcal{K})$ such that $V \in \Upsilon(A_n)$, $Z_n(A(V)) = A_n(V)$ ($n \geqslant N$) and $Z_n P_{A(V)} \to P_{A(V)}$ ($n \to \infty$).

(b) *$\Lambda(GAL^{-1}) = \Lambda(A)$ for each $G \in \mathcal{G}(\mathcal{K})$ and $L \in \mathcal{G}(\mathcal{H})$.*

(c) *$\overline{\mathcal{O}}{}_G^G(A) = \overline{\mathcal{O}}{}_G^U(A) = \overline{\mathcal{O}}{}_U^G(A)$ and $\overline{\mathcal{O}}{}_G^G(A)$ coincides with the set of all $C \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that $\Lambda(C) \subset \Lambda(A)$.*

Proof. (a): Suppose $V \in \Upsilon(A)$. Let $P\colon \mathcal{K} \to A(V)$ be the orthogonal projection. Then we have $PA_n\big|_V \to A\big|_V \in \mathcal{G}(V, A(V))$ and thus $PA_n\big|_V \in \mathcal{G}(V, A(V))$ as well for $n \geqslant N$. This implies that $V \in \Upsilon(A_n)$,

$$A_n(V) \in \Upsilon(P) \quad \text{and} \quad P(A_n(V)) = A(V) \tag{3-1}$$

and $\mathcal{N}(P) + A_n(V)$ is closed. So, $(\alpha,\beta,\gamma_n) := \xi_{A_n}(V) \in \Lambda(A_n)$ for $n \geqslant N$. Now let F_n denote the orthogonal complement (in $\mathcal{K}$) of $\mathcal{N}(P) + A_n(V)$. We see that $\gamma_n = \dim(\mathcal{K} \ominus A_n(V)) = \dim(\mathcal{N}(P) \oplus F_n) = \dim(\mathcal{N}(P) \oplus P(F_n)) = \dim(\mathcal{K} \ominus P(A_n(V))) = \dim(\mathcal{K} \ominus A(V))$ which shows that

$$\dim(\mathcal{K} \ominus A_n(V)) = \dim(\mathcal{K} \ominus A(V)) \ (n \geqslant N) \tag{3-2}$$

and $\xi_A(V) \in \Lambda(A_n)$. From now on, $n \geqslant N$. Let P_n be the orthogonal projection of $\mathcal{K}$ onto $A_n(V)$. Since $P_n A_n P_V = A_n P_V \to AP_V$ $(n \to \infty)$ and simultaneously $\lim_{n\to\infty} P_n(A_n - A)P_V = 0$, we get $\lim_{n\to\infty} P_n A P_V = AP_V$. Since $A\big|_V \in \mathcal{G}(V, A(V))$, the latter relation is equivalent to

$$P_n P_{A(V)} \to P_{A(V)} \ (n \to \infty). \tag{3-3}$$

Put $T_n = P_n\big|_{A(V)} \in \mathcal{B}(A(V), A_n(V))$. A straightforward calculation shows that $T_n^* = P_{A(V)}\big|_{A_n(V)}$ (we compute T_n^* as the adjoint of a member of $\mathcal{B}(A(V), A_n(V))$). By (3-1), $T_n^* \in \mathcal{G}(A_n(V), A(V))$ and hence $T_n \in \mathcal{G}(A(V), A_n(V))$. Let $T_n = Q_n|T_n|$ be the polar decomposition of T_n; that is, $Q_n \in \mathcal{U}(A(V), A_n(V))$ and $|T_n| \in \mathcal{B}_+(A(V))$. Now by (3-2), there is a unitary operator Z_n on $\mathcal{K}$ which extends Q_n. It remains to prove that

$$\lim_{n\to\infty} Z_n P_{A(V)} = P_{A(V)}. \tag{3-4}$$

It follows from (3-3) that $\lim_{n\to\infty} P_{A(V)} P_n P_{A(V)} = P_{A(V)}$. We conclude from this that $\lim_{n\to\infty} T_n^* T_n = I_{A(V)}$ and thus $\lim_{n\to\infty} |T_n| = I_{A(V)}$ and $|T_n|^{-1} \to I_{A(V)}$ $(n \to \infty)$ as well. So, $Z_n P_{A(V)} = Q_n P_{A(V)} = T_n|T_n|^{-1} P_{A(V)} = (P_n P_{A(V)})|T_n|^{-1} P_{A(V)} \to P_{A(V)}$ $(n \to \infty)$, which finishes the proof of (a).

The point (b) is immediate. To prove (c), suppose $\Lambda(C) \subset \Lambda(A)$. For each $\varepsilon > 0$ we shall find $U \in \mathcal{U}(\mathcal{K})$ and $G \in \mathcal{G}(\mathcal{H})$ such that

$$\|C - UAG^{-1}\| \leqslant \varepsilon. \tag{3-5}$$

By Proposition 3.4, there is $V \in \Upsilon(C)$ such that

$$\|C - CP_V\| \leqslant \frac{1}{2}\varepsilon. \tag{3-6}$$

But then $\xi_C(V) \in \Lambda(C) \subset \Lambda(A)$, so we may find $W \in \Upsilon(A)$ such that $\xi_A(W) = \xi_C(V)$. We conclude from this that there are unitary operators U_0 on $\mathcal{H}$ and U on $\mathcal{K}$ with $U_0(W) = V$ and $U(A(W)) = C(V)$. For $n \geqslant 1$ define $G_n \in \mathcal{G}(\mathcal{H})$ by $G_n = (C\big|_V)^{-1} UA$ on W and $G_n = nU_0$ on $\mathcal{H} \ominus W$. Observe that $UAG_n^{-1} = C$ on V and $\|UAG_n^{-1}\big|_{\mathcal{H}\ominus V}\| \leqslant \frac{1}{n}\|A\|$. These properties combined with (3-6) yield (3-5) with $G = G_N$ for some $N \geqslant 1$. This shows that $C \in \overline{\mathcal{O}}{}_G^U(A)$ provided $\Lambda(C) \subset \Lambda(A)$. But the inclusion $\overline{\mathcal{O}}{}_G^U(A) \subset \overline{\mathcal{O}}{}_G^G(A)$ is immediate and the implication '$C \in \overline{\mathcal{O}}{}_G^G(A) \implies \Lambda(C) \subset \Lambda(A)$' follows from (a) and (b). So, $\overline{\mathcal{O}}{}_G^U(A) = \overline{\mathcal{O}}{}_G^G(A)$. Finally, if $C \in \overline{\mathcal{O}}{}_G^G(A)$, then $C^* \in \overline{\mathcal{O}}{}_G^G(A^*)$ and therefore $C^* \in \overline{\mathcal{O}}{}_G^U(A^*)$ which yields $C \in \overline{\mathcal{O}}{}_U^G(A)$. □

Now for an operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ we define the following indices:

(I1) *range* index of A: $\iota_r(A) = \dim \overline{\mathcal{R}}(A)$ (cardinal),
(I2) *initial* index of A: $\iota_i(A) = \dim \mathcal{N}(A) + \mathrm{IC}(\mathcal{R}(A))$ (cardinal),
(I3) *final* index of A: $\iota_f(A) = \dim(\mathcal{K} \ominus \overline{\mathcal{R}}(A)) + \mathrm{IC}(\mathcal{R}(A))$ (cardinal),

(I4) *binary* index of A: $\iota_b(A) = \mathfrak{b}(\mathcal{R}(A))$ (0 or 1).

For example, note that:

(P1) if A is a closed range operator, then $\iota_i(A)$ and $\iota_f(A)$ are the well known indices: nullity and defect (respectively), compare e.g. [1],

(P2) $\iota_i(A^*) = \iota_f(A) = \iota_i(|A^*|) = \iota_f(|A^*|)$, $\iota_f(A^*) = \iota_i(A) = \iota_i(|A|) = \iota_f(|A|)$, $\iota_r(A^*) = \iota_r(A) = \iota_r(|A|) = \iota_r(|A^*|)$ and $\iota_b(A^*) = \iota_b(A) = \iota_b(|A|) = \iota_b(|A^*|)$ (see Proposition 2.4),

(P3) $\iota_i(A)+\iota_r(A) = \dim \mathcal{H}$, $\iota_f(A)+\iota_r(A) = \dim \mathcal{K}$ (to see this, consider separately the cases when $\iota_r(A)$ is finite-dimensional or not),

(P4) if $\mathcal{K}$ and $\mathcal{H}$ are separable and $\mathcal{R}(A)$ is nonclosed, $\iota_i(A) = \iota_f(A) = \aleph_0$.

Using only these four indices we shall characterize all operators belonging to $\overline{\mathcal{O}}^{G}_{G}(A)$ (see Theorem 3.9). To do this, we need

3.6. **Lemma.** *For $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $\Lambda(A)$ consists precisely of all the triples of the form $(\iota_i(A) + \nu, \mu, \iota_f(A) + \nu)$ where μ and ν are cardinal numbers satisfying the conditions:*

(a) $\mu + \nu = \iota_r(A)$,

(b) *if $\iota_b(A) = 0$, then $\mu < \iota_r(A)$.*

Proof. Suppose $V \in \Upsilon(A)$ and let $(\alpha, \beta, \alpha') = \xi_A(V)$ and $\mu = \dim V$. Let W be the orthogonal complement (in $\mathcal{H}$) of the (closed) subspace $\mathcal{N}(A) + V$ and put $\nu = \dim W$. Then A restricted to $V + W$ is a (continuous) monomorphism onto $\mathcal{R}(A)$ and $A(V)$ is closed in $\mathcal{R}(A)$. By Lemma 3.3, $\nu = \dim(\overline{\mathcal{R}}(A) \ominus A(V))$, which gives $\mu+\nu = \iota_r(A)$ and $\nu \geqslant \mathrm{IC}(\mathcal{R}(A))$. Now Proposition 2.3–(a) yields that $\nu = \mathrm{IC}(\mathcal{R}(A)) + \nu$ and thus $\alpha = \iota_i(A) + \nu$ and $\alpha' = \iota_f(A) + \nu$. The condition (c) follows from the definition of $\iota_b(A)$ and the relation $\mu = \dim A(V)$.

Now assume that μ and ν satisfy (a)–(c). If $\mu < \iota_r(A)$, then ν is uniquely determined by (a), $\nu = \nu + \mathrm{IC}(\mathcal{R}(A))$ (because $\nu = \iota_r(A)$ if $\iota_r(A)$ is infinite and otherwise $\mathrm{IC}(\mathcal{R}(A)) = 0$) and there is a complete subspace E of $\mathcal{R}(A)$ with

$$\dim E = \mu \tag{3-7}$$

(cf. Proposition 2.3–(d)). Then automatically

$$\nu + \mathrm{IC}(\mathcal{R}(A)) = \dim(\overline{\mathcal{R}}(A) \ominus E). \tag{3-8}$$

If $\mu = \iota_r(A)$, then $\iota_b(A) = 1$ and (by Proposition 2.3–(c)) there is a complete subspace W of $\mathcal{R}(A)$ such that $\dim W = \mu$ and $\dim(\overline{\mathcal{R}}(A) \ominus W) = \mathrm{IC}(\mathcal{R}(A))$. Thanks to (a) we may find a closed subspace F of W such that $\dim(W \ominus F) = \mu$ and $\nu = \dim F$. Now putting $E = W \ominus F \subset \mathcal{R}(A)$, we see that relations (3-7) and (3-8) are fulfilled.

To end the proof, let $V = A^{-1}(E) \cap \overline{\mathcal{R}}(A^*)$. Note that $V \in \Upsilon(A)$ and $A(V) = E$. By (3-8) and Lemma 3.3, $\nu + \mathrm{IC}(\mathcal{R}(A)) = \dim(\overline{\mathcal{R}}(A^*) \ominus V)$ and thus $(\iota_i(A) + \nu, \mu, \iota_f(A) + \nu) = \xi_A(V) \in \Lambda(A)$. □

The next result is simply deduced from the previous one.

3.7. **Corollary.** *Let* $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

(i) *If* $\iota_r(A)$ *is finite, then*
$$\Lambda(A) = \{(\dim \mathcal{H} - j, j, \dim \mathcal{K} - j)\colon\ j = 0, \ldots, \iota_r(A)\}$$
where $\mathfrak{m} - j = \mathfrak{m}$ *when* $\mathfrak{m}$ *is infinite and* j *is finite.*

(ii) *If* $\iota_b(A) = 0$*, then*
$$\Lambda(A) = \{(\dim \mathcal{H}, \beta, \dim \mathcal{K})\colon\ 0 \leqslant \beta < \iota_r(A)\}.$$

(iii) *If* $\iota_r(A)$ *is infinite and* $\iota_b(A) = 1$*, then*
$$\Lambda(A) = \{(\dim \mathcal{H}, \beta, \dim \mathcal{K})\colon\ 0 \leqslant \beta < \iota_r(A)\} \cup \{(\iota_i(A) + \alpha, \iota_r(A), \iota_f(A) + \alpha)\colon\ 0 \leqslant \alpha \leqslant \iota_r(A)\}.$$

Now the points (a) and (c) of Lemma 3.5 combined with Corollary 3.7 yield

3.8. **Theorem.** *Every operator* $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *with* $\iota_b(A) = 1$ *has a neighbourhood* $\mathcal{X}$ *such that* $A \in \overline{\mathcal{O}}_G^G(T)$ *for all* $T \in \mathcal{X}$.

Proof. Notice that $(\iota_i(A), \iota_r(A), \iota_f(A)) \in \Lambda(A)$ and argue by contradiction: suppose there is a sequence of bounded operators $A_1, A_2, \ldots$ which converge to A and are such that $A \notin \overline{\mathcal{O}}_G^G(A_n)$ for each n. But then Lemma 3.5–(a) gives $(\iota_i(A), \iota_r(A), \iota_f(A)) \in \Lambda(A_n)$ for large n and therefore $\Lambda(A) \subset \Lambda(A_n)$ for this n (by the formula for $\Lambda(X)$ given in Corollary 3.7). This denies the point (c) of Lemma 3.5. □

The main result of the paper is the following consequence of Corollary 3.7 and Lemma 3.5–(c). Its proof is omitted.

3.9. **Theorem.** *For* $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\overline{\mathcal{O}}_G^G(A)$ *consists precisely of those operators* $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *which satisfy the following three conditions:*

(a) $\iota_r(C) \leqslant \iota_r(A)$,

(b) *if* $\iota_b(A) = 0$ *and* $\iota_b(C) = 1$*, then* $\iota_r(C) < \iota_r(A)$,

(c) *there is a cardinal* α *for which* $\iota_i(C) = \iota_i(A) + \alpha$ *and* $\iota_f(C) = \iota_f(A) + \alpha$.

What is more, if
$$(\star) \qquad \iota_i(C) = \dim \mathcal{H} \quad \text{and} \quad \iota_f(C) = \dim \mathcal{K},$$
then (c) *is fulfilled for any* $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

3.10. *Remark.* The description of $\overline{\mathcal{O}}_G^G(A)$ may be given in terms of only three indices: both the indices ι_r and ι_b may be 'included' in one index ι_R defined by the rule: $\iota_R(X)$ is equal to $\iota_r(X)$ iff $\iota_b(X) = 0$, otherwise $\iota_R(X)$ is the direct successor of $\iota_r(X)$ (in other words, $\iota_R(X)$ is the least cardinal α such that $\mathcal{R}(X)$ contains no complete subspace of dimension α). Using Proposition 2.3–(e), one may show that for $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ the points (a) and (b) of Theorem 3.9 are fulfilled iff $\iota_R(B) \leqslant \iota_R(A)$.

So, $\overline{\mathcal{O}}^{\,G}_{G}(A)$ may be described by means of ι_R, ι_i and ι_f. However, it seems to us that the index ι_R is rather unnatural, because $\iota_R(X)$ may be uncountable even if X acts on a separable Hilbert space.

Moreover, using (P3), it may be shown that when $\dim\mathcal{H} \neq \dim\mathcal{K}$, the condition (c) of Theorem 3.9 may be simplified to '$\iota_m(C) \geqslant \iota_m(A)$' where $\iota_m(X) = \min(\iota_i(X), \iota_f(X))$ for $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$. So, one needs only two indices (ι_R and ι_m) for the description of $\overline{\mathcal{O}}^{\,G}_{G}$ provided $\mathcal{H}$ and $\mathcal{K}$ have different dimensions. The case when $\dim\mathcal{H} = \dim\mathcal{K}$ may simply be reduced to the one when $\mathcal{K} = \mathcal{H}$. This case will be investigated in Section 5.

3.11. **Corollary.** *For $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $\overline{\mathcal{O}}^{\,G}_{G}(A) = \overline{\mathcal{O}}^{\,G}_{G}(B)$ iff $\iota_r(A) = \iota_r(B)$, $\iota_i(A) = \iota_i(B)$, $\iota_f(A) = \iota_f(B)$ and $\iota_b(A) = \iota_b(B)$.*

3.12. **Corollary.** *Let $A \in \mathcal{B}(\mathcal{H})$.*

- (I) *If A is a finite rank operator, $\overline{\mathcal{O}}^{\,G}_{G}(A) = \{B \in \mathcal{B}(\mathcal{H})\colon\ \iota_r(B) \leqslant \iota_r(A)\}$.*
- (II) *If A is compact and of infinite rank, $\overline{\mathcal{O}}^{\,G}_{G}(A)$ coincides with the class of all compact operators on $\mathcal{H}$.*
- (III) *Suppose $\mathcal{H}$ is infinite-dimensional and separable.*
  - (i) *If A is noncompact and nonsemi-Fredholm, $\overline{\mathcal{O}}^{\,G}_{G}(A)$ coincides with the class of all nonsemi-Fredholm operators on $\mathcal{H}$,*
  - (ii) *If A is semi-Fredholm, $\overline{\mathcal{O}}^{\,G}_{G}(A)$ consists of all nonsemi-Fredholm operators on $\mathcal{H}$ and of precisely those semi-Fredholm operators $B \in \mathcal{B}(\mathcal{H})$ for which* $\operatorname{ind}(B) = \operatorname{ind}(A)$ *and*

$$\min(\iota_i(B), \iota_f(B)) \geqslant \min(\iota_i(A), \iota_f(A)).$$

From Corollary 3.12 and Theorem 3.8 one may conclude the classical theorem that in a separable Hilbert space all semi-Fredholm operators of the same (arbitrarily fixed) index form a connected open set (which in fact is the interior of the closure of the orbit $\mathcal{O}^G_G$ of a one semi-Fredholm operator which is a monomorphism or an epimorphism). It may also be easily infered that all of these open sets have the same boundary, which was first shown by Mbekhta [4]. For details and generalization see Section 5.

Since $\mathcal{O}^G_G(I_{\mathcal{H}}) = \mathcal{G}(\mathcal{H})$, we obtain the following generalization of the result of Izumino and Kato [3].

3.13. **Corollary.** *The norm closure of the group of all invertible operators on a Hilbert space $\mathcal{H}$ is the set of all $A \in \mathcal{B}(\mathcal{H})$ such that $\iota_i(A) = \iota_f(A)$.*

Our last purpose of this section is to describe $\overline{\mathcal{O}}^{\,U}_{U}(A)$. We shall do this with use of the closures $\overline{\mathcal{O}\mathcal{U}}(A^*A)$ and $\overline{\mathcal{O}\mathcal{U}}(AA^*)$ of the orbits $\mathcal{O}\mathcal{U}(A^*A)$ and $\mathcal{O}\mathcal{U}(AA^*)$ where $\mathcal{O}\mathcal{U}(X) = \{UXU^{-1}\colon\ U \in \mathcal{U}(\mathcal{H})\}$ for

$X \in \mathcal{B}(\mathcal{H})$. The characterization of the members of $\overline{\mathcal{O}\mathcal{U}}(X)$ will be made in a subsequent paper.

3.14. **Proposition.** *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$.*
(a) *If $\iota_i(A) \leqslant \iota_f(A)$, then*

$$\overline{\mathcal{O}}_U^U(A) = \{B \in \mathcal{B}(\mathcal{H},\mathcal{K})\colon\ B^*B \in \overline{\mathcal{O}\mathcal{U}}(A^*A) \text{ and } \iota_f(B) = \iota_f(A)\}.$$

(b) *If $\iota_i(A) \geqslant \iota_f(A)$, then*

$$\overline{\mathcal{O}}_U^U(A) = \{B \in \mathcal{B}(\mathcal{H},\mathcal{K})\colon\ BB^* \in \overline{\mathcal{O}\mathcal{U}}(AA^*) \text{ and } \iota_i(B) = \iota_i(A)\}.$$

Proof. (a): If $B = \lim_{n\to\infty} V_n A U_n^{-1}$ with unitary U_n's and V_n's, then $B^*B = \lim_{n\to\infty} U_n A^* A U_n^{-1}$ and $A \in \overline{\mathcal{O}}_U^U(B)$ which gives $\overline{\mathcal{O}}_G^G(B) = \overline{\mathcal{O}}_G^G(A)$. Thus we infer from Corollary 3.11 that

$$\iota_f(B) = \iota_f(A). \tag{3-9}$$

Conversely, suppose (3-9) holds true and $B^*B = \lim_{n\to\infty} U_n A^* A U_n^{-1}$ for some $U_n \in \mathcal{U}(\mathcal{H})$. Then

$$U_n|A|U_n^{-1} \to |B|\ (n \to \infty) \tag{3-10}$$

as well. This implies that $\overline{\mathcal{O}}_G^G(|B|) = \overline{\mathcal{O}}_G^G(|A|)$ and hence $\iota_f(|B|) = \iota_f(|A|)$ (cf. Corollary 3.11). The latter connection combined with (P2) gives

$$\iota_i(B) = \iota_i(A). \tag{3-11}$$

Fix $\varepsilon > 0$ and take, using Proposition 3.4, $W \in \Upsilon(B)$ such that $|B|(W) = W$ and

$$\|B - BP_W\| < \frac{\varepsilon}{4}. \tag{3-12}$$

Since $W \in \Upsilon(|B|)$, from (3-10) and Lemma 3.5–(a) it follows that $W \in \Upsilon(|A|U_n^{-1}) = \Upsilon(AU_n^{-1})$ for all but finitely many n's. Passing to a subsequence, we may assume that this is true for all n's. Put $W_n = U_n|A|U_n^{-1}(W)$. Again by Lemma 3.5–(a), there is a sequence of unitary operators $Z_1, Z_2, \ldots$ on $\mathcal{H}$ such that $Z_n(W) = W_n$ and $\lim_{n\to\infty} Z_n P_W = P_W$. We infer from this that $\lim_{n\to\infty} Z_n|B|P_W = |B|P_W$ and therefore

$$Z_n^{-1}U_n|A|U_n^{-1}P_W \to |B|P_W\ (n \to \infty) \tag{3-13}$$

(because $\|Z_n^{-1}U_n|A|U_n^{-1}P_W - |B|P_W\| = \|U_n|A|U_n^{-1}P_W - Z_n|B|P_W\|$).

Further, let $B = Q|B|$ be the polar decomposition of B. Since $\iota_f(A) \geqslant \iota_i(A)$, there is a cardinal α for which $\iota_f(A) = \iota_i(A) + \alpha$. We claim that there is an isometry $V \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that $V\big|_W = Q\big|_W$ and $\dim(\mathcal{K} \ominus \mathcal{R}(V)) = \alpha$. Indeed, since W is a complete subspace of $\mathcal{R}(|B|)$, $\dim(\overline{\mathcal{R}}(|B|) \ominus W) \geqslant \mathrm{IC}(\mathcal{R}(|B|)) = \mathrm{IC}(\mathcal{R}(B))$ and hence there are orthogonal closed subspaces E and F of $\mathcal{H}$ such that $\overline{\mathcal{R}}(|B|) \ominus W = E \oplus F$ and $\dim E = \mathrm{IC}(\mathcal{R}(B))$. Then $\mathcal{K} \ominus Q(W) = [(\mathcal{K} \ominus \overline{\mathcal{R}}(B)) \oplus Q(E)] \oplus Q(F)$ and $\dim[(\mathcal{K} \ominus \overline{\mathcal{R}}(B)) \oplus Q(E)] = \dim(\mathcal{K} \ominus \overline{\mathcal{R}}(B)) + \mathrm{IC}(\mathcal{R}(B)) = \iota_f(B) = \iota_i(B) + \alpha$, thanks to (3-9) and (3-11). Similarly, $\mathcal{H} \ominus W =$

$(\mathcal{N}(B) \oplus E) \oplus F$ and $\dim(\mathcal{N}(B) \oplus E) = \iota_i(B)$. This means that we may find suitable V in such a way that it extends $Q\big|_{W \oplus F}$.

Now observe that $\iota_f(VZ_n^{-1}U_n|A|U_n^{-1}) = \alpha + \iota_f(|A|) = \iota_f(A) = \iota_f(AU_n^{-1})$ and $(VZ_n^{-1}U_n|A|U_n^{-1})^*(VZ_n^{-1}U_n|A|U_n^{-1}) = (AU_n^{-1})^*(AU_n^{-1})$. From Theorem 6.1 (see Section 6) it follows that $VZ_n^{-1}U_n|A|U_n^{-1} \in \overline{\mathcal{O}}^{\,U}(AU_n^{-1})$ and thus there is $V_n \in \mathcal{U}(\mathcal{K})$ for which

$$\|V_nAU_n^{-1} - VZ_n^{-1}U_n|A|U_n^{-1}\| < \frac{\varepsilon}{4}. \tag{3-14}$$

To this end, note that $\|V_nAU_n^{-1}(I_{\mathcal{H}} - P_W)\| = \|U_n|A|U_n^{-1}(I_{\mathcal{H}} - P_W)\| \to \|\,|B|(I_{\mathcal{H}} - P_W)\| < \frac{\varepsilon}{4}$ (by (3-10) and (3-12)) and therefore for large n's one has $\|(V_nAU_n^{-1} - B)(I_{\mathcal{H}} - P_W)\| \leqslant \frac{\varepsilon}{2}$. While on the other hand, first making use of (3-14) and next of (3-13),

$$\begin{aligned}\|(V_nAU_n^{-1} - B)P_W\| &\leqslant \|(V_nAU_n^{-1} - VZ_n^{-1}U_n|A|U_n^{-1})P_W\| \\ &\quad + \|VZ_n^{-1}U_n|A|U_n^{-1}P_W - QP_W|B|\,\| \\ &\leqslant \frac{\varepsilon}{4} + \|VZ_n^{-1}U_n|A|U_n^{-1}P_W - VP_W|B|\,\| \to \frac{\varepsilon}{4}\ (n \to \infty)\end{aligned}$$

which clearly shows that for some large n we have $\|V_nAU_n^{-1} - B\| \leqslant \varepsilon$.

To prove (b), pass to adjoints and apply (a) (using (P2)). $\square$

3.15. **Example.** Let $\mathcal{H}$ be separable infinite-dimensional, $A \in \mathcal{B}_+(\mathcal{H})$ be a noninvertible operator with dense range and $V \in \mathcal{B}(\mathcal{H})$ be a nonunitary isometry. By (P4), $\iota_f(VA) = \iota_f(A) = \aleph_0$. Moreover, $(VA)^*(VA) = A^2 = A^*A$ and thus, by Proposition 3.14, $VA \in \overline{\mathcal{O}}_U^{\,U}(A)$ which easily gives $VA^2V^* = (VA)(VA)^* \in \mathcal{U}\overline{\mathcal{O}}(A^2)$. On the other hand, $VA^2V^* \notin \mathcal{O}\mathcal{U}(A^2)$ (since $\overline{\mathcal{R}}(VA^2V^*) \neq \mathcal{H}$). This implies that both the operators $B := VA^2V^* + I_{\mathcal{H}}$ and $C := A^2 + I_{\mathcal{H}}$ are nonnegative, invertible, non-unitarily equivalent, but $B \in \overline{\mathcal{O}\mathcal{U}}(C)$. The example shows that the orbit $\mathcal{O}\mathcal{U}(X)$ is not closed in general (even when X is invertible and nonnegative) and that the description of its closure seems to be much more difficult than in case of the orbits investigated in this paper.

4. Application: ideals of $\mathcal{B}(\mathcal{H})$

With use of Theorem 3.9, we may easily point out all closed two-sided ideals of $\mathcal{B}(\mathcal{H})$ for nonseparable Hilbert space $\mathcal{H}$. For each infinite cardinal $\alpha \leqslant \dim \mathcal{H}$ let J_α be the set of all operators $A \in \mathcal{B}(\mathcal{H})$ such that $\iota_r(A) < \alpha$ or $\iota_r(A) = \alpha$ and $\iota_b(A) = 0$ (notice that $J_{\aleph_0}$ is consists precisely of all compact operators). Our aim is to show that J_α's are the only nontrivial ideals in $\mathcal{B}(\mathcal{H})$.

4.1. **Lemma.** *For $A \in \mathcal{B}(\mathcal{H})$ the following conditions are equivalent:*

(a) $\iota_b(A) = 0$,

(b) *A is the limit of a sequence $(A_n)_{n=1}^{\infty} \in \mathcal{B}(\mathcal{H})$ such that $\iota_r(A_n) < \iota_r(A)$ for each n.*

Proof. From Proposition 3.4 we conclude that (a) implies (b). The inverse implication follows from Lemma 3.5–(a) and Corollary 3.7. $\square$

The next result is probably known.

4.2. **Theorem.** *Let $\mathcal{H}$ be a nonseparable Hilbert space. For each infinite $\alpha \leqslant \dim \mathcal{H}$, J_α is a closed two-sided ideal in $\mathcal{B}(\mathcal{H})$, $J_\alpha \neq J_{\alpha'}$ if $\alpha \neq \alpha'$ and every nonzero proper closed two-sided ideal in $\mathcal{B}(\mathcal{H})$ coincides with some J_α.*

Proof. That J_α is a closed two-sided ideal it may easily be infered from Lemma 4.1. It is also immediate that J_α uniquely determines α. Let J be a nonzero proper closed two-sided ideal of $\mathcal{B}(\mathcal{H})$. Observe that $J \subset J_{\dim \mathcal{H}}$. (Indeed, otherwise there would be $A \in J$ and $V \in \Upsilon(A)$ such that $\dim V = \dim \mathcal{H}$. But then $I_{\mathcal{H}}$ would belong to J since $I = XAY$ for suitable $X, Y \in \mathcal{B}(\mathcal{H})$.) Let $\alpha \geqslant \aleph_0$ be the least cardinal for which $J \subset J_\alpha$. We shall show that $J = J_\alpha$. To do this, thanks to Lemma 4.1 it is enough to prove that $C \in J$ provided $\iota_r(C) < \alpha$. A standard argument shows that a nonzero ideal contains all finite rank operators. So, we may assume that $\iota_r(C)$ is infinite. Since $J \not\subset J_\beta$ with $\beta = \iota_r(C)$, there is $A \in J$ such that $\iota_r(A) > \iota_r(C)$ or $\iota_r(A) = \iota_r(C)$ and $\iota_b(A) = 1$. But then, by Theorem 3.9, $C \in \overline{\mathcal{O}}_G^G(A) \subset J$ (note that $(\star)$ is fulfilled for C since $\iota_r(C) < \dim \mathcal{H}$). $\square$

5. Indices ind and ι_m

In this section $\mathcal{H}$ denotes an infinite-dimensional Hilbert space. Our aim is to define $\operatorname{ind}(A)$ for certain operators $A \in \mathcal{B}(\mathcal{H})$ in such a way that this new index extends the well known one (denoted in the same way) for semi-Fredholm operators.

Let $\mathfrak{D}$ be the class of all pairs (α, β) of cardinals such that either $\alpha = \beta < \aleph_0$ or α and β are different. For $(\alpha, \beta) \in \mathfrak{D}$ we define $\alpha - \beta$ as a cardinal or the negative of a cardinal in a very natural way:

- if α and β are finite, $\alpha - \beta$ is the difference of α and β treated as natural numbers,
- if $\alpha > \beta$ and α is infinite, $\alpha - \beta := \alpha$,
- if $\alpha < \beta$ and β is infinite, $\alpha - \beta := -\beta$.

Additionally, for simplicity, let us agree with the following notation: $|\alpha| = |-\alpha| := \alpha$ for every cardinal α.

Now for any $A \in \mathcal{B}(\mathcal{H})$ such that $(\iota_i(A), \iota_f(A)) \in \mathfrak{D}$ let $\operatorname{ind}(A) = \iota_i(A) - \iota_f(A)$. Notice that when $\iota_i(A) = \iota_f(A) \geqslant \aleph_0$, $\operatorname{ind}(A)$ is undefined.

For every γ such that $|\gamma| \leqslant \dim \mathcal{H}$ denote by $\operatorname{Ind}_\gamma(\mathcal{H})$ the set of all operators $A \in \mathcal{B}(\mathcal{H})$ for which $\operatorname{ind}(A)$ is defined and $\operatorname{ind}(A) = \gamma$, and let $\overline{\operatorname{Ind}}_\gamma(\mathcal{H})$ be the closure of $\operatorname{Ind}_\gamma(\mathcal{H})$. Finally, let $\operatorname{Uind}(\mathcal{H})$ stand for the set of all operators for which ind is undefined. Recall also (see Remark 3.10) that $\iota_m(A) = \min(\iota_i(A), \iota_f(A))$.

We leave this as a simple exercise that $\iota_b(A) = 1$ for $A \in \mathcal{B}(\mathcal{H}) \setminus \operatorname{Uind}(\mathcal{H})$.

The following is a reformulation of Theorem 3.9:

5.1. **Corollary.** *Let $A \in \mathcal{B}(\mathcal{H})$.*

(I) *If $A \in \operatorname{Uind}(\mathcal{H})$, $\overline{\mathcal{O}}{}^G_G(A)$ is the set of all $B \in \operatorname{Uind}(\mathcal{H})$ such that:*
  - (i) *$\iota_r(B) \leqslant \iota_r(A)$; and $\iota_r(B) < \iota_r(A)$ provided $\iota_b(A) = 0$ and $\iota_b(B) = 1$,*
  - (ii) *$\iota_m(B) \geqslant \iota_m(A)$.*

(II) *If $A \notin \operatorname{Uind}(\mathcal{H})$ and $\gamma := \operatorname{ind}(A)$, $\overline{\mathcal{O}}{}^G_G(A)$ is the set of all $B \in \operatorname{Ind}_\gamma(\mathcal{H})$ for which $\iota_r(B) \leqslant \iota_r(A)$ and $\iota_m(B) \geqslant \iota_m(A)$ and of all $C \in \operatorname{Uind}(\mathcal{H})$ such that $\iota_r(C) \leqslant \iota_r(A)$ and $\iota_m(C) \geqslant |\gamma|$.*

The main result of the section is

5.2. **Theorem.** *For every γ with $|\gamma| \leqslant \dim \mathcal{H}$ the set $\operatorname{Ind}_\gamma(\mathcal{H})$ is connected and open in $\mathcal{B}(\mathcal{H})$ and it coincides with the interior of its closure. The boundary of $\operatorname{Ind}_\gamma(\mathcal{H})$ is connected as well and consists precisely of all $A \in \operatorname{Uind}(\mathcal{H})$ such that $\iota_m(A) \geqslant |\gamma|$.*

Proof. Let us first show that

$$\overline{\operatorname{Ind}}_\gamma(\mathcal{H}) = \operatorname{Ind}_\gamma(\mathcal{H}) \cup \{A \in \operatorname{Uind}(\mathcal{H})\colon\ \iota_m(A) \geqslant |\gamma|\}. \tag{5-1}$$

Let Z be a closed range operator which is a monomorphism or an epimorphism and for which $\operatorname{ind}(Z) = \gamma$. Notice that $\iota_r(Z) = \dim \mathcal{H}$, $\iota_m(Z) = 0$ and $\mathcal{O}^G_G(Z) \subset \operatorname{Ind}_\gamma(\mathcal{H})$. What is more, we infer from Corollary 5.1 that $\overline{\mathcal{O}}{}^G_G(Z)$ coincides with the right hand side expression of (5-1). This shows that (5-1) holds true and that $\operatorname{Ind}_\gamma(\mathcal{H})$ is connected (since $\mathcal{O}^G_G(Z)$ is connected, by the connectedness of $\mathcal{G}(\mathcal{H})$).

Further, by (5-1), $\overline{\operatorname{Ind}}_\gamma(\mathcal{H}) \cap \overline{\operatorname{Ind}}_k(\mathcal{H}) = \{B \in \operatorname{Uind}(\mathcal{H})\colon\ \iota_m(B) \geqslant |\gamma|\}$ for each integer $k \neq \gamma$ and thus the latter set is contained in the boundary of $\overline{\operatorname{Ind}}_\gamma(\mathcal{H})$. So, to end the proof, it is enough to show that $\operatorname{Ind}_\gamma(\mathcal{H})$ is open.

Fix $A \in \operatorname{Ind}_\gamma(\mathcal{H})$. Since $\iota_b(A) = 1$, by Theorem 3.8 there is a neighbourhood $\mathcal{X}$ of A such that

$$A \in \overline{\mathcal{O}}{}^G_G(X) \tag{5-2}$$

for any $X \in \mathcal{X}$. Note that $\operatorname{Uind}(\mathcal{H})$ is closed (by (5-1): $\operatorname{Uind}(\mathcal{H}) = \overline{\operatorname{Ind}}_0(\mathcal{H}) \cap \overline{\operatorname{Ind}}_1(\mathcal{H})$) and therefore we may assume that $\mathcal{X}$ is disjoint from $\operatorname{Uind}(\mathcal{H})$. But then, thanks to (5-2) and Corollary 5.1, $\operatorname{ind}(A) = \operatorname{ind}(X)$ for $X \in \mathcal{X}$ and hence $\mathcal{X} \subset \operatorname{Ind}_\gamma(\mathcal{H})$.

Finally, to show that the boundary of $\operatorname{Ind}_\gamma(\mathcal{H})$ is connected, take a closed range operator $T \in \operatorname{Uind}(\mathcal{H})$ such that $\iota_r(T) = \dim \mathcal{H}$ and $\iota_m(T) = \max(\aleph_0, |\gamma|)$ and observe, applying again Corollary 5.1, that the boundary coincides with $\overline{\mathcal{O}}{}^G_G(T)$. $\square$

The above result shows that $\operatorname{Uind}(\mathcal{H})$ is closed, nowhere dense and connected. Notice also that $\operatorname{Ind}_\gamma(\mathcal{H})$ consists of closed range operators iff $\gamma \in \mathbb{Z} \cup \{-\aleph_0, \aleph_0\}$. So, in case of a nonseparable Hilbert space $\mathcal{H}$ we may define a *semi-Fredholm* operator on $\mathcal{H}$ as a bounded operator $A \notin \operatorname{Uind}(\mathcal{H})$ such that $\operatorname{ind}(A) \in \mathbb{Z} \cup \{-\aleph_0, \aleph_0\}$. Under such a definition, semi-Fredholm operators automatically have closed ranges. Observe also that in a separable Hilbert space the class Uind coincides with the class of all non-semi-Fredholm operators. So, Theorem 5.2 generalizes the result of Mbekhta [4].

With use of Corollary 5.1 we are also able to show

5.3. **Proposition.** *For each γ with $|\gamma| \leqslant \dim \mathcal{H}$ and a positive cardinal $\mathfrak{m}$ the set $\operatorname{Ind}_\gamma^{\mathfrak{m}}(\mathcal{H})$ of all $A \in \operatorname{Ind}_\gamma(\mathcal{H})$ for which $\iota_m(A) < \mathfrak{m}$ is open and dense in $\operatorname{Ind}_\gamma(\mathcal{H})$.*

Proof. First of all note that if $\mathfrak{m} \geqslant \max(\aleph_0, |\gamma|)$, then $\operatorname{Ind}_\gamma^{\mathfrak{m}}(\mathcal{H}) = \operatorname{Ind}_\gamma(\mathcal{H})$. So, we may assume that

$$\mathfrak{m} < \max(\aleph_0, |\gamma|). \tag{5-3}$$

Let Z be as in the proof of Theorem 5.2. Observe that $\mathcal{O}_G^G(Z) \subset \operatorname{Ind}_\gamma^{\mathfrak{m}}(\mathcal{H})$ and therefore the latter set is dense in $\operatorname{Ind}_\gamma(\mathcal{H})$. What is more, thanks to (5-3), there exists a closed range operator $Z_{\mathfrak{m}} \in \mathcal{B}(\mathcal{H}) \cap \operatorname{Ind}_\gamma(\mathcal{H})$ such that $\iota_m(Z_{\mathfrak{m}}) = \mathfrak{m}$ and $\iota_r(Z_{\mathfrak{m}}) = \dim \mathcal{H}$. Now by Corollary 5.1, $\operatorname{Ind}_\gamma^{\mathfrak{m}}(\mathcal{H}) = \operatorname{Ind}_\gamma(\mathcal{H}) \setminus \overline{\mathcal{O}}_G^G(Z_{\mathfrak{m}})$ which finishes the proof. □

5.4. **Corollary.** *The closure of $\mathcal{O}_G^G(A)$ has nonempty interior iff A or A^* is an epimorphism.*

Proof. The sufficiency is clear ($\mathcal{O}_G^G(A)$ is open provided A or A^* is an epimorphism). To see the necessity, first note that the nonemptiness of the interior of $\overline{\mathcal{O}}_G^G(A)$ implies that $A \notin \operatorname{Uind}(\mathcal{H})$, and it suffices to show that $\mathfrak{m} := \iota_m(A) = 0$ (the latter condition is equivalent to the epimorphicity of A or A^*). Suppose, for the contrary that $\mathfrak{m} > 0$. Then $\mathcal{O}_G^G(A) \subset \overline{\operatorname{Ind}}_\gamma(\mathcal{H}) \setminus \operatorname{Ind}_\gamma^{\mathfrak{m}}(\mathcal{H})$ where $\gamma = \operatorname{ind}(A)$. Now it follows from Proposition 5.3 that $\mathcal{O}_G^G(A)$ is nowhere dense. A contradiction. □

5.5. *Remark.* The index ind may clearly be defined by the same formula in spaces $\mathcal{B}(\mathcal{H}, \mathcal{K})$. All the results of the section have their (natural) counterparts in such spaces when $\dim \mathcal{H} = \dim \mathcal{K} \geqslant \aleph_0$, that is, when $(\dim \mathcal{H}, \dim \mathcal{K}) \notin \mathfrak{D}$. In the opposite, when $(\dim \mathcal{H}, \dim \mathcal{K}) \in \mathfrak{D}$, one may easily prove (using (P3)) that $(\iota_i(X), \iota_f(X)) \in \mathfrak{D}$ and $\operatorname{ind}(X) = \dim \mathcal{H} - \dim \mathcal{K}$ for every $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. So, the restriction (in this section) of our investigations to operators acting on a one Hilbert space was reasonable and justified.

6. One-sided actions

6.1. **Theorem.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.*

(a) *$\overline{\mathcal{O}}^U(A)$ is the set of all $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that*

$$B^*B = A^*A \quad \text{and} \quad \iota_f(B) = \iota_f(A). \tag{6-1}$$

(b) $\overline{\mathcal{O}}_U(A) = \{B \in \mathcal{B}(\mathcal{H},\mathcal{K})\colon\ BB^* = AA^* \text{ and } \iota_i(B) = \iota_i(A)\}$.

Proof. (a): First of all, note that $B \in \overline{\mathcal{O}}^U(A)$ iff $A \in \overline{\mathcal{O}}^U(B)$. So, thanks to Corollary 3.11, $\iota_f(B) = \iota_f(A)$ for $B \in \overline{\mathcal{O}}^U(A)$. It is also clear that $B^*B = A^*A$ for such B.

Conversely, take B satisfying (6-1) and put $D = |A|$ $(= |B|)$. Fix $\varepsilon > 0$ and take a subspace $V \in \Upsilon(D)$ contained in $\mathcal{R}(D)$ such that

$$\|D - DP_V\| \leqslant \varepsilon \tag{6-2}$$

(cf. Proposition 3.4). Since $D(V)$ is a complete subspace of $\mathcal{R}(D)$, $\dim(\overline{\mathcal{R}}(D) \ominus D(V)) \geqslant \mathrm{IC}(\mathcal{R}(D))$ and thus there are mutually orthogonal closed subspaces E and F of $\mathcal{H}$ such that $\overline{\mathcal{R}}(D) \ominus D(V) = E \oplus F$ and $\dim E = \mathrm{IC}(\mathcal{R}(D))$. Recall that $\mathrm{IC}(\mathcal{R}(A)) = \mathrm{IC}(\mathcal{R}(D)) = \mathrm{IC}(\mathcal{R}(B))$. Let $A = Q_A D$ and $B = Q_B D$ be the polar decompositions of A and B, respectively. Observe that $Q_A(D(V) \oplus E \oplus F) = A(V) \oplus Q_A(E) \oplus Q_A(F)$ and hence $\dim(\mathcal{K} \ominus A(V)) = \dim F + \iota_f(A)$. For the same reason, $\dim(\mathcal{K} \ominus B(V)) = \dim F + \iota_f(B)$. So, by (6-1), there is a unitary operator U_0 of $\mathcal{K} \ominus A(V)$ onto $\mathcal{K} \ominus B(V)$. Now it suffices to define $U \in \mathcal{U}(\mathcal{K})$ by: $U = Q_B(Q_A\big|_{D(V)})^{-1}$ on $A(V)$ and $U = U_0$ on the orthogonal complement of $A(V)$. Finally we have $UA\big|_V = B|_V$ and therefore $\|UA - B\| \leqslant 2\varepsilon$ (by (6-2)).

In order to prove (b), pass to adjoint operators and apply (a). $\square$

6.2. **Corollary.** *Let $\mathcal{H}$ be a separable Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(A)$ is nonclosed. Then $\overline{\mathcal{O}}^U(A) = \{B \in \mathcal{B}(\mathcal{H})\colon\ B^*B = A^*A\}$ and $\overline{\mathcal{O}}_U(A) = \{B \in \mathcal{B}(\mathcal{H})\colon\ BB^* = AA^*\}$.*

The case of the closures of orbits $\mathcal{O}_G$ and $\mathcal{O}^G$ is much more complicated. For need of their descriptions, let us define $L_+(A)$ for $A \in \mathcal{B}_+(\mathcal{H})$ as the set of all $B \in \mathcal{B}_+(\mathcal{H})$ such that $B \leqslant cA$ for some scalar $c > 0$, and let $\mathcal{L}_+(A)$ be the closure of $L_+(A)$. By Theorem 2.1 of [2], for an operator $B \in \mathcal{B}_+(\mathcal{H})$,

$$B \in L_+(A) \iff \mathcal{R}(\sqrt{B}) \subset \mathcal{R}(\sqrt{A}), \tag{6-3}$$

iff $\sqrt{B} = \sqrt{A}T$ for some $T \in \mathcal{B}(\mathcal{H})$. Observe that the latter condition gives $B = \sqrt{A}TT^*\sqrt{A}$. Conversely, if $B = \sqrt{A}C\sqrt{A}$ for some $C \in \mathcal{B}_+(\mathcal{H})$, then $B \leqslant \|C\|A$, that is, $B \in L_+(A)$. Thus we have obtained that, whenever $A, B \in \mathcal{B}_+(\mathcal{H})$:

$$B \in L_+(A) \iff \exists C \in \mathcal{B}_+(\mathcal{H})\colon\ B = \sqrt{A}C\sqrt{A}. \tag{6-4}$$

It is clear that $\mathcal{L}_+(A)$ is a cone (i.e. $tB + sC \in \mathcal{L}_+(A)$ whenever $B, C \in \mathcal{L}_+(A)$ and $t, s \geqslant 0$). Other properties of $\mathcal{L}_+(A)$ are established in the following

6.3. **Proposition.** *Let $A \in \mathcal{B}_+(\mathcal{H})$.*

(a) *$B \in \mathcal{L}_+(A) \implies \overline{\mathcal{R}}(B) \subset \overline{\mathcal{R}}(A)$.*

(b) *$B \in \mathcal{L}_+(A) \implies \mathcal{L}_+(B) \subset \mathcal{L}_+(A)$.*

(c) *Let* $B \in \mathcal{B}_+(\mathcal{H})$ *and* $E\colon \mathfrak{B}(\mathbb{R}_+) \to \mathcal{B}(\mathcal{H})$ *be the spectral measure of* B. *Then* $B \in \mathcal{L}_+(A)$ *iff* $E([\frac{1}{n}, +\infty)) \in \mathcal{L}_+(A)$ *for each* $n \geqslant 1$.
(d) *Let* P *be an orthogonal projection.* $P \in \mathcal{L}_+(A)$ *iff there is a sequence* $P_1, P_2, \ldots$ *of orthogonal projections which converge to* P *and whose ranges are contained in the range of* $\sqrt{A}$.
(e) *If* $B \in \mathcal{B}_+(\mathcal{H})$ *is compact and* $\mathcal{R}(B) \subset \overline{\mathcal{R}}(A)$, *then* $B \in \mathcal{L}_+(A)$.

Proof. The point (a) follows from (6-4) and the connection $\overline{\mathcal{R}}(A) = \overline{\mathcal{R}}(\sqrt{A})$.

(b): Suppose $B = \lim_{n\to\infty} B_n$ with $B_n \in L_+(A)$. Then $\sqrt{B} = \lim_{n\to\infty} \sqrt{B_n}$. Now if $C \in L_+(B)$, $C = \sqrt{B}D\sqrt{B}$ for some $D \in \mathcal{B}_+(\mathcal{H})$ (by (6-4)). So, $C = \lim_{n\to\infty} \sqrt{B_n}D\sqrt{B_n}$. But (again by (6-4)) $\sqrt{B_n}D\sqrt{B_n} \in L_+(B_n) \subset L_+(A)$. This shows that $L_+(B) \subset \mathcal{L}_+(A)$ and we are done.

(c): Let $P_n = E([\frac{1}{n}, +\infty))$. Note that $BP_n \leqslant B$, $BP_n \to B$ $(n \to \infty)$ and $\frac{1}{n}P_n \leqslant BP_n \leqslant \|B\|P_n$. So, it suffices to apply (b).

(d): The sufficiency follows from (6-3). To prove the necessity, take a sequence $A_1, A_2, \ldots \in \mathcal{L}_+(A)$ convergent to P. Let $V = \mathcal{R}(P) \in \Upsilon(P)$ and let $N \geqslant 1$ and $Z_N, Z_{N+1}, \ldots$ be as in Lemma 3.5–(a) for $\mathcal{K} := \mathcal{H}$ and $A := P$. We may assume that $N = 1$. Put $P_n = P_{A_n(V)}$. Observe that $\mathcal{R}(P_n) \subset \mathcal{R}(\sqrt{A})$ (since $A_n \in L_+(A)$ and thanks to (6-3)). Finally, $P_n = Z_n P Z_n^{-1}$ (because $Z_n(V) = A_n(V)$) and therefore $\lim_{n\to\infty} P_n = \lim_{n\to\infty}(Z_nP)(Z_nP)^* = P \cdot P^* = P$.

(e): Thanks to (c), it suffices to show that every finite rank orthogonal projection whose image is contained in $\overline{\mathcal{R}}(A)$ is a member of $\mathcal{L}_+(A)$ which we leave as a simple exercise. □

6.4. **Corollary.** *For a compact operator* $A \in \mathcal{B}_+(\mathcal{H})$, $\mathcal{L}_+(A)$ *consists of all compact operators* $B \in \mathcal{B}_+(\mathcal{H})$ *such that* $\mathcal{R}(B) \subset \overline{\mathcal{R}}(A)$.

6.5. **Example.** Let $(\mathcal{H}, \langle\cdot,-\rangle)$ be infinite-dimensional and separable, and let $(e_n)_{n=1}^\infty$ be an orthonormal basis of $\mathcal{H}$. Put $A\colon \mathcal{H}\oplus\mathcal{H} \ni (x,y) \mapsto (x, \sum_{n=1}^\infty \frac{\langle y,e_n\rangle}{2^n} e_n) \in \mathcal{H}\oplus\mathcal{H}$, $V = \{(x,y) \in \mathcal{H}\oplus\mathcal{H}\colon\ x = 0\}$ and $U\colon \mathcal{H}\oplus\mathcal{H} \ni (x,y) \mapsto (y,x) \in \mathcal{H}\oplus\mathcal{H}$. Observe that $A \in \mathcal{B}_+(\mathcal{H}\oplus\mathcal{H})$, $V \subset \bar{\mathcal{R}}(A) = \mathcal{H}\oplus\mathcal{H}$, $U \in \mathcal{U}(\mathcal{H}\oplus\mathcal{H})$ and $U(V) \subset \mathcal{R}(A)$. However, $P_V \notin \mathcal{L}_+(A)$. The example shows that if A is noncompact and the range of A is nonclosed, the description of $\mathcal{L}_+(A)$ is not so easy as stated in Corollary 6.4. This issue will be investigated elsewhere.

As the next result shows, the cones $\mathcal{L}_+(AA^*)$ and $\mathcal{L}_+(A^*A)$ play an important role in the description of $\overline{\mathcal{O}}_G(A)$ and $\overline{\mathcal{O}}^G(A)$.

6.6. **Theorem.** *Let* $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$.
(a) $\overline{\mathcal{O}}_G(A)$ *consists of precisely those* $B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ *such that* $BB^* \in \mathcal{L}_+(AA^*)$ *and*

$$\iota_i(B) = \iota_i(A) + \iota_i(B^*\big|_{\overline{\mathcal{R}}(A)}). \tag{6-5}$$

(b) $\overline{\mathcal{O}}^{G}(A)$ *consists of precisely those* $B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ *such that* $B^*B \in \mathcal{L}_+(A^*A)$ *and* $\iota_f(B) = \iota_f(A) + \iota_i(B\big|_{\overline{\mathcal{R}}(A^*)})$.

Proof. Since (b) may be infered from (a) by passing to adjoints, we only need to show (a).

First suppose that $B \in \overline{\mathcal{O}}_G(A)$. This means that $B = \lim_{n\to\infty} AG_n$ for some $G_n \in \mathcal{G}(\mathcal{H})$ and thus $BB^* = \lim_{n\to\infty} |A^*|QG_nG_n^*Q^*|A^*|$ where Q is the partial isometry appearing in the polar decomposition of A. So, it follows from (6-4) that $BB^* \in \mathcal{L}_+(AA^*)$. What is more, the latter implies that $\overline{\mathcal{R}}(B) \subset \overline{\mathcal{R}}(A)$. For simplicity, put $\mathcal{K}_0 = \overline{\mathcal{R}}(A)$ and think of A and B as members of $\mathcal{B}(\mathcal{H},\mathcal{K}_0)$. Under such a consideration, $B \in \overline{\mathcal{O}}_G^G(A)$ and hence Theorem 3.9 implies that $\iota_i(B) = \iota_i(A)+\alpha$ and $\iota_f(B) = \iota_f(A) + \alpha$ for some cardinal α where all the indices which appear in both the equations are computed in the space $\mathcal{B}(\mathcal{H},\mathcal{K}_0)$. Notice that then $\iota_f(A) = \mathrm{IC}(\mathcal{R}(A))$, so $\iota_f(B) = \mathrm{IC}(\mathcal{R}(A)) + \alpha$ and (by Proposition 2.3–(a)) $\iota_i(B) = \iota_i(A) + \mathrm{IC}(\mathcal{R}(A)) + \alpha = \iota_i(A) + \iota_f(B)$. It suffices to observe that (still in the space $\mathcal{B}(\mathcal{H},\mathcal{K}_0)$) $\iota_f(B) = \dim(\overline{\mathcal{R}}(A) \ominus \overline{\mathcal{R}}(B)) + \mathrm{IC}(\mathcal{R}(B)) = \iota_i(B^*\big|_{\overline{\mathcal{R}}(A)})$ which finally gives (6-5).

Now suppose that $BB^* \in \mathcal{L}_+(AA^*)$ and (6-5) is satisfied. As in the first part of the proof, notice that then $\overline{\mathcal{R}}(B) \subset \overline{\mathcal{R}}(A)$ and thus $\iota_i(B^*\big|_{\overline{\mathcal{R}}(A)}) = \dim(\overline{\mathcal{R}}(A) \ominus \overline{\mathcal{R}}(B)) + \mathrm{IC}(\mathcal{R}(B))$. So, (6-5) is equivalent to

$$\iota_i(B) = \iota_i(A) + \mathrm{IC}(\mathcal{R}(B)) + \dim(\overline{\mathcal{R}}(A) \ominus \overline{\mathcal{R}}(B)). \tag{6-6}$$

Fix $\varepsilon > 0$ and take $V \in \Upsilon(|B^*|)$ such that $|B^*|(V) = V$ and $\| \, |B^*| - |B^*|P_V\| \leqslant \varepsilon$. Since V is a complete subspace of $\mathcal{R}(|B^*|) = \mathcal{R}(B)$, we see that $\dim(\overline{\mathcal{R}}(B) \ominus V) \geqslant \mathrm{IC}(\mathcal{R}(B))$ and thus

$$\dim(\overline{\mathcal{R}}(B) \ominus V) = \dim(\overline{\mathcal{R}}(B) \ominus V) + \mathrm{IC}(\mathcal{R}(B)). \tag{6-7}$$

Let $B = Q|B|$ be the polar decompositions of B. Then also $B = |B^*|Q$. Put $W = Q^{-1}(V) \cap \overline{\mathcal{R}}(B^*)$. Note that $W \in \Upsilon(B)$, $QP_W = P_VQ$,

$$B(W) = V \qquad \text{and} \qquad \|B - BP_W\| < \varepsilon. \tag{6-8}$$

Further, since $BB^* \in \mathcal{L}_+(AA^*)$, by (6-4), there is a sequence $T_1, T_2, \ldots$ of bounded nonnegative operators on $\mathcal{K}$ such that

$$BB^* = \lim_{n\to\infty} |A^*|T_n|A^*|.$$

We conclude from the relations $V \in \Upsilon(|B^*|)$ and $|B^*|(V) = V$ that $V \in \Upsilon(BB^*)$ as well and therefore, thanks to Lemma 3.5–(a), after omitting finitely many entries of $(|A^*|T_n|A^*|)_{n=1}^{\infty}$, there is a sequence $(Z_n)_{n=1}^{\infty}$ of unitary operators on $\mathcal{K}$ such that

$$Z_n(V) = V_n \text{ for each } n \tag{6-9}$$

where $V_n := |A^*|T_n|A^*|(V)$ is a closed subspace of $\mathcal{K}$, and

$$Z_nP_V \to P_V \ (n \to \infty). \tag{6-10}$$

What is more, since we might restrict our argument (when taking Z_n) to $\overline{\mathcal{R}}(A)$ (and work in $\mathcal{B}(\mathcal{H}, \overline{\mathcal{R}}(A))$ and $\mathcal{U}(\overline{\mathcal{R}}(A))$), we may also assume that

$$Z_n(\overline{\mathcal{R}}(A)) = \overline{\mathcal{R}}(A) \text{ for every } n. \tag{6-11}$$

Observe that $V_n \subset \mathcal{R}(|A^*|) = \mathcal{R}(A)$ and thus $W_n \in \Upsilon(A)$ where $W_n := A^{-1}(V_n) \cap \overline{\mathcal{R}}(A^*)$ and

$$\dim(\overline{\mathcal{R}}(A) \ominus V_n) = \dim(\overline{\mathcal{R}}(A) \ominus V_n) + \mathrm{IC}(\mathcal{R}(A)) \tag{6-12}$$

(compare the proof of (6-7)). We have:

$$A(W_n) = V_n \tag{6-13}$$

and, by (6-9),

$$\dim W_n = \dim V_n = \dim Z_n(V) = \dim V = \dim W. \tag{6-14}$$

Now Lemma 3.3 (applied twice) combined with (6-13), (6-12), (6-11), (6-9), (6-7) (twice) and (6-6) yields

$$\begin{aligned}
\dim(\mathcal{H} \ominus W_n) &= \dim \mathcal{N}(A) + \dim(\overline{\mathcal{R}}(A^*) \ominus W_n) \\
&= \dim \mathcal{N}(A) + \dim(\overline{\mathcal{R}}(A) \ominus V_n) \\
&= \dim \mathcal{N}(A) + \mathrm{IC}(\mathcal{R}(A)) + \dim(Z_n(\overline{\mathcal{R}}(A)) \ominus Z_n(V)) \\
&= \iota_i(A) + \dim(\overline{\mathcal{R}}(A) \ominus V) \\
&= \iota_i(A) + \dim(\overline{\mathcal{R}}(A) \ominus \overline{\mathcal{R}}(B)) + \dim(\overline{\mathcal{R}}(B) \ominus V) \\
&= \iota_i(A) + \dim(\overline{\mathcal{R}}(A) \ominus \overline{\mathcal{R}}(B)) + \mathrm{IC}(\mathcal{R}(B)) + \dim(\overline{\mathcal{R}}(B) \ominus V) \\
&= \iota_i(B) + \dim(\overline{\mathcal{R}}(B) \ominus V) = \dim \mathcal{N}(B) + \mathrm{IC}(\mathcal{R}(B)) + \dim(\overline{\mathcal{R}}(B) \ominus V) \\
&= \dim \mathcal{N}(B) + \dim(\overline{\mathcal{R}}(B) \ominus V) = \dim \mathcal{N}(B) + \dim(\overline{\mathcal{R}}(B^*) \ominus W) \\
&= \dim(\mathcal{H} \ominus W).
\end{aligned}$$

The above connection and (6-14) imply that there is $U_n \in \mathcal{U}(\mathcal{H})$ for which $U_n(W) = W_n$. Now define $G_n \in \mathcal{G}(\mathcal{H})$ by:

$$G_n\big|_W = (A\big|_{W_n})^{-1} Z_n B\big|_W \in \mathcal{G}(W, W_n)$$

(use (6-8), (6-9) and (6-13) to see that $G_n\big|_W$ is well defined) and $G_n\big|_{\mathcal{H}\ominus W} = \frac{1}{n} U_n\big|_{\mathcal{H}\ominus W} \in \mathcal{G}(\mathcal{H} \ominus W, \mathcal{H} \ominus W_n)$. We claim that

$$AG_n \to BP_W \ (n \to \infty). \tag{6-15}$$

Indeed, $\lim_{n\to\infty} AG_n(I_{\mathcal{H}} - P_W) = \lim_{n\to\infty} \frac{1}{n} AU_n\big|_{\mathcal{H}\ominus W} = 0 = BP_W(I_{\mathcal{H}} - P_W)$ and, thanks to (6-8) and (6-10),

$$\lim_{n\to\infty} AG_n P_W = \lim_{n\to\infty} Z_n B P_W = \lim_{n\to\infty} Z_n P_V B P_W = P_V B P_W = BP_W.$$

Finally, we infer from (6-8) and (6-15) that $\|AG_n - B\| \leqslant \varepsilon$ for some n, which finishes the proof. □

The next result has its natural counterpart for the closures of $\mathcal{O}^G$.

6.7. **Corollary.** *Let* $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$*.*

(I) $\overline{\mathcal{O}}_G(A) = \overline{\mathcal{O}}_G(B)$ *iff* $\mathcal{L}_+(AA^*) = \mathcal{L}_+(BB^*)$ *and* $\iota_i(B) = \iota_i(A)$*.*

(II) *Suppose* A *is compact.*

(a) $\overline{\mathcal{O}}_G(A)$ *constists of all compact operators* $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *such that* $\mathcal{R}(C) \subset \overline{\mathcal{R}}(A)$*.*

(b) $\overline{\mathcal{O}}_G(A) = \overline{\mathcal{O}}_G(B)$ *iff* B *is compact and* $\overline{\mathcal{R}}(B) = \overline{\mathcal{R}}(A)$*.*

Proof. The point (I) follows from Theorem 6.6 and the fact that $\overline{\mathcal{R}}(B) = \overline{\mathcal{R}}(A)$ and $\iota_i(B^*\big|_{\overline{\mathcal{R}}(B)}) = \mathrm{IC}(\mathcal{R}(B))$ provided $\mathcal{L}_+(BB^*) = \mathcal{L}_+(AA^*)$.

To see (II), it suffices to apply Corollary 6.4 after observing that when A and B are compact and $\mathcal{R}(B) \subset \overline{\mathcal{R}}(A)$, then (6-5) is fulfilled (consider separately the cases when $\iota_r(A)$ is finite; $\mathcal{H}$ is separable and $\mathcal{R}(A)$ is nonclosed; and $\mathcal{H}$ is nonseparable). $\square$

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