PROBLEM WITH ALMOST EVERYWHERE EQUALITY

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ABSTRACT. A topological space Y is said to have (AEEP) if the following condition is fulfilled. Whenever $(X, \mathfrak{M})$ is a measurable space and $f, g\colon X \to Y$ are two measurable functions, then the set $\Delta(f,g) = \{x \in X\colon\ f(x) = g(x)\}$ is a member of $\mathfrak{M}$. It is shown that a metrizable space Y has (AEEP) iff the cardinality of Y is no greater than $2^{\aleph_0}$.

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1. Introduction

In several aspects of mathematics dealing with measurability (such as measure theory, descriptive set theory, stochastic processes, ergodic theory, study of L^p-spaces, etc.) the idea of identifying functions which are equal almost everywhere is quite natural. The experience gained from real-valued functions may lead to an oversight that the set on which two measurable functions (taking values in an arbitrary topological space) coincide is always measurable. It is well-known and quite easy to prove that this happens when functions take values in a space with countable base or, if they have separable images lying in a metrizable space. However, it is not true in general. The reason for this is that $\mathfrak{B}(Y) \otimes \mathfrak{B}(Y)$ differs (in general) from $\mathfrak{B}(Y \times Y)$ where $\mathfrak{B}(Y)$ is the σ-algebra of all Borel subsets of a topological space Y. Let us say that a topological space Y has *almost everywhere equality property* (briefly, (AEEP)) if Y satisfies the condition stated in Abstract. It is easily seen (see Lemma 2.1 below) that Y has (AEEP) iff the diagonal of Y belongs to $\mathfrak{B}(Y) \otimes \mathfrak{B}(Y)$. So, it may turn out that Y has (AEEP) but still $\mathfrak{B}(Y) \otimes \mathfrak{B}(Y) \neq \mathfrak{B}(Y \times Y)$. The aim of this short note is to prove that a metrizable space has (AEEP) iff $\operatorname{card}(Y) \leqslant 2^{\aleph_0}$. Thus, among metrizable spaces only those whose topological weight is no greater than $2^{\aleph_0}$ have (AEEP). It may seem surprising that not only separable spaces appear in this characterization.

Nonseparable metric spaces are widely investigated in functional analysis and operator theory. In fact, the Banach algebra of all bounded linear operators acting on a separable Banach space is usually nonseparable. Also infinite-dimensional von Neumann algebras are nonseparable. A special and a very important in theory of geometry of Banach spaces example of them is the Banach space l^∞ of all bounded sequences. So, our result may find applications in investigations of these spaces.

Notation and terminology. For every topological space Y, $\mathfrak{B}(Y)$ stands for the σ-algebra of all Borel subsets of Y. That is, $\mathfrak{B}(Y)$ is the smallest σ-algebra containing all open sets. Whenever $(\Omega,\mathfrak{M})$ and $(\Lambda,\mathfrak{N})$ are two measurable spaces, a function $f\colon(\Omega,\mathfrak{M})\to(\Lambda,\mathfrak{N})$ is said to be *measurable* iff $f^{-1}(B)\in\mathfrak{M}$ for each $B\in\mathfrak{N}$. $\mathfrak{M}\otimes\mathfrak{N}$ denotes the product σ-algebra of $\mathfrak{M}$ and $\mathfrak{N}$, i.e. $\mathfrak{M}\otimes\mathfrak{N}$ is the smallest σ-algebra on $\Omega\times\Lambda$ which contains all sets of the form $A\times B$ with $A\in\mathfrak{M}$ and $B\in\mathfrak{N}$. If $g\colon(\Omega,\mathfrak{M})\to Y$ where Y is a topological space, g is measurable if $g^{-1}(U)\in\mathfrak{M}$ for any open set $U\subset Y$ or, equivalently, if $g\colon(\Omega,\mathfrak{M})\to(Y,\mathfrak{B}(Y))$ is measurable. For two functions $u,v\colon D\to E$, $\Delta(f,g)$ stands for the set $\{x\in D\colon\ u(x)=v(x)\}$. Additionally, for every set E, Δ_E denotes the diagonal of E, i.e. $\Delta_E=\{(x,x)\colon\ x\in E\}$; and $\operatorname{card}(E)$ is the cardinality of E.

2. The result

We begin with a simple

2.1. **Lemma.** *For a topological space Y the following conditions are equivalent:*

(i) Y *has* (AEEP),
(ii) $\Delta_Y\in\mathfrak{B}(Y)\otimes\mathfrak{B}(Y)$.

Proof. If $f,g\colon(\Omega,\mathfrak{M})\to Y$ are two measurable functions, then the function $h\colon(\Omega,\mathfrak{M})\ni\omega\mapsto(f(\omega),g(\omega))\in(Y\times Y,\mathfrak{B}(Y)\otimes\mathfrak{B}(Y))$ is measurable as well and thus $\Delta(f,g)=h^{-1}(\Delta_Y)$ is a member of $\mathfrak{M}$. This shows that (i) follows from (ii). To see the converse, notice that the natural projections $p_j\colon(Y\times Y,\mathfrak{B}(Y)\otimes\mathfrak{B}(Y))\ni(y_1,y_2)\mapsto y_j\in Y$ $(j=1,2)$ are measurable and that $\Delta(p_1,p_2)=\Delta_Y$ which finishes the proof. □

2.2. **Lemma.** *For an arbitrary topological space Y, every member F of $\mathfrak{B}(Y)\otimes\mathfrak{B}(Y)$ may be written in the form*

$$F=\bigcup_{t\in[0,1]}(A_t\times B_t) \tag{2-1}$$

where $A_t,B_t\in\mathfrak{B}(Y)$ $(t\in[0,1])$.

Proof. Let $\mathcal{A}$ be the family of all subsets of $Y\times Y$ which are finite unions of sets of the form $A\times B$ with $A,B\in\mathfrak{B}(Y)$. It is easily seen that $\mathcal{A}$ is an algebra of subsets of $Y\times Y$. Hence, by the Monotone Class Theorem (cf. e.g. Theorem 1.3 of [3]), $\mathfrak{B}(Y)\otimes\mathfrak{B}(Y)$ is the smallest family $\mathcal{F}$ such that $\mathcal{A}\subset\mathcal{F}$ and

$$F_1,F_2,\ldots\in\mathcal{F}\implies\bigcap_{n=1}^{\infty}F_n,\bigcup_{n=1}^{\infty}F_n\in\mathcal{F}. \tag{2-2}$$

Now let $\mathcal{F}$ consists of all sets F which may be written in the form (2-1) (with $A_t, B_t \in \mathfrak{B}(Y)$). Since $\varnothing \in \mathcal{F}$, $\mathcal{A} \subset \mathcal{F}$. So, it remains to check that $\mathcal{F}$ satisfies (2-2). Let $F_j = \bigcup_{t\in[0,1]} A_t^{(j)} \times B_t^{(j)}$ for $j = 1, 2, \dots$ It is clear that $\bigcup_{j=1}^{\infty} F_j \in \mathcal{F}$. To this end, put $\Lambda = [0,1]^{\mathbb{N}}$ (here $0 \notin \mathbb{N}$) and observe that

$$\bigcap_{j=1}^{\infty} F_j = \bigcup_{\xi\in\Lambda} (\bigcap_{j=1}^{\infty} A_{\xi(j)}^{(j)}) \times (\bigcap_{j=1}^{\infty} B_{\xi(j)}^{(j)})$$

which finishes the proof since there is a bijection between Λ and $[0,1]$. □

As an immediate consequence of the above results, we obtain

2.3. **Corollary.** *If a topological space Y has* (AEEP)*, then*

$$\operatorname{card}(Y) \leqslant 2^{\aleph_0}.$$

Now we want to prove the converse of Corollary 2.3 for metrizable Y. For need of this, we recall some classical notion in topology. The *topological cone* $C(Y)$ over a topological space Y is the set $(Y \times (0,1]) \cup \{\omega_Y\}$ equipped with the topology such that $Y \times (0,1]$ is open in $C(Y)$, the topology on $Y \times (0,1]$ inherited from the one of $C(Y)$ coincides with the product topology, and the sets $(Y \times (0,t)) \cup \{\omega_Y\}$ with $t \in (0,1)$ form a basis of open neighbourhoods of ω_Y in $C(Y)$. The next result is elementary and we omit its proof.

2.4. **Lemma.** *If $u\colon X \to Y$ is a continuous function between topological spaces X and Y, then the function $\widehat{u}\colon C(X) \to C(Y)$ given by $\widehat{u}((x,t)) = (u(x),t)$ and $\widehat{u}(\omega_X) = \omega_Y$ is continuous as well.*

An important example of a topological cone is the so-called *hedgehog* space ([1, Example 4.1.5]). The hedgehog $J(\mathfrak{m})$ of spininess $\mathfrak{m} \geqslant \aleph_0$ is the topological cone over a discrete space of cardinality $\mathfrak{m}$. Its importance is justified by the following result of Kowalsky [2] (see also [1, Theorem 4.4.9]; it is already known that $[J(\mathfrak{m})]^{\aleph_0}$ with infinite $\mathfrak{m}$ is homeomorphic to the Hilbert space of Hilbert space dimension $\mathfrak{m}$, see [4, 5] or Remark in Exercise 4.4.K of [1]).

2.5. **Theorem.** *Every metrizable space of topological weight no greater than $\mathfrak{m}$ (where $\mathfrak{m} \geqslant \aleph_0$) is homeomorphic to a subset of $[J(\mathfrak{m})]^{\aleph_0}$.*

As a colorrary of Theorem 2.5, we obtain the following result, which may be interesting in itself.

2.6. **Proposition.** *Every metrizable space Y such that* $\operatorname{card}(Y) \leqslant 2^{\aleph_0}$ *admits a continuous one-to-one function of Y into a separable metrizable space.*

Proof. Let $D = [0,1]$ and $T = [0,1]$ be equipped with, respectively, the discrete and the natural topology. By Lemma 2.4, the function $C(D) \ni z \mapsto z \in C(T)$ is continuous. This means that there is a one-to-one continuous function $u\colon J(2^{\aleph_0}) \to W$ where W is a separable metrizable space. But then the function $[J(2^{\aleph_0})]^{\aleph_0} \ni (x_n)_{n=1}^{\infty} \mapsto (u(x_n))_{n=1}^{\infty} \in W^{\aleph_0}$ is continuous and one-to-one as well. Now it suffices to apply Theorem 2.5. □

We are now able to prove the main result of the paper.

2.7. **Theorem.** *For a metrizable space Y the following conditions are equivalent:*

(i) *Y has* (AEEP),
(ii) *the topological weight of Y is no greater than $2^{\aleph_0}$,*
(iii) $\operatorname{card}(Y) \leqslant 2^{\aleph_0}$.

Proof. The equivalence of (ii) and (iii) follows since $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. So, thanks to Lemma 2.1 and Corollary 2.3, we only have to show that $\Delta_Y \in \mathfrak{B}(Y) \otimes \mathfrak{B}(Y)$ provided $\operatorname{card}(Y) \leqslant 2^{\aleph_0}$.

Assume (iii) is fulfilled. We infer from Proposition 2.6 that there is a separable metrizable space X and a continuous one-to-one function $u\colon Y \to X$. Then $v = u \times u\colon (Y \times Y, \mathfrak{B}(Y) \otimes \mathfrak{B}(Y)) \to (X \times X, \mathfrak{B}(X) \otimes \mathfrak{B}(X))$ is measurable ($v(x,y) = (u(x), u(y))$ for $x, y \in Y$). Since X is separable, $\mathfrak{B}(X) \otimes \mathfrak{B}(X) = \mathfrak{B}(X \times X)$ and therefore $\Delta_Y = v^{-1}(\Delta_X) \in \mathfrak{B}(Y) \otimes \mathfrak{B}(Y)$ and we are done. □

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