

GRAPH BOOTSTRAP PERCOLATION

JÓZSEF BALOGH, BÉLA BOLLOBÁS, AND ROBERT MORRIS

ABSTRACT. Graph bootstrap percolation is a deterministic cellular automaton which was introduced by Bollobás in 1968, and is defined as follows. Given a graph H , and a set $G \subset E(K_n)$ of initially ‘infected’ edges, we infect, at each time step, a new edge e if there is a copy of H in K_n such that e is the only not-yet infected edge of H . We say that G percolates in the H -bootstrap process if eventually every edge of K_n is infected.

The extremal questions for this model, when H is the complete graph K_r , were solved (independently) by Alon, Kalai and Frankl almost thirty years ago. In this paper we study the random questions, and determine the critical probability $p_c(n, K_r)$ for the K_r -process up to a poly-logarithmic factor. In the case $r = 4$ we prove a stronger result, and determine the threshold for $p_c(n, K_4)$.

1. INTRODUCTION

Cellular automata, which were introduced by von Neumann (see [30]) after a suggestion of Ulam [32], are dynamical systems (defined on a graph G) whose update rule is homogeneous and local. We shall study a particular cellular automaton, called H -bootstrap percolation, which was introduced over 40 years ago by Bollobás [13]. This model is a substantial generalization of r -neighbour bootstrap percolation (see below), an extensively studied model related to statistical physics. We shall determine the critical probability for K_r -percolation up to a poly-logarithmic factor for every $r \geq 4$ and moreover, using a completely different method, we shall determine the threshold for percolation in the case $r = 4$.

Given a graph H , we define H -bootstrap percolation (or H -edge-bootstrap percolation) as follows. Given a set $G \subset E(K_n)$ of initially ‘infected’ edges on vertex set $[n]$ (that is, given a graph), we set $G_0 = G$ and define, for each $t \geq 0$,

$$G_{t+1} := G_t \cup \left\{ e \in E(K_n) : \exists H \text{ with } e \in H \subset G_t \cup \{e\} \right\}.$$

In words, this says that an edge e becomes infected at time $t + 1$ if there exists a copy of H in K_n for which e is the only uninfected edge at time t . Let $\langle G \rangle_H = \bigcup_t G_t$ denote the closure of G under the H -bootstrap process, and say that G percolates (or H -percolates) in K_n if $\langle G \rangle_H = E(K_n)$.

The H -bootstrap process was introduced over 40 years ago by Bollobás [13] (see also [15]), under the name ‘weak saturation’. He conjectured that if a graph G percolates in the K_r -process, then G has at least $\binom{n}{2} - \binom{n-r+2}{2}$ edges, and, building on work in [12], proved his conjecture when $r \leq 7$. For general r , the conjecture was proved using linear algebraic

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methods by Alon [1], Frankl [22] and Kalai [28]. See [9] for more recent extremal results, on a closely related process, using such methods.

In this paper, we shall study the H -bootstrap process in the random setting, i.e., when the initial graph G is chosen to be $G_{n,p}$. Apart from its intrinsic interest, this question is motivated by the following, closely related cellular automaton, which was introduced in 1979 by Chalupa, Leath and Reich [17] in the context of disordered magnetic systems, and for which our process is named. Given an underlying graph G , an integer r and a set of infected vertices $A \subset V(G)$, set $A_0 = A$ and let

$$A_{t+1} := A_t \cup \{v \in V(G) : |N(v) \cap A_t| \geq r\}$$

for each $t \geq 0$; that is, we infect a vertex if it has at least r already-infected neighbours. Say that the set A *percolates* if the entire vertex set is eventually infected. This process is known as r -neighbour bootstrap percolation, and has been extensively studied by mathematicians (see, for example, [3, 5, 16, 25, 26, 31]), physicists (see [2], and the references therein) and sociologists [23, 33], amongst others. It has moreover found applications in the Glauber Dynamics of the Ising model (see [21, 29]).

The r -neighbour bootstrap model is usually studied in the random setting, where the main question is to determine the *critical threshold* at which percolation occurs. To be precise, if $V(G) = [n]$ and the elements of $A \subset V(G)$ are chosen independently at random, each with probability p , then one aims to determine the value p_c of $p = p(n)$ at which percolation becomes likely. Sharp bounds on p_c have recently been determined in several cases of particular interest, such as $[n]^d$ (see [5, 6, 7, 8, 24, 25]), on a large family of ‘two-dimensional’ graphs [18], on trees [10, 20], and on various types of random graph [11, 27]. In each case, it was shown that the critical probability has a sharp threshold.

Motivated by these results, let us define the critical threshold for H -bootstrap percolation on K_n as follows:

$$p_c(n, H) := \inf \left\{ p : \mathbb{P}(\langle G_{n,p} \rangle_H = K_n) \geq 1/2 \right\},$$

where $G_{n,p}$ is the Erdős-Rényi random graph, obtained by choosing each edge independently with probability p . (For background on the theory of Random Graphs, see [14].) Our aim is to determine $p_c(n, H)$ for every graph H . Here we shall study the case $H = K_r$, the complete graph; our main theorems partially solve Problem 1 of [15].

In order to aid the reader’s intuition, let us first consider the case $H = K_3$, which is trivial. Indeed, it is easy to see that G percolates in the K_3 -process if and only if G is connected. It is well-known (see [14]) that, with high probability, $G_{n,p}$ is connected if and only if it has no isolated vertex; thus, a straightforward calculation gives the following theorem of Erdős and Rényi [19], which was one of the first results on random graphs:

$$p_c(n, K_3) = \frac{\log n}{n} + \Theta\left(\frac{1}{n}\right).$$

In fact Erdős and Rényi proved even more: that if $p = (\log n + c)/n$, then the probability that $G_{n,p}$ percolates in the K_3 -process converges to $e^{-e^{-c}}$ as $n \rightarrow \infty$. We remark that the same result holds for the C_k -process, for any $k \geq 3$, see Section 5.

For $r \geq 4$, the problem is more challenging, since there seems to be no simple description of the closed sets under the K_r -process. Set

$$\lambda(r) := \frac{\binom{r}{2} - 2}{r - 2}.$$

The following theorem is our main result.

Theorem 1. *For every $r \geq 4$, there exists a constant $c = c(r) > 0$ such that*

$$\frac{n^{-1/\lambda(r)}}{c \log n} \leq p_c(n, K_r) \leq n^{-1/\lambda(r)} \log n$$

for every sufficiently large $n \in \mathbb{N}$.

In fact we shall prove slightly stronger bounds (see Propositions 3 and 8); however, we do not expect either of our bounds to be sharp. The proof of the lower bound in Theorem 1 is based on an extremal result on graphs which cause a given edge to be infected (see Lemma 9). Although it is not long, the proof of this lemma is delicate, and does not seem to extend easily to other graphs. The upper bound, on the other hand, holds for a much wider family of graphs H (see Section 2), which we call ‘balanced’.

In the case $r = 4$ we shall prove the following stronger result, which determines $p_c(n, K_4)$ up to a constant factor.

Theorem 2. *If n is sufficiently large, then*

$$\frac{1}{4} \sqrt{\frac{1}{n \log n}} \leq p_c(n, K_4) \leq 5 \sqrt{\frac{1}{n \log n}}.$$

The proof of Theorem 2 is completely different from that of Theorem 1, and is based on ideas from two-neighbour bootstrap percolation on $[n]^d$. Although we have not done so, it seems plausible to us that one could prove a sharp threshold in the case $H = K_4$.

The rest of the paper is organized as follows. In Sections 2 and 3 we shall prove the upper and lower bounds in Theorem 1, respectively. In Section 4 we shall prove Theorem 2, and in Section 5 we shall discuss other graphs H , and state some open problems.

2. AN UPPER BOUND FOR BALANCED GRAPHS

In this section we shall prove the upper bound in Theorem 1; in fact we prove a stronger bound for a more general family of graphs, H . Throughout, we shall assume that $v(H) \geq 4$, since otherwise the problem is trivial. We make the following definition.

Definition 1 (Balanced graphs). We call a graph H *balanced* if $e(H) \geq 2v(H) - 2$, and

$$\frac{e(F) - 1}{v(F) - 2} \leq \lambda(H) := \frac{e(H) - 2}{v(H) - 2}$$

for every proper subgraph $F \subset H$ with $v(F) \geq 3$.

It is straightforward to check that the complete graph K_r is balanced for every $r \geq 4$. Thus, the upper bound in Theorem 1 follows immediately from the following proposition.

Proposition 3. *If H is a balanced graph, then*

$$p_c(n, H) \leq C \left(\frac{\log n}{\log \log n} \right)^{2/\lambda(H)} n^{-1/\lambda(H)},$$

for some constant $C > 0$.

Note that $\lambda(K_r) = \lambda(r)$ satisfies

$$\frac{r}{2} \leq \lambda(r) \leq \frac{r+1}{2}$$

if $r \geq 4$ (we shall use these bounds several times during the proof), so Proposition 3 actually implies the following slightly stronger upper bound than that stated in Theorem 1:

$$p_c(n, K_r) \leq n^{-1/\lambda(H)} (\log n)^{4/r}.$$

We begin by sketching the proof of Proposition 3. We shall define for each $d \in \mathbb{N}$ a rooted graph, i.e., a pair (H_d, e) where H_d is a graph and $e \in (V(H_d))$ is its ‘root’, with the following properties:

$$(a) v(H_d) = (v(H) - 2)d + 2 \quad (b) e(H_d) = (e(H) - 2)d + 1 \quad (c) e \in \langle H_d \rangle_H.$$

That is, if H_d occurs in $G_{n,p}$, then its root e is infected in the H -bootstrap process.

To define H_d , choose a sequence of edges (e_1, e_2, \dots) of H , with $e_1 = e$, such that for every $j \in \mathbb{N}$, e_j and e_{j+1} do not share an endpoint. Let (V_1, V_2, \dots) be a sequence of vertex sets with $|V_j| = |V(H)|$ for each $j \in \mathbb{N}$, such that $|V_i \cap V_j| = 2$ if $|i - j| = 1$, and $|V_i \cap V_j| = 0$ otherwise. We remark that although the definition of H_d will depend on the choice of (e_1, e_2, \dots) , the proof below will work for any such sequence.

Definition 2 (The graph H_d). For each $d \in \mathbb{N}$, let (H_d, e) denote the rooted graph with root $e = e_1$, vertex set $V_1 \cup \dots \cup V_d$, and edge set $E(H_d) = E(H_d[V_1]) \cup \dots \cup E(H_d[V_d])$, such that

$$H_d[V_j] = H - \{e_j, e_{j+1}\}$$

and $V_j \cap V_{j+1} = e_{j+1}$ for every $1 \leq j \leq d - 1$, and $H_d[V_d] = H - \{e_d\}$.

In other words, we place a copy of H on each V_j , in such a way that $V_j \cap V_{j+1} = e_{j+1}$ for each $j \in [d - 1]$, and then remove the edges e_1, \dots, e_d .

Observation 4. *Properties (a), (b) and (c) hold for H_d .*

Let $X_d(e)$ be the random variable which counts the number of copies of H_d in $G_{n,p}$, rooted at a given edge e . It is straightforward (using property (a)) to show that the expected value of X_d is large if $p^{\lambda(H)} n \geq (\log n)^2$ (see Lemma 5); the main challenge will be to bound the variance of X_d . The key step is therefore Lemma 6, below, which controls the number of edges in the intersection of two copies of H_d with the same root. Having bounded the variance of X_d , the proposition follows easily by Chebychev’s inequality.

We begin by bounding the expected value of $X_d(e)$.

Lemma 5. *Let H be a balanced graph, and $e \in E(K_n)$. If $p = p(n)$ and $d = d(n)$ are chosen so that $p^{\lambda(H)}n \geq \omega v(H)d$ and $\omega^{(v(H)-2)d} \geq n$ for some function $\omega = \omega(n)$, and $pn \rightarrow \infty$, then*

$$\mathbb{E}(X_d(e)) \rightarrow \infty$$

as $n \rightarrow \infty$.

Proof. Recall that H_d has $(v(H) - 2)d + 2$ vertices and $(e(H) - 2)d + 1$ edges. Thus

$$\mathbb{E}(X_d(e)) \geq \binom{n}{v(H_d) - 2} p^{e(H_d)} \geq \left(\frac{n}{v(H)d} \right)^{(v(H)-2)d} p^{(e(H)-2)d+1}.$$

Since $e(H) - 2 = \lambda(H)(v(H) - 2)$, and using our bounds on ω , it follows that

$$\mathbb{E}(X_d(e)) \geq p \left(\frac{p^{\lambda(H)}n}{v(H)d} \right)^{(v(H)-2)d} \geq p \cdot \omega^{(v(H)-2)d} \geq pn \rightarrow \infty,$$

as required. \square

Let $\text{root}(H_d)$ denote the root of H_d , and recall that $e(H_d) = (v(H_d) - 2)\lambda(H) + 1$. We shall next prove that the variance of $X_d(e)$ is small; the following lemma is the key step.

Lemma 6. *Let H be a balanced graph, and let $d \in \mathbb{N}$. If $F \subsetneq H_d$ and $\text{root}(H_d) \subset V(F)$, then*

$$e(F) \leq (v(F) - 2)\lambda(H).$$

Proof. We shall use induction on d . The case $d = 1$ is trivial, since $e = \text{root}(H_d) \subset V(F)$, so either $v(F) = 2$ and $e(F) = 0$, or $v(F) \geq 3$ and $F + e \subsetneq H$, and so the bound follows by Definition 1. Let $d \geq 2$, and assume that the result holds for every $d' < d$. For each $j \in [d]$, let $V(F_j) = V(F) \cap V_j$ and let $F_j = H[V(F_j)]$ be the subgraph of H induced by these vertices.

Suppose first that $v(F_j) \leq 1$ for some $j \in [d]$, and let F' and F'' be the subgraphs of H_d induced by $V(F) \cap (V_1 \cup \dots \cup V_{j-1})$ and $V(F) \cap (V_{j+1} \cup \dots \cup V_d)$, respectively. Applying the induction hypothesis to F' , we see that

$$e(F) = e(F') + e(F'') \leq (v(F') - 2)\lambda(H) + e(F''),$$

so it will suffice to prove that $e(F'') \leq v(F'')\lambda(H)$. Now, applying the induction hypothesis to F^* , the subgraph of H induced by $V(F'') \cup (V_j \cap V_{j+1})$, we get either

$$e(F^*) \leq (v(F^*) - 2)\lambda(H) \leq v(F'')\lambda(H)$$

as required, or $V(F^*) = V_{j+1} \cup \dots \cup V_d$. But in the latter case $e(F^*) - e(F'') \geq 1$ (since $v(F_j) \leq 1$ and H is balanced), so

$$e(F'') \leq e(F^*) - 1 = (v(F^*) - 2)\lambda(H) \leq v(F'')\lambda(H),$$

as required. Hence we may assume that $v(F_j) \geq 2$ for every $j \in [d]$. Moreover, a similar easy calculation proves the lemma if $V(F) = V(H_d)$, so we may assume that $v(F) < v(H_d)$.

Now, for each $j \in [d-1]$, let E_j denote the event that $V_j \cap V_{j+1} \subset V(F)$, and let $\mathbf{1}[\cdot]$ denote the indicator function. Then, recalling that $F_j = H[V(F_j)]$, we have

$$e(F) \leq \left(\sum_{j=1}^d e(F_j) \right) - 1 - 2 \sum_{j=1}^{d-1} \mathbf{1}[E_j],$$

by the definition of H_d , and since $\text{root}(H_d) \subset V(F)$. We next claim that, since H is balanced and $F \neq H_d$, it follows that

$$\begin{aligned} e(F) &\leq \sum_{j=1}^d \left((v(F_j) - 2)\lambda(H) + 2 \right) - 2 - 2 \sum_{j=1}^{d-1} \mathbf{1}[E_j] \\ &= \left(\sum_{j=1}^d v(F_j) - 2d \right) \lambda(H) + (2d - 2) - 2 \sum_{j=1}^{d-1} \mathbf{1}[E_j]. \end{aligned}$$

To see this, observe that $e(F_j) \leq (v(F_j) - 2)\lambda(H) + 1$ holds for every $F_j \subsetneq H$ with $v(F_j) \geq 2$, by Definition 1, and that $F_j \neq H$ for some $j \in [d]$, since $v(F) < v(H_d)$.

Finally, observe that

$$v(F) \geq \sum_{j=1}^d v(F_j) - (d-1) - \sum_{j=1}^{d-1} \mathbf{1}[E_j],$$

and so

$$e(F) \leq \left(v(F) - d - 1 + \sum_{j=1}^{d-1} \mathbf{1}[E_j] \right) \lambda(H) + (2d - 2) - 2 \sum_{j=1}^{d-1} \mathbf{1}[E_j].$$

But

$$\left(d - 1 - \sum_{j=1}^{d-1} \mathbf{1}[E_j] \right) \lambda(H) \geq 2d - 2 - 2 \sum_{j=1}^{d-1} \mathbf{1}[E_j],$$

since $\lambda(H) \geq 2$, by Definition 1. Hence

$$e(F) \leq (v(F) - 2)\lambda(H)$$

for every $F \subsetneq H$, as required. \square

It is now straightforward to deduce the required bound on the variance of $X_d(e)$.

Lemma 7. *Let H be a balanced graph, and $e \in E(K_n)$. If $p = p(n)$ and $d = d(n)$ are chosen so that $v(H_d)^{-2} p^{\lambda(H)} n \rightarrow \infty$ as $n \rightarrow \infty$, then*

$$\frac{\text{Var}(X_d(e))}{\mathbb{E}(X_d(e))^2} \rightarrow 0$$

as $n \rightarrow \infty$.

Proof. Let $\ell(H_d)$ denote the number of copies of H_d , rooted at e , which share the same vertex set. Then

$$\mathbb{E}(X_d(e)) = \binom{n}{v(H_d) - 2} \ell(H_d) \cdot p^{e(H_d)}.$$

Moreover, we claim that

$$\text{Var}(X_d(e)) \leq \sum_{m=1}^{v(H_d)-2} \ell(H_d)^2 \binom{n}{m} \binom{n}{v(H_d) - m - 2}^2 p^{2e(H_d) - \lambda(H)m}. \quad (1)$$

To see this, we simply count (ordered) pairs (A, B) , where A and B are copies of H_d in $G_{n,p}$ with root e . Let $F = A \cap B$ and $m = |V(A) \cap V(B)| - 2$, and note that we expect at most $\mathbb{E}(X_d(e))^2$ such pairs (A, B) with $m = 0$.

By Lemma 6, if $A \neq B$ then $e(F) \leq \lambda(H)m$, and so $e(A \cup B) \geq 2e(H_d) - \lambda(H)m$. Moreover, given m , there are at most

$$\ell(H_d)^2 \binom{n}{m} \binom{n}{v(H_d) - m - 2}^2$$

choices for A and B . This proves (1).

Combining the bounds above, and setting $k = v(H_d)$, we obtain

$$\frac{\text{Var}(X_d(e))}{\mathbb{E}(X_d(e))^2} \leq \sum_{m=1}^{k-2} \frac{2^m (k-2)!^2}{m! (k-m-2)!^2} (p^{\lambda(H)} n)^{-m} \leq \sum_{m=1}^{k-2} 2 \left(k^{-2} p^{\lambda(H)} n \right)^{-m} \rightarrow 0$$

as $n \rightarrow \infty$, as required. \square

We can now deduce Proposition 3 using Chebychev's inequality and sprinkling.

Proof of Proposition 3. Let H be a balanced graph, suppose that $p \gg \left(\frac{\log n}{\log \log n} \right)^{2/\lambda(H)} n^{-1/\lambda(H)}$, and let

$$d(n) = \left\lfloor \frac{\log n}{\log \log n} \right\rfloor.$$

We claim that $p(n)$ and $d(n)$ satisfy the conditions of Lemmas 5 and 7. Indeed, setting $\omega(n) = d(n)$ we have $p^{\lambda(H)} n \gg \omega d$ and $\omega^{2d} \gg n$, so Lemma 5 holds, and $p^{\lambda(H)} n \gg d^2$, so Lemma 7 holds. Thus, by Chebychev's inequality,

$$\mathbb{P}(X_d(e) = 0) \leq \frac{\text{Var}(X_d(e))}{\mathbb{E}(X_d(e))^2} \rightarrow 0$$

as $n \rightarrow \infty$. Moreover, if $X_d(e) \neq 0$ then $e \in \langle G_{n,p} \rangle_H$, since if e is the root of some copy of H_d then it is infected after at most d steps of the H -process. Hence, by Markov's inequality, if $p^{\lambda(H)} n \gg \left(\frac{\log n}{\log \log n} \right)^2$ then, with high probability, all but $o(n^2)$ edges of K_n are infected in the H -process on $G_{n,p}$.

To finish the proof, we shall show that by sprinkling $O(n \log n)$ extra edges, we shall infect all of the remaining edges, with high probability. We use the following easy claim.

Claim: If $e(\langle G_{n,p} \rangle_H) \geq \binom{n}{2} - o(n^2)$, then there is a clique of size $n - o(n)$ in $G = \langle G_{n,p} \rangle_H$.

Proof of Claim. Let $0 < c < 1$ be arbitrary. By the pigeonhole principle, there are $o(n)$ pairs $\{x, y\}$ such that $d_G(x) + d_G(y) < (2 - c)n$. We claim that every other edge is in G .

Indeed, if $d_G(x) + d_G(y) \geq (2 - c)n$ then, by Turán's Theorem, there is a $(v(H) - 2)$ -clique in $N_G(x) \cap N_G(y)$, since $o(n^2)$ edges are missing. But then $xy \in \langle G \rangle$, which implies that $xy \in G$, since G is closed under the H -bootstrap process. Thus $e(G) \geq \binom{n}{2} - o(n)$, and so the claim follows. \square

Finally, let us sprinkle edges with density p ; that is, let us take a second copy of $G_{n,p}$ and consider the union of the two random graphs. We obtain a random graph G_{n,p^*} of density $p^* = 1 - (1 - p)^2 < 2p$. Let K be the clique found in the claim, and observe that if every vertex outside K has at least $v(H) - 1$ neighbours in K (in the second copy of $G_{n,p}$) then G_{n,p^*} will percolate. Since $pn \gg \log n$, this occurs with high probability, and hence

$$p_c(n, H) \leq C \left(\frac{\log n}{\log \log n} \right)^{2/\lambda(H)} n^{-1/\lambda(H)},$$

if C is sufficiently large, as required. \square

3. LOWER BOUND FOR K_r -PERCOLATION

In this section we shall prove the following proposition, which shows that, if $r \geq 4$ and $(p \log n)^{\lambda(r)} n \leq 1$, then with high probability $o(n^2)$ edges are infected in the K_r -bootstrap process with initial set $G_{n,p}$.

Proposition 8. *Let $r \geq 4$, and let $e \in E(K_n)$. If $pn^{1/\lambda(r)} \log n \leq 1/(2e)$, then*

$$\mathbb{P}(e \in \langle G_{n,p} \rangle_{K_r}) \rightarrow 0$$

as $n \rightarrow \infty$.

The idea of the proof is as follows. If $e \in \langle G \rangle_{K_r}$ for some graph G , then there must exist a 'witness set' of edges of G which caused e to be infected. We shall describe an algorithm which finds such a set $F = F(e)$ of edges, and show that this set has two useful properties:

- (a) $e(F) \geq \lambda(r)(v(F) - 2) + 1$ (see Lemma 9).
- (b) If $e(F) \geq \binom{r}{2}L$, then $L \leq e(F(f)) \leq \binom{r}{2}L$ for some $f \in \langle G \rangle_{K_r}$ (see Lemma 13).

Property (a) will allow us to bound the expected number of such sets when $G = G_{n,p}$ and $e(F) = O(\log n)$; combining it with property (b) will allow us to do so when $e(F)$ is larger than this.

3.1. Extremal results. Let $r \geq 4$ be fixed for the remainder of this section, and let G be an arbitrary graph. We begin by describing the algorithm which finds $F(e)$.

The Witness-Set Algorithm. We assign a graph $F = F(e) \subset G$ to each edge $e \in \langle G \rangle_{K_r}$ as follows:

1. If $e \in G$ then set $F(e) = \{e\}$.
2. Choose an order in which to infect the edges of $\langle G \rangle_{K_r}$, and at each step identify which r -clique was completed (if more than one is completed then choose one).

3. Infect the edges one by one. If e is infected by the r -clique K , then set

$$F(e) := \bigcup_{e \neq e' \in K} F(e').$$

We call the graph $F(e)$ a *witness set* for the event $e \in \langle G \rangle_{K_r}$.

Since every $e \neq e' \in K$ is either in G , or was infected earlier in the process, the algorithm is well-defined. Note that the graphs $F(e)$ depend on the order in which we chose to infect the edges (that is, they depend on Step 2 of the algorithm); the results below hold for every possible such choice.

We shall say that a graph F is an r -*witness set* if there exists a graph G , an edge e , and a realization of the Witness-Set Algorithm (i.e., a choice as in Step 2) such that $F = F(e)$. The key lemma in the proof of Proposition 8 is the following extremal result.

Lemma 9. *Let F be a graph and $r \geq 4$, and suppose that F is an r -witness set. Then*

$$e(F) \geq \lambda(r)(v(F) - 2) + 1.$$

We shall prove Lemma 9 using induction; in order to do so, we shall need to state a more general version of it (see Lemma 10, below). The statement is slightly technical, and we shall need some preparatory definitions. We shall use the following algorithm, which is simply a restatement of the Witness-Set Algorithm.

The Red Edge Algorithm. Let G be a graph, let $r \geq 4$, and let $e \in \langle G \rangle_{K_r} \setminus G$.

1. Run the Witness-Set Algorithm until edge e is infected.
2. Let (e_1, e_2, \dots, e_m) be the edges infected in the process with $F(e_j) \subset F(e)$ and $e_j \notin G$, written in the order in which they are infected, where $e_m = e$.
3. For each $1 \leq j \leq m$, let $K^{(j)}$ be the r -clique which is completed by e_j .
4. Colour the edges $\{e_1, \dots, e_m\}$ red, and note that $e_j \in K^{(j)} \setminus (K^{(1)} \cup \dots \cup K^{(j-1)})$.

The key observation is that $F(e) = (K^{(1)} \cup \dots \cup K^{(m)}) \setminus \{e_1, \dots, e_m\}$, or, in words, $F(e)$ consists of all the non-red edges of the cliques. We shall bound the number of non-red edges after t steps of the Red Edge Algorithm. Thus, given a realization of the algorithm and $t \in [m]$, define

$$B_t := (K^{(1)} \cup \dots \cup K^{(t)}) \setminus \{e_1, \dots, e_t\}.$$

We shall only use the following properties of the Red Edge Algorithm: at step j an r -clique is added, one of the edges e_j of $K^{(j)}$ is coloured red, and $e_j \notin K^{(i)}$ for every $i < j$.

In order to state Lemma 10, we need to define two more parameters of the model, which will both play a key role in the induction step.

Definition 3 (ℓ and k). Let \mathcal{G}_t denote the graph, obtained using the Red Edge Algorithm, whose vertices are the cliques $\{K^{(1)}, \dots, K^{(t)}\}$, and in which two cliques are adjacent if they share at least two vertices.

Let $\ell = \ell_t$ denote the number of components of \mathcal{G}_t , let $c(v) = c_t(v)$ denote the number of components of \mathcal{G}_t containing the vertex $v \in V(G)$, and set

$$k = k_t = \sum_{v \in V(G)} (c_t(v) - 1).$$

Here, and throughout, we treat C_1 as a component in \mathcal{G}_t , and also as a subset of $V(G)$, and trust that this will not cause confusion.

The following lemma easily implies Lemma 9, since when $t = m$ we have $\ell = 1$ and hence $k = 0$.

Lemma 10. $e(B_t) \geq \left(\frac{\binom{r}{2} - 2}{r - 2}\right)(v(B_t) + k - \ell r) + \ell \left(\binom{r}{2} - 1\right)$.

We shall prove Lemma 10 by induction on t . The induction step will be relatively straightforward when $\ell_t \geq \ell_{t-1}$; when $\ell_t < \ell_{t-1}$ we shall need the following lemma.

Say that a (multi-)family of sets \mathcal{A} is a double cover of X if every element of X is in at least two members of \mathcal{A} .

Lemma 11. *Let $m \geq 2$ and $r \geq 4$, and let \mathcal{A} be a multi-family of subsets of $[m]$. If \mathcal{A} is a double cover of $[m]$, and $|\mathcal{A}| \leq r$, then*

$$\left| \left\{ \{A, B\} \in \binom{\mathcal{A}}{2} : A \cap B \neq \emptyset \right\} \right| \leq \lambda(r) \left(\sum_{A \in \mathcal{A}} |A| - 2m \right) + m. \quad (2)$$

Proof of Lemma 11. We shall use induction on m . Suppose first that $m = 2$, and let \mathcal{A} consist of x sets of size two and y sets of size one. If $x \geq 2$, then we have

$$\binom{x}{2} + xy + \binom{y}{2} = \binom{x+y}{2} = \lambda(x+y)(x+y-2) + 2 \leq \lambda(r)(2x+y-4) + 2,$$

since $2x+y \geq 4$ and $x+y \leq r$. Similarly, if $x = 1$ then $y + \binom{y-1}{2} \leq \lambda(r)(y-2) + 2$ for every $3 \leq y \leq r-1$, and if $x = 0$, then $1 + \binom{y-2}{2} \leq \lambda(r)(y-4) + 2$ for every $4 \leq y \leq r$.

So let $m \geq 3$, and let \mathcal{A} be a multi-family as described, let $T = \{A \in \mathcal{A} : m \in A\}$, and apply the induction hypothesis to the multi-family \mathcal{A}' obtained by removing m from each element of T . Letting $t = |T|$, assume first that $t < r$. This gives

$$\begin{aligned} \left| \left\{ \{A, B\} \in \binom{\mathcal{A}}{2} : A \cap B \neq \emptyset \right\} \right| &\leq \left| \left\{ \{A, B\} \in \binom{\mathcal{A}'}{2} : A \cap B \neq \emptyset \right\} \right| + \binom{t}{2} \\ &\leq \lambda(r) \left(\sum_{A \in \mathcal{A}'} |A| - 2m + 2 \right) + (m-1) + \binom{t}{2} \\ &= \lambda(r) \left(\sum_{A \in \mathcal{A}} |A| - 2m \right) + (m-1) + \binom{t}{2} - \lambda(r)(t-2), \end{aligned}$$

so it will suffice to show that $\lambda(r)(t-2) \geq \binom{t}{2} - 1$. But $\frac{\binom{t}{2}-1}{t-2} = \frac{t+1}{2}$, and $\lambda(r) \geq \frac{r}{2}$ if $r \geq 4$. Hence we are done unless $t = r$.

Finally, suppose that $t = r$. Then the left-hand side of (2) is equal to $\binom{r}{2}$, and the right-hand side is at least

$$\lambda(r)(r + 2(m-1) - 2m) + m = \binom{r}{2} - 2 + m \geq \binom{r}{2},$$

since \mathcal{A} is a double cover of $[m]$ and $m \geq 2$. The induction step, and hence the lemma, follows. \square

In fact, the following reformulation of Lemma 11 will be more convenient for us in the proof below. Here $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $\mathcal{P}(m)$ denotes the non-empty subsets of $[m]$.

Lemma 12. *Let $m \geq 2$ and $r \geq 4$. Given any function $a : \mathcal{P}(m) \rightarrow \mathbb{N}_0$ such that $\sum_S a_S \leq r$ and $\sum_{S \ni j} a_S \geq 2$ for every $j \in [m]$, we have*

$$\sum_{S \in \mathcal{P}(m)} \binom{a_S}{2} + \sum_{\{S, T\} \in \mathcal{J}} a_S a_T \leq \lambda(r) \left(\sum_{S \in \mathcal{P}(m)} a_S |S| - 2m \right) + m, \quad (3)$$

where $\mathcal{J} = \{\{S, T\} \in \binom{\mathcal{P}(m)}{2} : S \cap T \neq \emptyset\}$.

Proof. We apply Lemma 11 to the multi-family \mathcal{A} which contains exactly a_S copies of S for each $S \subset [m]$. The condition $\sum_{S \ni j} a_S \geq 2$ implies that \mathcal{A} is a double cover, and $\sum_S a_S \leq r$ implies that $|\mathcal{A}| \leq r$. Thus (2) holds, which is clearly equivalent to (3). \square

We can now deduce Lemma 10.

Proof of Lemma 10. We shall prove the lemma by induction on t . When $t = 1$ we have $v(B_1) = r$ and $e(B_1) = \binom{r}{2} - 1$. Clearly $\ell_1 = 1$ and $k_1 = 0$, and

$$e(B_1) = \binom{r}{2} - 1 = \lambda(r)(v(B_1) - r) + \binom{r}{2} - 1,$$

so in fact equality holds. For the induction step we divide into three cases. Let $t \geq 2$, and assume that the lemma holds for smaller values of t .

Case 1: $\ell_t = \ell_{t-1} + 1$.

Since \mathcal{G}_t has one more component than \mathcal{G}_{t-1} , it follows that $K^{(t)}$ intersects every other clique in at most one vertex. Hence all of the edges of $K^{(t)}$ are new, and so

$$e(B_t) = e(B_{t-1}) + \binom{r}{2} - 1.$$

Now let b be the number of vertices of $K^{(t)}$ which are not new, and hence intersect other components of M . Then $v(B_t) = v(B_{t-1}) + r - b$ and $k_t = k_{t-1} + b$, so, by the induction hypothesis for $t - 1$,

$$\begin{aligned} e(B_t) &\geq \left(\frac{\binom{r}{2} - 2}{r - 2} \right) (v(B_{t-1}) + k_{t-1} - \ell_{t-1} r) + (\ell_{t-1} + 1) \left(\binom{r}{2} - 1 \right) \\ &= \left(\frac{\binom{r}{2} - 2}{r - 2} \right) (v(B_t) + k_t - \ell_t r) + \ell_t \left(\binom{r}{2} - 1 \right) \end{aligned}$$

as required.

Case 2: $\ell_t = \ell_{t-1}$.

Since \mathcal{G}_t and \mathcal{G}_{t-1} have the same number of components, it follows that $K^{(t)}$ must intersect some component, C_1 , in at least two vertices, and intersects every clique not in C_1 in at most one vertex. Thus, the only edges of $K^{(t)}$ which are not new have both endpoints in C_1 . Hence, letting $a = |K^{(t)} \cap C_1|$, we have

$$e(B_t) \geq e(B_{t-1}) + \binom{r}{2} - \binom{a}{2} - 1.$$

Now, let b be the number of vertices of $K^{(t)} \setminus C_1$ which are not new, and hence intersect other components of \mathcal{G}_t . Then $v(B_t) = v(B_{t-1}) + r - a - b$ and $k_t = k_{t-1} + b$, so, by the induction hypothesis for $t - 1$,

$$\begin{aligned} e(B_t) &\geq \left(\frac{\binom{r}{2} - 2}{r - 2}\right) (v(B_{t-1}) + k_{t-1} - \ell_{t-1}r) + (\ell_{t-1} + 1) \left(\binom{r}{2} - 1\right) - \binom{a}{2} \\ &\geq \left(\frac{\binom{r}{2} - 2}{r - 2}\right) (v(B_t) + k_t - \ell_t r - r + a) + (\ell_t + 1) \left(\binom{r}{2} - 1\right) - \binom{a}{2}. \end{aligned}$$

If $a \leq r - 1$ then

$$\binom{r}{2} - 1 - \binom{a}{2} - (r - a) \left(\frac{\binom{r}{2} - 2}{r - 2}\right) \geq 0,$$

since the worst cases are the extremes ($a = 2$ and $a = r - 1$), and using the fact that $r \geq 4$. But if $a = r$, then our bound on $e(B_t)$ can be improved to $e(B_t) \geq e(B_{t-1})$ (which is trivial since we are not allowed to colour edges of B_{t-1} red), and $v(B_t) = v(B_{t-1})$, so we are done in this case as well.

Case 3: $\ell_t < \ell_{t-1}$.

This case is more difficult, and we shall need to use Lemma 12. Set $m = \ell_{t-1} - \ell_t + 1$, and observe that $m \geq 2$, and that $K^{(t)}$ intersects m components C_1, \dots, C_m in at least two vertices each, and intersects every clique not in these components in at most one vertex. Define, for each $S \in \mathcal{P}(m)$,

$$a_S = |\{v \in K^{(t)} : v \in C_j \Leftrightarrow j \in S\}|,$$

and set

$$e(A) = \sum_{S \in \mathcal{P}(m)} \binom{a_S}{2} + \sum_{\{S, T\} \in \mathcal{J}} a_S a_T,$$

where $\mathcal{J} = \{\{S, T\} \in \binom{\mathcal{P}(m)}{2} : S \cap T \neq \emptyset\}$, as in Lemma 12. Then

$$e(B_t) \geq e(B_{t-1}) + \binom{r}{2} - e(A) - 1.$$

Set $a = \sum_{S \in \mathcal{P}(m)} a_S$, and let b denote the number of vertices of $K^{(t)} \setminus (C_1 \cup \dots \cup C_m)$ which intersect other components of \mathcal{G}_t . Then $v(B_t) = v(B_{t-1}) + r - a - b$. Also, let $c = \sum_{S \in \mathcal{P}(m)} a_S(|S| - 1)$, and observe that $k_t \leq k_{t-1} + b - c$.

Thus, by the induction hypothesis for $t - 1$,

$$\begin{aligned} e(B_t) &\geq \left(\frac{\binom{r}{2} - 2}{r - 2}\right) \left(v(B_{t-1}) + k_{t-1} - \ell_{t-1}r\right) + (\ell_{t-1} + 1) \left(\binom{r}{2} - 1\right) - e(A) \\ &\geq \left(\frac{\binom{r}{2} - 2}{r - 2}\right) \left(v(B_t) + k_t - \ell_t r - mr + a + c\right) + (\ell_t + m) \left(\binom{r}{2} - 1\right) - e(A). \end{aligned}$$

Note that $a \leq r$ and $\sum_{S \ni j} a_S = |K^{(t)} \cap C_j| \geq 2$. Hence, by Lemma 12,

$$e(A) \leq \lambda(r) \left(\sum_{S \in \mathcal{P}(m)} a_S |S| - 2m \right) + m = \lambda(r)(a + c) - m \left(\lambda(r)r - \binom{r}{2} + 1 \right),$$

since $2\lambda(r) - 1 = \lambda(r)r - \binom{r}{2} + 1$. Thus

$$e(B_t) \geq \left(\frac{\binom{r}{2} - 2}{r - 2}\right) \left(v(B_t) + k_t - \ell_t r\right) + \ell_t \left(\binom{r}{2} - 1\right),$$

as required. This completes the induction step, and hence the proof of the lemma. \square

For completeness, let us quickly note formally that Lemma 9 follows immediately from Lemma 10.

Proof of Lemma 9. Let F be an r -witness set for the graph G and the edge e , and run the Red Edge Algorithm. We claim that the graph \mathcal{G}_m is connected. To see this, consider the component C of \mathcal{G}_m which contains the edge e , and run the process backwards. A little thought reveals that every edge of $F(e)$ must lie in some clique in C , and so the component C must span the entire graph \mathcal{G}_m , as claimed.

Thus $\ell_m = 1$, and so $c(v) = 1$ for every $v \in V(F)$, which means that $k_m = 0$. Hence, by Lemma 10,

$$e(F) \geq \left(\frac{\binom{r}{2} - 2}{r - 2}\right) (v(F) - r) + \left(\binom{r}{2} - 1\right) = \lambda(r)(v(F) - 2) + 1,$$

as required. \square

3.2. Bootstrap methods. To deduce Proposition 8, we shall borrow only one simple idea from the theory of bootstrap percolation. The following lemma is based on an idea of Aizenman and Lebowitz [3].

Lemma 13. *Let F be an r -witness set on a graph G , and let $L \in \mathbb{N}$. If $e(F) \geq \binom{r}{2}L$, then there exists an edge $f \in E(G)$ with*

$$L \leq e(F(f)) \leq \binom{r}{2}L$$

in the same realization of the Witness-Set Algorithm.

Proof. Run the Witness-Set Algorithm, and observe that the maximum size of $e(F(f))$, over all infected edges, increases by at most a factor of $\binom{r}{2}$ at each step of the process. It follows immediately that a graph $F(f)$ as described must have been created, at some point in the process. Moreover, such a graph exists with $F(f) \subset F$. \square

We have finally finished with our deterministic preliminaries, and it is time to reintroduce randomness. There is, however, little left to do: the bound we require will follow easily from Lemmas 9 and 13 by Markov's inequality.

For each $m \in \mathbb{N}$ and every $e \in E(K_n)$, let

$$Y_m(e) := \left| \left\{ S \subset [n] : e \subset S, \text{ and } e(G_{n,p}[S]) \geq m \geq \lambda(r)(|S| - 2) + 1 \right\} \right|$$

be the random variable which counts the number of sets S which contain e , and also at least $m \geq \lambda(r)(|S| - 2) + 1$ edges of $G_{n,p}$. We first bound the expected size of $Y_m(e)$.

Lemma 14. *For every $r \geq 4$, there exists a $C(r) > 0$ such that the following holds. If $n \in \mathbb{N}$ and $p > 0$ satisfy $pn^{1/\lambda(r)} \log n \leq 1/(2e)$, and n is sufficiently large, then*

$$\mathbb{E}(Y_m(e)) \leq \left(\frac{m + C(r)}{2 \binom{r}{2} \log n} \right)^{m - \lambda(r)}$$

for every $e \in E(K_n)$ and every $\lambda(r) + 1 \leq m \leq \binom{r}{2} \log n$.

Proof. Let $\ell \in \mathbb{N}$ be maximal such that $m \geq \lambda(r)(\ell - 2) + 1$. Then $|S| \leq \ell$, and hence

$$\mathbb{E}(Y_m(e)) \leq 2 \binom{n}{\ell - 2} \binom{\ell^2/2}{m} p^m \leq \frac{2}{(\ell - 2)!} \left(\frac{ep\ell^2}{2m} \right)^{m - \lambda(r)(\ell - 2)} \left(\frac{ep\ell^2 n^{1/\lambda(r)}}{2m} \right)^{\lambda(r)(\ell - 2)}.$$

Note that $1 \leq m - \lambda(r)(\ell - 2) \leq \lambda(r)$, and that $\frac{2}{(\ell - 2)!} \left(\frac{ep\ell^2}{2m} \right)^\alpha \leq 1$ for every $\alpha \in [\lambda(r)]$, assuming n is sufficiently large. Thus it suffices to observe that

$$\frac{\ell^2}{4m} \leq \frac{m + C(r)}{2 \binom{r}{2}}$$

if $C(r)$ is sufficiently large, since $m(m + C(r)) \geq (m + 2\lambda(r))^2 \geq (\lambda(r)\ell)^2 \geq \frac{r^2\ell^2}{4}$. \square

We can now easily deduce Proposition 8.

Proof of Proposition 8. Let $r \geq 4$, let $n \in \mathbb{N}$, and let $p = p(n) > 0$ satisfy $pn^{1/\lambda(r)} \log n \leq 1/(2e)$. We claim that, for every $e \in E(K_n)$,

$$\mathbb{P}(e \in \langle G_{n,p} \rangle_{K_r}) \rightarrow 0$$

as $n \rightarrow \infty$. Indeed, suppose that $e \in \langle G_{n,p} \rangle_{K_r}$, run the Witness-Set Algorithm, and consider the graph $F = F(e) \subset G_{n,p}$.

Suppose first that $e(F) \leq \binom{r}{2} \log n$. By Lemma 9, we have

$$e(F) \geq \lambda(r)(v(F) - 2) + 1,$$

and thus either $e \in G_{n,p}$, or $Y_m(e) \geq 1$ for some

$$\lambda(r) + 1 \leq m \leq \binom{r}{2} \log n.$$

By Lemma 14, this has probability at most

$$p + \sum_{m=\lambda(r)+1}^{\binom{r}{2} \log n} \mathbb{E}(Y_m(e)) \leq p + \sum_{m=\lambda(r)+1}^{\binom{r}{2} \log n} \left(\frac{m + O(r)}{2 \binom{r}{2} \log n} \right)^{m-\lambda(r)} \rightarrow 0,$$

as $n \rightarrow \infty$, as claimed.

So suppose next that $e(F) \geq \binom{r}{2} \log n$. By Lemma 13, there must exist an edge f in K_n such that $\log n \leq e(F(f)) \leq \binom{r}{2} \log n$, which means that $Y_m(f) \geq 1$ for some $\log n \leq m \leq \binom{r}{2} \log n$. By Lemma 14, the expected number of such edges f is at most

$$\binom{n}{2} \sum_{m=\log n}^{\binom{r}{2} \log n} \left(\frac{m + C(r)}{2 \binom{r}{2} \log n} \right)^{m-\lambda(r)} \leq n^2 \left(\frac{2}{r^2} \right)^{\log n - \lambda(r)} \rightarrow 0,$$

as $n \rightarrow \infty$, since $r \geq 4$. This proves the proposition. \square

We finish by noting that Theorem 1 follows immediately from Propositions 3 and 8.

Proof of Theorem 1. By Proposition 3, we have

$$p_c(n, H) \ll (\log n)^{2/\lambda(H)} n^{-1/\lambda(H)}$$

for every balanced graph H . Moreover K_r is balanced, since

$$\frac{\binom{r-1}{2} - 1}{r-3} \leq \frac{\binom{r}{2} - 2}{r-2}$$

for every $r \geq 4$, and $\lambda(K_r) = \lambda(r) \geq 2$, so the upper bound follows.

For the lower bound, suppose that $pn^{1/\lambda(r)} \log n \leq 1/(2e)$, and n is sufficiently large. By Proposition 8, we have

$$\mathbb{P}(e \in \langle G_{n,p} \rangle_{K_r}) \rightarrow 0$$

for every edge $e \in E(K_n)$. Thus $G_{n,p}$ does not percolate, with high probability, as required. \square

4. THE THRESHOLD FOR K_4 -PERCOLATION

In this section we shall prove Theorem 2, which determines the threshold for K_4 -percolation on K_n . The proof is quite different from that of Theorem 1, and uses ideas from the study of 2-neighbour bootstrap percolation on $[n]^d$; see in particular [3, 4, 24].

We begin with a simple but key observation. A hypergraph \mathcal{H} is said to be triangle-free if there do not exist distinct vertices v_1, v_2, v_3 and edges $e_1, e_2, e_3 \in E(\mathcal{H})$ such that $v_1 \in e_2 \cap e_3$, $v_2 \in e_1 \cap e_3$ and $v_3 \in e_1 \cap e_2$.

Observation 15. *For every graph G , the graph $\langle G \rangle_{K_4}$ consists of a collection of edge-disjoint cliques. Moreover, the hypergraph defined by these cliques is triangle-free.*

Proof. For the first part, simply note that if two cliques R_1 and R_2 share more than one vertex, then the closure $\langle R_1 \cup R_2 \rangle_{R_4}$ is a clique on vertex set $V(R_1) \cup V(R_2)$. To prove the second part, observe that if R_1 , R_2 and R_3 form a triangle, then the closure $\langle R_1 \cup R_2 \cup R_3 \rangle_{R_4}$ is a clique on vertex set $V(R_1) \cup V(R_2) \cup V(R_3)$. \square

Say that a clique K is *internally spanned* by a graph G if $\langle G \cap K \rangle_{K_4} = K$. We shall study, for each $\ell \in \mathbb{N}$ and $p > 0$, the probability

$$P(\ell, p) := \mathbb{P}\left(K_\ell \text{ is internally spanned by } G_{n,p}\right).$$

The following bounds are both straightforward.

Lemma 16. *For every $3 \leq \ell \in \mathbb{N}$ and $p \in (0, 1)$ with $p\ell^2 \leq 1$,*

$$\left(\frac{1}{2e^2}\right)^\ell (\ell p)^{2\ell-3} \leq P(\ell, p) \leq (4e)^3 \left(\frac{3}{4}\right)^{2\ell} (\ell p)^{2\ell-3}.$$

Proof. For the lower bound, simply count the graphs on vertex set $[\ell]$, and with $2\ell - 3$ edges, in which every vertex $j \geq 3$ sends two edges ‘backwards’ in the order induced by \mathbb{Z} . It is easy to see, by induction on t , that the clique K_t with vertex set $[t]$ is internally spanned, for each $t \in [\ell]$. The number of such graphs is

$$\prod_{j=3}^{\ell} \binom{j-1}{2} \geq \frac{(\ell!)^2}{2^\ell \ell^3} \geq \frac{2\pi \ell^{2\ell-3}}{(2e^2)^\ell},$$

by Stirling’s formula, and each is an *induced* subgraph of $G_{n,p}$ with probability at least $p^{2\ell-3}(1-p)^{\ell^2} \geq p^{2\ell-3}/2\pi$. Since these events are mutually exclusive, the lower bound follows.

For the upper bound, recall that if a graph G internally spans K_ℓ , then $e(G) \geq 2\ell - 3$. (This was first proved in [13]; see also Lemma 18 for a short proof.) It follows that

$$P(\ell, p) \leq \binom{\ell^2/2}{2\ell-3} p^{2\ell-3} \leq \left(\frac{e\ell}{4\ell-6}\right)^{2\ell-3} (\ell p)^{2\ell-3} \leq (4e)^3 \left(\frac{3}{4}\right)^{2\ell} (\ell p)^{2\ell-3},$$

as required. \square

4.1. The lower bound. The following lemma, like Lemma 13, it is based on an idea of Aizenman and Lebowitz [3], who proved the corresponding result in the context of two-neighbour bootstrap percolation on $[n]^d$. The lower bound in Theorem 2 will follow by combining it with Lemma 16.

Lemma 17. *Suppose that $\langle G \rangle_{K_4} = K_n$, and let $1 \leq L \leq n$. There exists a clique $K \subset K_n$ which is internally spanned by G , with*

$$L \leq v(K) \leq 3L.$$

We remark that this result does not generalize to K_r -percolation for $r \geq 5$. In fact, it is not hard to construct a graph G for which $\langle G \rangle_{K_r} = K_n$, but no clique K_ℓ with $r < \ell < n$ is internally spanned.

In order to prove Lemma 17 we shall introduce a simple algorithm for filling K_n , which we call the Clique-Process. This algorithm will also provide a short proof of the bound $e(G) \geq$

$2\ell - 3$ for a graph G which internally spans K_ℓ . It is analogous to the ‘rectangle process’ in two-neighbour bootstrap percolation on $[n]^d$ (see Proposition 30 of [25] or Theorem 11 of [4]).

The Clique Process. Let G be a graph on n vertices, and run the K_4 -process as follows:

0. At each step of the process, we will maintain a collection $(R_1, A_1), \dots, (R_m, A_m)$, where R_j is a clique and $A_j \subset E(G)$, such that $\langle A_j \rangle_{K_4} = R_j$ for each $j \in [m]$.
1. At time zero, set $R_j = A_j = \{e_j\}$ for each $j \in [m]$, where $E(G) = \{e_1, \dots, e_m\}$.
2. At time $t \in 2\mathbb{Z}$, choose a pair $\{i, j\}$ such that $|R_i \cap R_j| \geq 2$, if such a pair exists. Delete (R_i, A_i) and (R_j, A_j) , and replace them with $(\langle A_i \cup A_j \rangle_{K_4}, A_i \cup A_j)$.
3. At time $t \in 2\mathbb{Z} + 1$, choose a triple $\{i, j, k\}$ such that R_i, R_j , and R_k form a triangle in the hypergraph defined by the cliques, if such a triple exists. Delete (R_i, A_i) , (R_j, A_j) and (R_k, A_k) , and replace them with $(\langle A_i \cup A_j \cup A_k \rangle_{K_4}, A_i \cup A_j \cup A_k)$.
4. Repeat steps 2 and 3 until all edges of $\langle G \rangle_{K_4}$ are infected.

The algorithm terminates by the proof of Observation 15. Observe moreover that the A_j are in fact *disjoint* sets of edges of G . We can now prove Lemma 17.

Proof of Lemma 17. Suppose that $\langle G \rangle_{K_4} = K_n$, and run the Clique Process for G . At each step of the process, the value of $\max_{j \in [m]} v(R_j)$ increases by a factor of at most three. Hence, for every $L \in [n]$, there exists a clique $K = R_j \subset K_n$, with

$$L \leq v(K) \leq 3L,$$

which is internally spanned by G , as claimed. \square

We can also easily deduce the following bound, which was first proved by Bollobás [13].

Lemma 18. *If G internally spans K_ℓ then $e(G) \geq 2\ell - 3$.*

Proof. We shall use induction on ℓ ; for $\ell \leq 3$ the result is trivial. Now suppose that G internally spans $R = K_\ell$, and run the Clique Process for G . At the penultimate step we have either two or three disjointly internally spanned proper sub-cliques of R , which together span R . If these cliques are $S = \langle A \rangle_{K_4}$ and $T = \langle B \rangle_{K_4}$, and $A \cap B = \emptyset$ then

$$e(G) \geq e(A) + e(B) \geq 2(v(S) + v(T)) - 6 \geq 2v(R) - 2,$$

since $|S \cap T| \geq 2$, so $v(S) + v(T) \geq v(R) + 2$. If they are $S = \langle A \rangle_{K_4}$, $T = \langle B \rangle_{K_4}$ and $U = \langle C \rangle_{K_4}$, with A, B, C pairwise disjoint, then we have

$$e(G) \geq e(A) + e(B) + e(C) \geq 2(v(S) + v(T) + v(U)) - 9 \geq 2v(R) - 3,$$

since (S, T, U) form a triangle, so $v(S) + v(T) + v(U) \geq v(R) + 3$. \square

We can now prove the lower bound on $p_c(n, K_4)$ in Theorem 2. It follows easily from Lemmas 16 and 17, using Markov’s inequality.

Proposition 19. *If $p^2 n \log n \leq 16/(9e^3)$, then*

$$\mathbb{P}\left(\langle G_{n,p} \rangle_{K_4} = K_n\right) \rightarrow 0$$

as $n \rightarrow \infty$.

Proof. Let $p^2 n \log n = 16/(9e^3)$ and $L = \log n$. By Lemma 17, if $\langle G_{n,p} \rangle_{K_4} = K_n$ then there exists an internally spanned clique R with $L \leq v(R) \leq 3L$. By Lemma 16, the expected number of such cliques is at most

$$(4e)^3 \sum_{\ell=L}^{3L} \binom{n}{\ell} \left(\frac{3}{4}\right)^{2\ell} (\ell p)^{2\ell-3} \leq \sum_{\ell=L}^{3L} \left(\frac{4e}{\ell p}\right)^3 \left(\frac{9e\ell p^2 n}{16}\right)^\ell \leq \sum_{\ell=L}^{3L} n^{3/2} \left(\frac{\ell}{e^2 \log n}\right)^\ell.$$

Thus

$$\sum_{\ell=L}^{3L} n^{3/2} \left(\frac{\ell}{e^2 \log n}\right)^\ell \leq 3L \cdot n^{3/2} e^{-2\ell} \rightarrow 0$$

as $n \rightarrow \infty$, as required. \square

4.2. The upper bound. We shall use the second moment method (and Lemma 16) in order to show that $G_{n,p}$ internally spans a clique of order $\sim \log n$ with high probability. We will then deduce the upper bound in Theorem 2 using sprinkling.

Let $X(\ell, p)$ denote the random variable which counts the number of copies of K_ℓ which are internally spanned by $G_{n,p}$. We first bound the expected value of $X(\ell, p)$.

Lemma 20. *For every $n \in \mathbb{N}$, $3 \leq \ell \in \mathbb{N}$ and $p \in (0, 1)$ with $p\ell^2 \leq 1$,*

$$\mathbb{E}(X(\ell, p)) \geq \left(\frac{p^2 n \ell}{2e^2}\right)^\ell \left(\frac{1}{\ell p}\right)^3.$$

Proof. By Lemma 16, we have

$$\mathbb{E}(X(\ell, p)) \geq \binom{n}{\ell} \left(\frac{1}{2e^2}\right)^\ell (\ell p)^{2\ell-3} \geq \left(\frac{p^2 n \ell}{2e^2}\right)^\ell \left(\frac{1}{\ell p}\right)^3$$

as required. \square

To bound the variance of $X(\ell, p)$, we shall use the following extension of Lemma 18. Given cliques $S \subset R$, let

$$D(S, R) := \left\{ \langle (G_{n,p} \cup S) \cap R \rangle_{K_4} = R \right\}$$

denote the event that R is internally spanned by $G_{n,p} \cup S$. Lemma 18 is equivalent to the case $v(S) = 3$ of the following lemma.

Lemma 21. *If $D(S, R)$ holds, then $e((G_{n,p} \setminus S) \cap R) \geq 2(v(R) - v(S))$.*

Proof. We shall use induction on $\ell = v(R)$. Suppose that $D(S, R)$ holds, and apply the Clique Process, except starting with the clique S already formed. Suppose at the penultimate step we have two disjointly internally spanned cliques, $T = \langle A \cup S \rangle_{K_4}$ and $U = \langle B \rangle_{K_4}$, where $A \cap S = A \cap B = \emptyset$. (That A and B may be taken to be disjoint follows by the comment after the Clique Process.) By the induction hypothesis, we have

$$e((G_{n,p} \setminus S) \cap R) \geq e(A) + e(B) \geq 2(v(T) - v(S)) + 2v(U) - 3 > 2(v(R) - v(S)),$$

since $|T \cap U| \geq 2$. The case of three cliques is similar, so we shall skip the details. \square

We can now bound the variance of $X(\ell, p)$. Let $P(k, \ell) = \mathbb{P}(D(K_k, K_\ell))$.

Lemma 22. *Let $n \in \mathbb{N}$, $4\ell \leq \log n$ and $p^2 n \log n \geq 1/4e$. Then*

$$\text{Var}(X(\ell, p)) \ll \mathbb{E}(X(\ell, p))^2$$

as $n \rightarrow \infty$.

Proof. We first claim that

$$\text{Var}(X(\ell, p)) \leq \sum_{k=2}^{\ell} \mathbb{E}(X(\ell, p)) \binom{\ell}{k} \binom{n}{\ell-k} P(k, \ell).$$

This follows by considering ordered pairs (S, T) of internally spanned ℓ -cliques which intersect in a k -clique. By Lemma 21, if $D(S \cap T, T)$ holds then there are at least $2(\ell - k)$ edges of $G_{n,p}$ in $T \setminus S$, and so

$$P(k, \ell) \leq \binom{(\ell^2 - k^2)/2}{2\ell - 2k} p^{2(\ell-k)} \leq \left(\frac{e(\ell+k)p}{4} \right)^{2(\ell-k)}.$$

Thus, by Lemma 20,

$$P(k, \ell) \leq \left(\frac{e(\ell+k)p}{4} \right)^{2(\ell-k)} \left(\frac{2e^2}{p^2 n \ell} \right)^{\ell} (\ell p)^3 \mathbb{E}(X(\ell, p)).$$

But, using the fact that $(\ell+k)^{\ell-k} \leq e^{2k}(\ell-k)^{\ell-k}$, an easy calculation gives

$$\binom{\ell}{k} \binom{n}{\ell-k} \left(\frac{e(\ell+k)p}{4} \right)^{2(\ell-k)} \left(\frac{2e^2}{p^2 n \ell} \right)^{\ell} \leq \left(\frac{e^5}{4} \right)^{\ell} \left(\frac{1}{8kp^2 n} \right)^k.$$

Hence, using the fact that $(1/Cx)^x \leq e^{1/Ce}$ and $(\ell p)^3 \left(\frac{e^5}{4} \right)^{\ell} \ll n^{-1/2-o(1)}$, we obtain

$$\text{Var}(X(\ell, p)) \leq \mathbb{E}(X(\ell, p))^2 (\ell p)^3 \left(\frac{e^5}{4} \right)^{\ell} \sum_{k=2}^{\ell} \left(\frac{1}{8kp^2 n} \right)^k \ll \mathbb{E}(X(\ell, p))^2,$$

as required. \square

Using Chebychev, and sprinkling, we can now deduce the following bound on $p_c(n, K_4)$.

Proposition 23. *If $p^2 n \log n \geq 18$, then*

$$\mathbb{P}(\langle G_{n,p} \rangle_{K_4} = K_n) \rightarrow 1$$

as $n \rightarrow \infty$.

Proof. Set $4\ell = \log n$ and $p^2 n \log n = 1 > 1/4e$, and observe that the conditions of Lemmas 20 and 22 are satisfied. By Lemma 20, we obtain

$$\mathbb{E}(X(\ell, p)) \geq \left(\frac{p^2 n \ell}{2e^2} \right)^{\ell} \left(\frac{1}{\ell p} \right)^3 \geq \left(\frac{1}{8e^2} \right)^{\log n/4} n^{3/2-o(1)} \rightarrow \infty.$$

Thus, by Lemma 22 and Chebychev's inequality, with high probability there exists a copy of K_{ℓ} which is internally spanned by $H := G_{n,p}$.

Now set $p_j = 2^{-j+4}p$ for each $j \in \mathbb{N}$, and let $G_j = G_{n,p_j}$ be independent random graphs with density p_j . We make the following claim.

Claim 1: Let $\varepsilon > 0$ be sufficiently small. If $2^{2j-2}\ell = t \leq \varepsilon n$, then $\langle G_j \cup K_t \rangle_{K_4}$ contains a clique of size $4t$, with probability at least $1 - e^{-t/8}$.

Proof of Claim 1. Observe that every vertex v that has at least two neighbours in K_t (in the graph G_j) is added to the clique in $\langle G_j \cup K_t \rangle$. It therefore suffices to show that there are at least $3t$ such vertices, with high probability. The expected number of such vertices is at least

$$\frac{3n}{4} \cdot \binom{t}{2} p_j^2 (1 - p_j)^{t-2} \geq 16tp^2 n \ell \geq 4t$$

since $p_j t = 2^{j+2} \ell p = 8p\sqrt{\ell t} = O(\sqrt{\varepsilon})$.

This event (having two neighbours) is independent for each vertex. Thus, by Chernoff's inequality, with probability at least $1 - e^{-t/8}$, the number of such vertices is at least $3t$, as required. \square

We apply the claim for each $j \geq 0$. It follows that, with high probability, $\langle H \cup \bigcup_{j=1}^{\infty} G_j \rangle_{K_4}$ contains a clique of order εn , for some $\varepsilon > 0$. Finally, let H' be another independent copy of $G_{n,p}$.

Claim 2: If $t \geq \varepsilon n$, then $\langle H' \cup K_t \rangle_{K_4} = K_n$ with high probability.

Proof of Claim 2. We apply the same argument as in the proof of Claim 1. Indeed, the probability that a vertex v has at most one neighbour in K_t is at most

$$(1 - p)^t + tp(1 - p)^{t-1} \leq (1 + 2tp)e^{-tp} \ll \frac{1}{n^2}$$

as $n \rightarrow \infty$. Hence, by Markov, the probability that there exists such a vertex is at most $1/n$, as required. \square

To complete the proof, we simply note that the graph $G = H \cup H' \cup \bigcup_{j=1}^{\infty} G_j$ is a random graph G_{n,p^*} of density

$$p^* \leq 2p + \sum_{j=1}^{\infty} 2^{-j+4} p \leq 18p,$$

and G_{n,p^*} percolates in the K_4 -process with high probability, as required. \square

Theorem 2 follows immediately from Propositions 19 and 23.

5. OTHER GRAPHS, AND OPEN PROBLEMS

In this section we shall mention some simple results for graphs other than K_r , and state several of the many open problems relating to this model. Since the results will all be fairly straightforward, we shall only sketch the proofs. We begin by stating a simple extension of the (trivial) result for the K_3 -process mentioned in the Introduction.

Proposition 24. *Let $H = C_k$ for some $k \geq 3$, or $H = K_{2,3}$. Then,*

$$p_c(n, H) = \frac{\log n}{n} + \Theta\left(\frac{1}{n}\right).$$

Sketch of proof. We shall show that, with high probability, the graph $G_{n,p}$ percolates in the H -bootstrap process if and only if it is connected. The bounds on $p_c(n, H)$ then follow by standard results, see [14].

Indeed, first let $H = C_k$ and consider a path of length at least k attached to a triangle; we claim that this graph spans a clique (on its vertex set). To see this, identify the vertices with $[\ell]$ so that the edges are $\{i(i+1) : i \in [\ell-1]\} \cup \{13\}$, and say that ij is a t -edge if $|i-j| = t$. The edges are infected in the following order: $(k-1)$ -edges, k -edges, 2-edges, 3-edges, 4-edges, and so on. Finally, observe that if $G_{n,p}$ is connected then, with high probability, every vertex has a path of length at least k leading to a triangle.

For $H = K_{2,3}$ the proof is similar. Let $x, y \in V(G_{n,p})$, and suppose that there exist vertex disjoint paths from x and y to opposite corners of a copy of C_4 . Then it is easy to see that the percolation process works its way along these paths and eventually infects the edge xy . This gives a large complete bipartite graph, and if there is an edge in each part then the closure is a complete graph. Since $G_{n,p}$ is connected, every vertex is eventually swallowed by this clique. \square

The case $H = K_{2,3}$ is the first we have seen for which $p_c(n, H) \neq n^{-1/\lambda(H)+o(1)}$. We shall now determine a large family of such graphs. Define

$$\lambda^*(H) := \min_{e \in E(H)} \max_{F \subset H-e} \left\{ \frac{e(F)}{v(F)} \right\}.$$

This parameter gives us a general lower bound on $p_c(n, H)$.

Proposition 25. *For every graph H , there exists a constant $c(H)$ such that*

$$p_c(n, H) \geq c(H)n^{-1/\lambda^*(H)}$$

for every $n \in \mathbb{N}$.

Sketch of proof. We shall show that if $p \leq c(H)n^{-1/\lambda^*(H)}$ then, with probability at least $1/2$, no new edges are infected in the H -bootstrap process. To do so, for each $e \in E(H)$ choose a subgraph $F = F(e) \subset H - e$ which maximizes $e(F)/v(F)$, and note that $e(F)/v(F) \geq \lambda^*(H)$. Thus, the expected number of copies of F in $G_{n,p}$ is at most

$$\binom{n}{v(F)} p^{e(F)} \leq c(H)n^{v(F)-e(H)/\lambda^*(H)} \leq c(H).$$

Summing over edges of H , we obtain

$$\mathbb{P}\left(F(e) \subset G_{n,p} \text{ for some } e \in E(H)\right) \leq e(H)c(H) < \frac{1}{2},$$

if $c(H)$ is sufficiently small. But if $F(e) \not\subset G_{n,p}$ for every $e \in E(H)$ then $H - e \not\subset G_{n,p}$ for every $e \in E(H)$, and hence no new edges are infected, as claimed. \square

We next show that Proposition 25 is sharp for a large class of graphs H .

Proposition 26. *If H has a leaf, then*

$$p_c(n, H) = \Theta\left(n^{-1/\lambda^*(H)}\right).$$

Sketch of proof. The lower bound follows from Proposition 25. For the upper bound, let $p \gg n^{-1/\lambda^*(H)}$ and recall (see [14]) that, with high probability, $H - e \subset G_{n,p}$ for some $e \in E(H)$. (To see this, let e and $F \subset H - e$ be such that $e(F)/v(F) = \max_{F' \subset H - e} e(F')/v(F') = \lambda^*(H)$, find a copy of F in $G_{n,p}$ by the second moment method, and then find $H - e$ by sprinkling.) Let v_1 be the neighbour of a leaf in H , and observe that we can infect every edge which is incident with v (and is not in our copy of $H - e$).

Now, take a second, independent copy of $G_{n,p}$, and apply the same argument inside the neighbourhood of v_1 . We find a vertex v_2 such that we can add (almost) all edges incident with v_2 . Repeating this process $v(H)$ times, we find (with high probability) a clique on $v(H)$ vertices in $\langle G_{n,p^*} \rangle_H$, where $p^* = v(H)p$.

Finally, observe that $\langle K_{v(H)} \rangle_H = K_n$, since we may add the remaining vertices to the clique one by one. Thus $p = O(n^{-1/\lambda^*(H)})$, as claimed. \square

A slightly less trivial case, which lies somewhere between a clique and a tree, also matches the general lower bound in Proposition 25. Say that H is an r -clique-tree if (for some $2 \leq \ell \in \mathbb{N}$) it is composed of ℓ disjoint copies of K_r , plus $\ell - 1$ extra edges, and is connected.

Proposition 27. *Let H be an r -clique tree. Then*

$$c(H)n^{-1/\lambda^*(H)} \leq p_c(n, H) \leq n^{-1/\lambda^*(H)} \log n$$

for some $c(H) > 0$.

Sketch of proof. The lower bound again follows by Proposition 25. For the upper bound, first observe that

$$\lambda^*(H) = \frac{\binom{r}{2}}{r} + \frac{\ell - 2}{\ell r},$$

where $v(H) = \ell r$. Assume first that $\ell \geq 3$, and let $p \gg n^{-1/\lambda^*(H)}$ (we shall prove a stronger result in this case). Note that, as in the previous proof, $H - e \subset G_{n,p}$ for some $e \in E(H)$ with high probability; in fact, there exist at least $v(H)$ copies of $H - e$. Moreover, setting $\varepsilon = \frac{\ell - 2}{\ell r}$, there exist at least n^ε copies of K_r in $G_{n,p}$. Let X denote the union of those copies of K_r which do not intersect a copy of $H - e$.

From each copy of $H - e$, pick a clique R which is the neighbour of a leaf (in the tree-structure of H), and observe that we may infect every edge between R and X . We thus obtain a complete bipartite graph, with parts of size $v(H)$ and n^ε . Moreover, each part consists of r -cliques, and thus these edges span a clique on the same vertex set.

Finally, sprinkling edges with density p , we see that every vertex in a copy of K_r minus an edge, and with a neighbour in X , is added to the clique. With high probability there are $n^{2\varepsilon}$ such vertices. Repeating this process $1/\varepsilon$ times, we infect the entire edge set, as required.

For the case $\ell = 2$ we prove the weaker bound in the statement. Let p be as above, and take $\log n$ copies of $G_{n,p}$. In the first we span a clique of order C , for some large constant C ; in the second a clique of order C^2 ; in the third C^3 , and so on. In the first step this is just the union of copies of K_r ; in later steps it is the union of copies of K_r minus an edge which have a neighbour in the clique formed in the previous step. The proposition now follows. \square

We give one final cautionary example, whose purpose is just to point out that $\lambda(H)$ and $\lambda^*(H)$ are not the only possible values of

$$-\limsup_{n \rightarrow \infty} \frac{\log n}{\log p_c(n, H)}.$$

Let DD_r denote the ‘double-dumbbell’, the graph consisting of two disjoint copies of K_r , plus two extra (disjoint) edges between the two cliques. Note that $\lambda(DD_r) = r/2$ and $\lambda^*(DD_r) = (2\binom{r}{2} + 1)/2r$, and therefore

$$\lambda^*(DD_r) < \frac{\binom{r}{2} + 1}{r} < \lambda(DD_r).$$

Proposition 28. *For every $r \geq 4$,*

$$-\lim_{n \rightarrow \infty} \frac{\log n}{\log p_c(n, DD_r)} = \frac{\binom{r}{2} + 1}{r}.$$

Sketch of proof. The key observation is that if $H = DD_r$ and $e \in E(DD_r)$, then $\langle H - e \rangle_H = K_{|H|}$, i.e., a copy of DD_r spans a clique on its vertex set. Moreover, two ($\geq 2r$)-cliques which overlap in two (or more) points span a clique on their union. We shall use these observations, plus the usual ‘critical droplet’ argument from bootstrap percolation on $[n]^d$.

Let’s begin with the (easier) upper bound. Let $n^r p^{\binom{r}{2}+1} \gg \log n$, and consider $m = \log n$ copies G_1, \dots, G_m of $G_{n,p}$. We claim that their union percolates with high probability. To see this, first observe that $G_{n,p}$ contains an r -clique R_1 with high probability. Next, note that the expected number of copies of K_r plus a pendant edge, with its endpoint in R_1 , is at least $|R_1| \binom{n-|R_1|}{r} p^{\binom{r}{2}+1} \gg \log n$. Using Chebychev, it follows that there exist at least $\log n$ such copies with high probability, and the closure of these is a clique R_2 on at least $\log n$ vertices. Now, simply repeat this procedure for each graph G_3, \dots, G_m . A straightforward calculation shows that, with high probability, at each step the clique R_j (at least) doubles in size, until it reaches size $1/p$. But now a positive fraction of the vertices have r neighbours in R_{m-2} , so $|R_{m-1}| \geq \varepsilon n$, and thus $|R_m| = n$ with high probability, as required.

To prove the lower bound, we define a process analogous to the Clique Process in Section 4. To be precise, we can break up the process into steps of the following two types: (a) if two ($\geq 2r$)-cliques share two vertices then merge them, and (b) if an edge is infected then consider the copy of H it completes, and merge the ($\geq 2r$)-cliques which provided the edges of $H - e$. To see that this works, recall that $\langle DD_r - e \rangle_{DD_r} = K_{2r}$.

Using this process, we can easily prove a result analogous to Lemma 17, except with 3 replaced by $e(H)$. Indeed, at each step the size of the largest clique increases by at most a factor of $e(H)$. Moreover, by considering the penultimate step of the process, as in Lemma 18, and using induction, we can prove the following extremal result: If $\langle G \rangle_{DD_r} = K_n$ and $n \geq r$, then

$$e(G) \geq \left(\frac{\binom{r}{2} + 1}{r} \right) n - 1.$$

The result now follows by a straightforward (and standard) calculation (using Markov), as in the proof of Proposition 19. \square

We now turn to some open problems. We begin by asking for sharper versions of Theorems 1 and 2.

Problem 1. Determine $p_c(n, K_r)$ up to a constant factor.

Problem 2. Find $1/4 \leq \alpha \leq 5$, if it exists, such that

$$p_c(n, K_4) = (1 + o(1)) \frac{\alpha}{\sqrt{n \log n}}.$$

A natural family of graphs for which we do not have good bounds on the critical probability are the complete bipartite graphs.

Problem 3. Determine $p_c(n, K_{s,t})$, at least up to a poly-logarithmic factor, for all $s, t \in \mathbb{N}$.

Another is the random graph, $G_{k,1/2}$.

Problem 4. Give bounds on $p_c(n, G_{k,1/2})$ which hold with high probability as $k \rightarrow \infty$.

Finally, we mention a substantial generalization of the problem we have considered in this paper. Given graphs G and H , define H -bootstrap percolation on G by only allowing edges of G to be infected, and say that a graph F percolates if, starting with F , eventually all edges of G are infected. It seems likely that there are many beautiful theorems to discover about this very general bootstrap process.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 W. GREEN STREET, URBANA, IL 61801, AND, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CA 92093

E-mail address: jobal@math.uiuc.edu

TRINITY COLLEGE, CAMBRIDGE CB2 1TQ, ENGLAND, AND, DEPARTMENT OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF MEMPHIS, MEMPHIS, TN 38152, USA

E-mail address: B.Bollobas@dpmms.cam.ac.uk

IMPA, ESTRADA DONA CASTORINA 110, JARDIM BOTÂNICO, RIO DE JANEIRO, RJ, BRASIL

E-mail address: rob@impa.br