A NOTE ON ANR'S

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Abstract. It is shown that if for a complete metric space (X, d) there is a constant $\varepsilon > 0$ such that the intersection $\bigcap_{j=1}^{n} B_d(x_j, r_j)$ of open balls is nonempty for every finite system $x_1, \ldots, x_n \in X$ of centers and a corresponding system of radii $r_1, \ldots, r_n > 0$ such that $d(x_j, x_k) \leqslant \varepsilon$ and $d(x_j, x_k) < r_j + r_k$ $(j, k = 1, \ldots, n)$, then X is an ANR; and if in the above one may put $\varepsilon = \infty$, the space X is an AR. A certain criterion for an incomplete metric space to be an A(N)R is presented.

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1. Introduction

A metrizable space X is an *absolute (neighbourhood) retract* (briefly, an A(N)R) if every map of a closed subset A of an arbitrary metric space Y into X is extendable to a map of a neighbourhood of A (of the whole space Y) into X. It is not difficult, using Hausdorff's theorem on extending metrics [6] (for other proof see [13] or [2, Theorem 3.2]), to ensure that a metric space (X, d) is an A(N)R iff every nonexpansive map of a closed subset A of a metric space (Y, ϱ) into (X, d) is extendable to a map of a neighbourhood (of the whole space Y) into X (cf. the proof of [1, §3, Corollary 4]; see also Proposition 2.1 below). On the other hand, there is a well-known example due to van Mill [9] of a separable metrizable space X which is not an ANR, but every map of a compact subset of an arbitrary separable metric space Y into X is extendable to a map of Y into X. Thus, it is natural to expect that the property of extending nonexpansive maps of closed subsets of compact metric spaces to nonexpansive maps with full domains is also insufficient for a given metric space to be an A(N)R. Surprisingly, such a supposition is false! We shall prove in the sequel that if for a metric space (X, d) (not necessarily complete) there is a constant $\varepsilon > 0$ such that

($\star$) *every nonexpansive map $g\colon L \to X$ of a closed subset L of a finite dimensional compact metric space (K, ϱ) of diameter no greater than ε is extendable to a Lipschitz map $G\colon K \to X$ with Lipschitz constant arbitrarily close to* 1, then X is an ANR; and if $(\star)$ is fullfiled with ε equal to the diameter of (X,d), then X is an AR. As a corollary of this theorem we shall prove the result stated in Abstract.

There is an important class of metric spaces for which $(\star)$ is fulfilled with $\varepsilon = \infty$, namely the class of the so-called *hyperconvex* metric spaces, introduced by Aronszajn and Panitchpakdi [1]. These spaces satisfy $(\star)$ with K an arbitrary metric space (possibly noncompact), L not necessarily closed, and the final function G nonexpansive. However, it is already known [1] that every hyperconvex space is an AR (see the proof of Proposition 2.1 below). More on hyperconvexity the reader may find in [5].

Probably the most famous metric space which is not hyperconvex but satisfies $(\star)$ (with $\varepsilon = \infty$) is the *Urysohn universal space* $\mathbb{U}$ ([15, 16], [7], [8], [17]) uniquely determined (up to isometry) by the following three conditions: $\mathbb{U}$ is separable and complete; every separable metric space admits an isometric embedding into $\mathbb{U}$ (universality); and every isometry between two finite subsets of $\mathbb{U}$ is extendable to an isometry of $\mathbb{U}$ onto itself (ω-homogeneity). It was proved by Uspenskij [17] that $\mathbb{U}$ is homeomorphic to the Hilbert space (and thus it is an AR). Here we will present another proof of this fact, based on the Dobrowolski-Toruńczyk theorem on separable complete ANR's admitting topological group structures [4]. Our proof that $\mathbb{U}$ is an AR is, more or less, based on an idea similar to Uspenskij's original proof. This fact, however, is an immediate consequence of the theorem stated in Abstract.

Notation and terminology. In this paper all topological spaces are metrizable. By a *map* we mean a continuous function. The least Lipschitz constant of a Lipschitz map f between metric spaces is denoted by $\operatorname{Lip}(f)$. The map f is *nonexpansive* iff $\operatorname{Lip}(f) \leqslant 1$.

The open ball in the metric space (X,d) with center at $x \in X$ and of radius $r > 0$ is denoted by $B_d(x,r)$. The diameter of a subset A of (X,d) (with respect to d) is denoted by $\operatorname{diam}_d(A)$; and $\partial(A)$ stands for the closure and the boundary of A (in X).

A topological space X is *homotopically trivial* if for every $n \geqslant 1$, each map $f\colon \partial([0,1]^n) \to X$ is extendable to a map $F\colon [0,1]^n \to X$. The empty space is homotopically trivial.

2. Main results

Let us begin with a simple and well-known

2.1. **Proposition.** *For a metric space (X,d) the following conditions are equivalent:*

(i) *X is an ANR (AR),*
(ii) *every nonexpansive map $f\colon A \to X$ of a closed subset of an arbitrary metric space (Y,ϱ) is extendable to a map $F\colon U \to X$ with $U \supset A$ open in Y (with $U = Y$).*

Proof. Of course, we only need to check that (i) is implied by (ii). To see this, let (Y,λ) be a metric space, A its closed subset and $f\colon A \to X$ a map. Let $\varrho_0\colon A \times A \to [0,\infty)$ be given by $\varrho_0(a,b) = \max(\lambda(a,b), d(f(a),f(b))$. Then ϱ_0 is a metric on A equivalent to $\lambda\big|_{A\times A}$ and f is nonexpansive as a map of (A,ϱ_0) into (X,d). By Hausdorff's theorem [6], there is a metric ϱ on Y which extends ϱ_0 and is equivalent to λ. Now it suffices to apply (ii) to get the assertion. □

Let $\mathfrak{M}$ be a class of metric spaces and let $M \in [0,\infty]$. A metric space (X,d) is said to be an *almost contractive extensor* for the pair $(\mathfrak{M}, M)$ (briefly, $(X,d) \in \mathrm{ACE}(\mathfrak{M}, M)$) iff every nonexpansive map $f\colon A \to X$ defined on a closed subset of $(Y,\varrho) \in \mathfrak{M}$ with $\mathrm{diam}_\varrho(Y) \leqslant M$ is extendable to a Lipschitz map $F\colon Y \to X$ with $\mathrm{Lip}(F)$ arbitrarily close to 1. If $(X,d) \in \mathrm{ACE}(\mathfrak{M},\infty)$, we shall write $(X,d) \in \mathrm{ACE}(\mathfrak{M})$. For simplicity, we shall write $(X,d) \in \mathrm{ACE}(M)$ ($(X,d) \in \mathrm{ACE}$) provided $(X,d) \in \mathrm{ACE}(\mathfrak{M}, M)$ ($(X,d) \in \mathrm{ACE}(\mathfrak{M})$) with $\mathfrak{M}$ being the class of all finite dimensional compact metric spaces, and $(X,d) \in \mathrm{ACE}_{loc}$ if $(X,d) \in \mathrm{ACE}(\varepsilon)$ for some $\varepsilon > 0$. Further, we denote by $\mathfrak{F}$ the class of all finite metric spaces.

The main tool of this paper is the following theorem due to Toruńczyk [14] (compare with [10, Corollary 4.2.18]).

2.2. **Theorem.** *A metrizable space is an ANR provided it has an (open) base β such that for every finite nonempty subfamily β_0 of β the set $\bigcap \beta_0$ is homotopically trivial.*

It is also well known (see e.g. [10, Theorem 4.2.20]) that a homotopically trivial nonempty ANR is an AR.

Now we are ready to prove

2.3. **Theorem.** *Let (X,d) be a metric space and $M = \mathrm{diam}_d(X)$. If $(X,d) \in \mathrm{ACE}_{loc}$ (respectively $(X,d) \in \mathrm{ACE}(M)$), then X is an ANR (an AR).*

Proof. We assume $M > 0$. If $(X,d) \in \mathrm{ACE}(M)$, put $\varepsilon = \infty$; otherwise take $\varepsilon \in (0,M)$ such that $(X,d) \in \mathrm{ACE}(\varepsilon)$. Notice that $(X,d) \in \mathrm{ACE}(\min(M,\varepsilon))$. Let $\beta = \{B_d(x,r)\colon\ 0 < r < \varepsilon/2\}$. Thanks to Theorem 2.2, it suffices to show that for every finite nonempty subfamily β_0 of β the set $\bigcap \beta_0$ is homotopically trivial (indeed, if $\varepsilon = \infty$, then every open ball will be homotopically trivial and thus so will be X). Let $u_1,\ldots,u_n \in X$ and $R_1,\ldots,R_n \in (0,\varepsilon/2)$, and assume that $C := \bigcap_{j=1}^n B_d(u_j,R_j)$ is nonempty. Fix $u_0 \in C$. We shall show that C is homotopically trivial. For this, let $m \geqslant 1$ and let $f\colon \partial([-1,1]^m) \to C$ be any map. Since the set $K := f(\partial([-1,1]^m)) \cup \{u_0\}$ is a compact subset of C, there are real numbers $r_j \in (0,R_j)$ ($j = 1,\ldots,n$) such that $K \subset B := \bigcap_{j=1}^n B_d(u_j,r_j)$. Notice that

$$\mathrm{diam}_d(B \cup \{u_1,\ldots,u_n\}) \leqslant \min(M,\varepsilon). \tag{2-1}$$

Let $a_0 = 0 \in [-1,1]^m$. Take distinct points $a_1, \ldots, a_n \in (0,1)^m$ and put $A = \partial([-1,1]^m) \cup \{a_0, a_1, \ldots, a_n\}$ and $f(a_j) = u_j$ for $j = 0, 1, \ldots, n$. Observe that $f\colon A \to X$ is continuous. Let ϱ_0 be any metric (compatible with the topology) on $[-1,1]^m$ such that for every $x, y \in [-1,1]^m$,

$$\varrho_0(x,y) \leqslant \min(r_1, \ldots, r_n, M, \varepsilon). \tag{2-2}$$

Define a new metric ϱ on A by $\varrho(x,y) = \max(\varrho_0(x,y), d(f(x), f(y)))$. Observe that

$$f\colon (A, \varrho) \to (X, d) \text{ is nonexpansive} \tag{2-3}$$

and $\operatorname{diam}_\varrho(A) \leqslant \min(\varepsilon, M)$ (thanks to (2-1) and (2-2)). Now let $L[A]$ consists of all functions $v\colon [0,1) \to A$ for which there is $t \in [0,1]$ and $x \in A$ such that

$$v\big|_{[0,t)} \equiv x \qquad \text{and} \qquad v\big|_{[t,1)} \equiv a_0 \tag{2-4}$$

(with convention that $[s,s) = \varnothing$). Let λ be a metric on $L[A]$ induced by ϱ, that is, $\lambda(v,w) = \int_0^1 \varrho(v(t), w(t))\,\mathrm{d}t$. For every $x \in X$ let $\widehat{x} \in L[A]$ be the function constantly equal to x. For $t \in [0,1]$ and $x \in A$ let '$t * x$' denote the function $v \in L[A]$ satisfying (2-4). We leave these as simple exercises that

(L0) $t * \widehat{a_0} = \widehat{a_0}$, $0 * \widehat{x} = \widehat{a_0}$ and $1 * \widehat{x} = \widehat{x}$ for every $t \in [0,1]$ and $x \in A$,
(L1) the function $[0,1] \times A \ni (t,x) \mapsto t * \widehat{x} \in L[A]$ is a continuous surjection,
(L2) the map $(0,1] \times (A \setminus \{a_0\}) \ni (t,x) \mapsto t * \widehat{x} \in L[A] \setminus \{\widehat{a_0}\}$ is a homeomorphism,
(L3) the map $(A, \varrho) \in x \mapsto \widehat{x} \in (L[A], \lambda)$ is isometric.

Now it follows from (L1) that $L[A]$ is compact and from (L2) that $L[A]$ is finite dimensional (in fact, $L[A]$ is the topological cone over $A \setminus \{a_0\}$ with vertex at $\widehat{a_0}$). Moreover, $\operatorname{diam}_\lambda(L[A]) = \operatorname{diam}_\varrho(A) \leqslant \min(\varepsilon, M)$.

Let $\delta = \min(R_1/r_1, \ldots, R_n/r_n) - 1 > 0$. Thanks to (2-3) and (L3) (and since $(X,d) \in \mathrm{ACE}(\min(M,\varepsilon))$), there is a Lipschitz map $\widehat{f}\colon L[A] \to X$ such that $\operatorname{Lip}(\widehat{f}\,) \leqslant 1 + \delta$ and for every $x \in A$,

$$\widehat{f}(\widehat{x}) = f(x). \tag{2-5}$$

Define $F\colon [-1,1]^m \to X$ by $F(a_0) = u_0 = f(a_0)$ and $F(x) = \widehat{f}(|x| * \widehat{x'})$ for $x \neq a_0 (= 0)$ where $|\cdot|_\infty$ is the maximum norm on $\mathbb{R}^m$ and $x' = x/|x| \in A$. By (2-5), $F(x) = f(x)$ for $x \in \partial([-1,1]^m)$. It is also clear that F is continuous, thanks to (L0) and (L1). It remains to show that $F([-1,1]^m) \subset B$. Fix $j \in \{1, \ldots, n\}$, $t \in (0,1]$ and $x \in \partial([-1,1]^m)$. It is enough to check that $d(\widehat{f}(t * \widehat{x}), u_j) < R_j$. By the definition of λ and by (2-5):

$$\begin{aligned} d(\widehat{f}(t * \widehat{x}), u_j) = d(\widehat{f}(t * \widehat{x}), \widehat{f}(\widehat{a_j})) &\leqslant (1+\delta)\lambda(t * \widehat{x}, \widehat{a_j}) \\ &= (1+\delta)[t\varrho(x, a_j) + (1-t)\varrho(a_0, a_j)]. \end{aligned}$$

So, taking into account the formula for δ, is suffices to show that $\varrho(x, a_j) < r_j$ and $\varrho(a_0, a_j) < r_j$. By (2-2) (and the definition of ϱ), we only need to check that $d(f(x), f(a_j)) < r_j$ and $d(f(a_0), f(a_j)) < r_j$. But these are fulfilled since $f(a_j) = u_j$ and both $f(x)$ and $f(a_0)$ belong to B. □

Uspenskij in [17] used a very similar technique to show that the Urysohn space $\mathbb{U}$ is an AR.

It is worth while to notice that in the above proof we only needed to extend a nonexpansive map defined on a space homeomorphic to the disjoint union of a sphere and a finite set. So, the condition $(X, d) \in \mathrm{ACE}_{loc}$ may be weaken.

2.4. **Example.** Let $X = [0, 2)$ and d be a metric on X given by $d(x, y) := \min(|x - y|, 2 - |x - y|)$. It may be easily shown that the function $X \ni t \mapsto e^{\pi i t} \in \mathbb{T} := \{z \in \mathbb{C}\colon\ |z| = 1\}$ is a homeomorphism and thus X is an ANR, but not an AR. We see that $\mathrm{diam}_d(X) = 1$. One may show that if $K \subset X$ is such that $\mathrm{diam}_d(K) < 1$, then K is contained in a set $J \subset X$ which is isometric (when equipped with the metric inherited from X) to $[0, 1]$. This implies that $(X, d) \in \mathrm{ACE}(\varepsilon)$ for every $\varepsilon \in [0, 1)$. However, since X is not an AR, $(X, d) \notin \mathrm{ACE}(1)$.

It turns out that in case of a complete metric space (X, d) the assumption of Theorem 2.3 may be weaken. Namely,

2.5. **Theorem.** *For a complete metric space (X, d) and $M \in (0, \infty]$ the following conditions are equivalent:*

(i) $(X, d) \in \mathrm{ACE}(t)$ *for every* $t < M$,
(ii) $(X, d) \in \mathrm{ACE}(\mathfrak{F}, t)$ *for every* $t < M$,
(iii) *whenever (Y, ϱ) is a separable metric space of diameter less than M, K is its compact subset and $f\colon K \to X$ is a nonexpansive map, there is a Lipschitz map $F\colon Y \to X$ extending f such that* $\mathrm{Lip}(F)$ *is arbitrarily close to* 1,
(iv) *whenever $x_1, \dots, x_n \in X$ and $r_1, \dots, r_n > 0$ are such that*

$$d(x_j, x_k) < M \qquad \text{and} \qquad d(x_j, x_k) < r_j + r_k$$

for $j, k = 1, \dots, n$, the set $\bigcap_{j=1}^n B_d(x_j, r_j) \neq \varnothing$.

Proof. Clearly, (i) implies (ii) and (i) follows from (iii). The equivalence of (ii) and (iv) is left as an exercise (cf. the proof of Proposition 3.1 below). We shall prove the most important part, i.e. that (iii) follows from (ii).

Suppose $(X, d) \in \mathrm{ACE}(\mathfrak{F}, t)$ for $t < M$ and (Y, ϱ), K and f are as in (iii). Fix $\varepsilon \in (0, 1)$ with

$$(1 + \varepsilon)\,\mathrm{diam}_\varrho(Y) < M. \tag{2-6}$$

We assume $K \neq Y$. Let $\{y_n\colon\ n \geqslant 1\}$ be a dense subset of $Y \setminus K$. Let $(\varepsilon_n)_{n=1}^{\infty}$ be a sequence of real numbers such that for all n,

$$0 < \varepsilon_n < \varepsilon_{n+1} < \varepsilon. \tag{2-7}$$

For every $n \geqslant 1$ define

$$\mu_n = \min(\operatorname{dist}_\varrho(y_1, K), \dots, \operatorname{dist}_d(y_n, K)) > 0$$

where $\operatorname{dist}_\varrho(y, K)$ denotes the distance of a point y from K. Let $(\delta_n)_{n=1}^{\infty}$ be a sequence of real numbers such that

$$0 < \delta_n < \frac{(\varepsilon_{n+1} - \varepsilon_n)\mu_n}{3} \tag{2-8}$$

and $\lim_{n\to\infty} \delta_n = 0$. Further, for each n let $A_n \subset K$ be a finite δ_n-net for K. We assume that $A_n \subset A_{n+1}$. For simplicity, put $B_n = A_n \cup \{y_1, \dots, y_n\}$.

We shall construct a sequence of Lipschitz maps $f_n\colon B_n \to X$ ($n = 1, 2, 3, \dots$) such that

(1_n) f_n coincides with f on A_n,
(2_n) if $n > 1$, f_n extends f_{n-1},
(3_n) $\operatorname{Lip}(f_n) \leqslant 1 + \varepsilon_n$.

The existence of f_1 follows from (ii). Suppose f_n is constructed. Define $f'_n\colon K \cup \{y_1, \dots, y_n\} \to X$ by: $f'_n(a) = f(a)$ for $a \in K$ and $f'_n(y_j) = f_n(y_j)$ for $j = 1, \dots, n$. We claim that

$$\operatorname{Lip}(f'_n) < 1 + \varepsilon_{n+1}. \tag{2-9}$$

Since f is nonexpansive and thanks to (2-7) and (3_n), it suffices to show that $d(f'_n(b), f(y_j)) \leqslant (1+\varepsilon'_n)\varrho(b, y_j)$ for $b \in K$ and $j \in \{1, \dots, n\}$ with $\varepsilon'_n < \varepsilon_{n+1}$. Take $a \in A_n$ for which $\varrho(a, b) \leqslant \delta_n$ and observe that (by (L1)):

$$\begin{aligned} d(f'_n(b), f'_n(y_j)) &\leqslant d(f(b), f(a)) + d(f_n(a), f_n(y_j)) \\ &\leqslant \varrho(a, b) + (1 + \varepsilon_n)\varrho(a, y_j) \leqslant \delta_n + (1 + \varepsilon_n)(\varrho(b, y_j) + \delta_n) \\ &\leqslant 3\delta_n + (1 + \varepsilon_n)\varrho(b, y_j) \leqslant \frac{3\delta_n}{\mu_n} \operatorname{dist}_\varrho(y_j, K) + (1 + \varepsilon_n)\varrho(b, y_j) \\ &\leqslant \Big(1 + \frac{3\delta_n}{\mu_n} + \varepsilon_n\Big)\varrho(b, y_j) =: (1 + \varepsilon'_n)\varrho(b, y_j). \end{aligned}$$

Now note that the above ε'_n is less than ε_{n+1}, thanks to (2-8). This shows (2-9).

Now put $c = \max(1, \operatorname{Lip}(f'_n))$ and $\varrho' := c\varrho$. We see that f'_n is nonexpansive with respect to ϱ'. What is more, (2-6), (2-7) and (2-9) imply that $\operatorname{diam}_{\varrho'}(Y) < M$. Consequently, according to (ii), there is $f_{n+1}\colon B_{n+1} \to X$ which coincides with f'_n on $B_n \cup A_{n+1}$ and

$$d(f_{n+1}(a), f_{n+1}(b)) \leqslant (1 + \varepsilon_{n+1})c^{-1}\varrho'(a, b)$$

for $a, b \in B_{n+1}$. It is clear that conditions (1_{n+1}), (2_{n+1}) and (3_{n+1}) are fulfilled.

To this end, let $D = \bigcup_{n=1}^{\infty} B_n$ and $F_0 = \bigcup_{n=1}^{\infty} f_n \colon D \to X$. Then F_0 is Lipschitz, $\mathrm{Lip}(F_0) \leqslant 1 + \varepsilon$ and F_0 coincides with f on $D \cap K$ which is dense in K. Since X is complete and D is dense in Y, F_0 admits a (unique) extension to a Lipschitz map $F \colon Y \to X$ with $\mathrm{Lip}(F) \leqslant 1+\varepsilon$ which necessarily extends f and we are done. □

As a consequence of Theorem 2.3 and Theorem 2.5 we obtain the main result of the paper

2.6. **Theorem.** *Let (X, d) be a complete metric space. If X satisfies condition* (iv) *of Theorem* 2.5 *with some $M > 0$ (respectively with $M = \infty$), then X is an ANR (an AR).*

Let us note that a (possibly incomplete) metric space fulfills condition (iv) of Theorem 2.5 with some $M > 0$ iff its (arbitrarily chosen) dense subset does so. We infer from this that in Theorem 2.6 we cannot omit the assumption of the completeness of the metric. The space of irrational numbers (with natural metric) is a simple counterexample (of a completely metrizable space) for this.

The reader will easily check that condition (iv) of Theorem 2.5 with $M = \infty$ is equivalent to hyperconvexity for a compact metric space X. And since the property of being an ANR is local, Theorem 2.6 in the version for ANR's remains true for an arbitrary locally compact metric space X (with possibly incomplete metric).

3. Urysohn universal space

Let us shortly prove that a Urysohn universal space is homeomorphic to the Hilbert space. This was first proved by Uspenskij [17]. (For more information on the topology of $\mathbb{U}$ see [12].)

Recall that a *Katětov map* on a metric space (X, d) is a function $f \colon X \to [0, \infty)$ such that $|f(x) - f(y)| \leqslant d(x, y) \leqslant f(x) + f(y)$ for all $x, y \in X$. Katětov maps corresponds to one-point extensions of metric spaces. A fundamental result on Urysohn space, due to Urysohn [15, 16] (see also [7] or [8]), states that a separable complete metric space (X, d) is Urysohn space iff for every Katětov map $f \colon A \to [0, \infty)$ defined on a finite nonempty subset A of X there is $x \in X$ for which $f(a) = d(x, a)$ for all $a \in A$.

Cameron and Vershik [3] have shown that the Urysohn universal space admits a topological group structure. (For other result in this direction see [11]). This is all we need to know in order to prove

3.1. **Proposition.** *The Urysohn universal space is homeomorphic to the Hilbert space.*

Proof. We have just mentioned that $\mathbb{U}$ is homeomorphic to a topological group. Since $\mathbb{U}$ is universal for separable metric spaces, it is non-locally compact. So, thanks to the result of Dobrowolski and Toruńczyk [4], it suffices to show that $\mathbb{U}$ is an AR. We will show this, applying

Theorem 2.6. Let d denote the metric of $\mathbb{U}$ and let $x_1, \ldots, x_n \in \mathbb{U}$ and $r_1, \ldots, r_n > 0$ be such that $d(x_j, x_k) < r_j + r_k$ for all j and k. Choose $s_j \in (0, r_j)$ in such a way that

$$d(x_j, x_k) \leqslant s_j + s_k. \tag{3-1}$$

Now put $A = \{x_1, \ldots, x_n\}$ and define $f\colon A \to [0, \infty)$ by $f(a) = \min\{s_j + d(a, x_j)\colon\ j = 1, \ldots, n\}$. The map f, as a minimum of nonexpansive functions, is nonexpansive as well. We see that $f(x_j) \leqslant s_j$. What is more, by (3-1), f is a Katětov map. So, there is $z \in \mathbb{U}$ such that $d(z, x_j) = f(x_j)$. But then $z \in \bigcap_{j=1}^{n} B_d(x_j, r_j)$ and we are done. □

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