MASS-CAPACITY INEQUALITIES FOR CONFORMALLY FLAT MANIFOLDS WITH BOUNDARY

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Abstract. In this paper we prove a mass-capacity inequality and a volumetric Penrose inequality for conformally flat manifolds, in arbitrary dimensions. As a by-product of the proofs, capacity and Aleksandrov-Fenchel inequalities for mean-convex Euclidean domains are obtained. For each inequality, the case of equality is characterized.

1. Introduction and Main Results

Inequalities between quasi-local quantities and global quantities have recently generated a fair amount of interest. Among those, the spacetime Penrose inequality stands out as one of the challenging open problems in mathematical relativity.

The Riemannian version of the Penrose inequality for three-dimensional manifolds was proved by Huisken and Ilmanen [7] using inverse mean curvature flow (for the case of connected horizons), and by Bray [1] using a conformal flow of the metric (for the general case). The argument of Bray uses the mass-capacity inequality in order to prove the monotonicity of the ADM mass along the conformal flow.

The proof of the mass-capacity inequality in Bray's work relies on the positive mass theorem and a modification of the reflection argument of [5]. A related reflection argument (implicitly involving a mass-capacity inequality) was later used by Bray and Lee [3] (also using the positive mass theorem) in order to prove the Riemannian Penrose inequality for dimensions less than eight. For the case of a connected boundary, but now only for dimension 3, Bray and Miao [4] gave a proof of the mass-capacity inequality which uses the monotonicity of the Hawking mass along the inverse mean curvature flow [7] instead of the positive mass theorem.

Our proof of the mass-capacity inequality (as well as the proof of the other two inequalities) uses only classical arguments and works in arbitrary dimensions.

Definition. A *conformally flat* manifold with boundary, or CF-manifold for short, is a manifold $(M^n, g), n \geq 3$, isometric to the complement of a smooth bounded open set (not necessarily connected) $\Omega \subset \mathbb{R}^n$ together with a conformally flat metric $g_{ij} = u^{\frac{4}{n-2}}\delta_{ij}$, where $u > 0$ is smooth, and so that:

- g is asymptotically flat, with non-negative scalar curvature (i.e. $\Delta_0 u \leq 0$), and normalized so that $u \to 1$ at infinity,
- $\Sigma = \partial\Omega$ is mean-convex with respect to the Euclidean metric (i.e. $H_0 > 0$),
- $\Sigma = \partial M$ is minimal with respect to the metric g (i.e. $H_g = 0$).

The main results of this paper are a mass-capacity inequality and a volumetric Penrose inequality for CF-manifolds (the latter is an improved version of the inequality of [12]), as well as a capacity and an Aleksandrov-Fenchel inequality for

Euclidean domains. The precise statements are the following. (See Section 2 for definitions.)

Theorem 1. *Let (M, g) be a CF-manifold as above, and let m denote its ADM mass.*

(a) *Mass-capacity inequality:*

$$m \geq C_g(\Sigma),$$

where $C_g(\Sigma)$ denotes the capacity of Σ in (M, g). Equality holds if and only if g the Riemannian Schwarzschild metric.

(b) *Volumetric Penrose inequality:*

$$m \geq 2 \left(\frac{V_0}{\beta_n} \right)^{\frac{n-2}{n}},$$

where V_0 is the Euclidean volume of Ω, and β_n is the volume of the Euclidean unit n-ball. Equality holds if and only if g is the Riemannian Schwarzschild metric.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain (not necessarily connected) with mean-convex boundary $\Sigma = \partial\Omega$. Denote by V_0 its volume, and by A_0, $H_0 > 0$ the area and mean curvature of Σ, respectively. Then*

(a) *Capacity inequality:*

$$C_0(\Sigma) \leq \frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0,$$

with equality achieved if and only if Ω is a round ball.

(b) *Aleksandrov-Fenchel inequality: Assume further that Σ is outer-minimizing. Then*

$$\frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0 \geq \left(\frac{A_0}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}},$$

with equality achieved if and only if Ω is a round ball.

The proof of the above results follows from Theorem 4 below, which relies on classical arguments including Huisken and Ilmanen's inverse mean curvature flow for arbitrary dimensions [7, 8].

The novelty in part (a) of Theorem 1 is that it does not use the positive mass theorem and applies in all dimensions. Compared to [12], the novelty in part (b) of Theorem 1 is that it is a sharp estimate and includes a rigidity statement. In the case of convex domains, part (a) of Theorem 2 is related to a classic result of Szegö [9]. Our result for mean-convex domains is more general. Part (b) of Theorem 2 is included in the family of classical Aleksandrov-Fenchel inequalities for the cross-sectional volumes of convex domains. These were generalized to the case of star-shaped k-convex domains in [6]. Our result for outer-minimizing, mean-convex domains (or 1-convex, $k = 1$) does not require the domain to be star-shaped (or even connected), hence it is more general.

2. Preliminaries

Let $(M^n, g), n \geq 3$ be a complete, non-compact Riemannian manifold with boundary $\Sigma = \partial M$. Here, we don't assume Σ is connected. For simplicity, let us assume M has only one end, $\mathcal{E}$. Such a manifold is said to be *asymptotically flat* if, outside a compact set, (M, g) is diffeomorphic to the complement of a ball in Euclidean space, and in the coordinates given by this diffeomorphism the metric has the asymptotic decay

$$|g - \delta| = O(|x|^{-p}), \quad |\partial g| = O(|x|^{-p-1}), \quad |\partial^2 g| = O(|x|^{-p-2}),$$

where $p > \frac{n-2}{2}$. Furthermore, we require (M, g) to have integrable scalar curvature $\int_M |R_g| dV_g < \infty$.

For these manifolds the ADM mass does not depend on the choice of asymptotically flat coordinates and is defined by

$$m = m_{ADM}(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \to \infty} \int_{S_r} \sum_{i,j} (\partial_j g_{ij} - \partial_i g_{jj}) \nu^j \, d\sigma_r^0.$$

Here, S_r is a Euclidean coordinate sphere, $d\sigma_r^0$ is Euclidean surface area.

There are several results in the literature which give lower bounds for the ADM mass in terms of geometric quantities. For example, the celebrated *positive mass theorem* [10, 13] (valid for asymptotically flat manifolds *without* boundary) asserts that if the scalar curvature of g is non-negative and either $3 \leq n \leq 7$ or M is spin, then

$$m \geq 0,$$

and $m = 0$ if and only if the manifold is Euclidean space.

Another well-known inequality is the *Riemannian Penrose inequality*, which can be thought of as a refinement of the positive mass theorem. It asserts that if M has non-negative scalar curvature and contains a compact outermost minimal hypersurface Σ, then

$$m \geq \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}},$$

where $|\Sigma|$ denotes the g-area of Σ and ω_{n-1} is the volume of the $(n-1)$-dimensional sphere. Rigidity also holds for the Riemannian Penrose inequality. More precisely, equality holds above if and only if the manifold is a *Riemannian Schwarzschild manifold* of mass $m > 0$

$$g_{ij} = \left(1 + \frac{m}{2r^{n-2}} \right)^{\frac{4}{n-2}} \delta_{ij},$$

where g is defined outside the ball of radius $R_s := (\frac{m}{2})^{\frac{1}{n-2}}$. This inequality was proved in [7] for $n = 3$ and connected Σ (using inverse mean-curvature flow and monotonicity of the Hawking mass), and in [1] for $n = 3$ without the connectedness assumption. The approach of [1] was generalized for $3 \leq n \leq 7$ in [3], although the rigidity statement requires the extra hypothesis that the manifold be spin.

It is natural to wonder if there is a proof of the Riemannian Penrose inequality in the general conformally flat case that uses only properties of superharmonic functions in $\mathbb{R}^n$ (see [2]). Our work provides evidence in this direction.

In what follows we will be using the notion of capacity of hypersurfaces. The precise definition is the following.

Definition. Let (M, g) be a complete, non-compact Riemannian manifold with compact boundary Σ and one end $\mathcal{E}$. The *capacity* of a hypersurface $\Sigma \subset (M, g)$ is

$$C_g(\Sigma) = \inf_{\varphi \in M_0^1} \left\{ \frac{1}{(n-2)\omega_{n-1}} \int_M |\nabla_g \varphi|_g^2 dV_g \right\},$$

where M_0^1 denotes the set of all smooth functions on M which are exactly 0 on Σ and approach 1 towards infinity in the end $\mathcal{E}$. We denote by $C_0(\Sigma)$ the Euclidean capacity of a hypersurface $\Sigma = \partial\Omega \subset \mathbb{R}^n$.

Remark 1. The normalization constant of the above definition is chosen so that $C_0(S_R) = R^{n-2}$, where $S_R = \partial\mathbb{B}_R$ in $\mathbb{R}^n$.

Remark 2. The infimum of the definition is attained by the unique g-harmonic function in M_0^1. If the ambient manifold is Euclidean space, it follows that the harmonic function which realizes the infimum has the asymptotic expansion

$$\varphi(x) = 1 - \frac{C_0(\Sigma)}{|x|^{n-2}} + O(|x|^{1-n}) \text{ as } x \to \infty.$$

Remark 3. Changing the boundary conditions we could also define (for $a \neq b$):

$$C_g^{(a,b)}(\Sigma) = \inf_{\varphi \in M_a^b} \left\{ \frac{1}{(n-2)\omega_{n-1}} \int_M |\nabla_g \psi|_g^2 dV_g \right\},$$

where M_a^b is defined as above. Since the map $\psi \mapsto \frac{a-\psi}{a-b}$ defines a bijection $M_a^b \to M_0^1$ which scales the integral of the square of the gradient by a constant, it follows that $C_g^{(a,b)}(\Sigma) = (a-b)^2 C_g(\Sigma)$.

In this paper we are interested in the case when (M, g) is a CF-manifold. Recall from its definition that this means that M is diffeomorphic to $\Omega^c := \mathbb{R}^n \setminus \Omega$, where $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain (not necessarily connected), and g is conformal to the Euclidean metric. That is, $g_{ij} = u^{\frac{4}{n-2}}\delta_{ij}$ with $u > 0$, and $u \to 1$ at Euclidean infinity.

In what follows we denote $\Sigma = \partial\Omega$. By a theorem of Schoen and Yau [10], up to changing m by an arbitrarily small amount (and g by a point-wise ratio arbitrarily close to 1), one may assume g is "harmonically flat at infinity". In our case, this becomes $\Delta_0 u = 0$ outside a sufficiently large Euclidean ball. Using an expansion in spherical harmonics we can further assume

$$u = 1 + \frac{m}{2r^{n-2}} + O(r^{1-n}), \quad u_r = -\frac{(n-2)m}{2} r^{1-n} + O(r^{-n}), \quad m = m_{ADM}(g).$$

Remark 4. For CF-manifolds which are harmonically flat at infinity the positive mass theorem follows easily for all $n \geq 3$.

Indeed, the transformation law formula for scalar curvature under conformal deformations gives that the scalar curvature of $g_{ij} = u^{\frac{4}{n-2}}\delta_{ij}$, denoted by R_g, is given by

$$R_g = u^{-\frac{n+2}{n-2}}\left(-\frac{4(n-1)}{n-2}\Delta_0 u + R_\delta u\right). \tag{1}$$

(Naturally, here $R_\delta \equiv 0$.) In particular, we obtain that $R_g \geq 0 \Leftrightarrow \Delta_0 u \leq 0$. The ADM integrand is easily computed in this case:

$$\sum_{i,j}(\partial_j g_{ij} - \partial_i g_{jj})\nu^j = -\frac{4(n-1)}{n-2}u^{\frac{6-n}{n-2}}u_r,$$

and since $u \to 1$ at infinity, we obtain

$$\int_{S_\rho} u^{\frac{6-n}{n-2}}u_r d\sigma^0_\rho \sim \int_{S_\rho} u_r d\sigma^0_\rho = \int_{\mathbb{B}_\rho}\Delta_0 u dx \leq 0.$$

Thus, $m \geq 0$, with equality if and only if u is positive harmonic on $\mathbb{R}^n$ with $u \to 1$ at infinity, i.e. $u \equiv 1$.

3. Model Case and Main Theorem

The motivation for this note is to investigate whether the mass-capacity inequality holds for CF-manifolds in all dimensions. The following transformation formula for the Laplacian plays a key role.

Lemma 3. *Let $g = u^{\frac{4}{n-2}}\delta$ and $f \in C^\infty(M)$. Then $\Delta_g f = u^{-\frac{n+2}{n-2}}(\Delta_0(uf) - f\Delta_0 u)$. In particular, if $\Delta_0 u = 0$, then $\Delta_0(uf) = 0$ if and only if $\Delta_g f = 0$.*

We use the Lemma in the following main example.

Model Case. Our prototypical example is the so-called Riemannian Schwarzschild manifold. (Compare with Theorem 9 of [1].) It is constructed as follows. For $R_s > 0$, denote $m = 2R_s^{n-2}$, and define on $\mathbb{B}^c_{R_s} = \mathbb{R}^n \setminus \mathbb{B}_{R_s}$ the function

$$u = 1 + \left(\frac{R_s}{r}\right)^{n-2} = 1 + \frac{m}{2}r^{2-n}. \tag{2}$$

Note that u is actually defined and harmonic ($\Delta_0 u = 0$) on $\mathbb{R}^n \setminus \{0\}$.

Now define

$$\varphi = \frac{1 - (\frac{R_s}{r})^{n-2}}{1 + (\frac{R_s}{r})^{n-2}}.$$

Then $\Delta_0(u\varphi) = 0$, so $\Delta_g \varphi = 0$ by Lemma 3 above. Moreover $\varphi \to 1$ as $r \to \infty$, and $\varphi_{|\Sigma} = 0$ for $\Sigma = \partial\mathbb{B}_{R_s}$. Thus, by direct integration we obtain

$$C_g(\Sigma) = \frac{1}{\omega_{n-1}(n-2)}\int_{\mathbb{B}^c_{R_s}} |\nabla_g\varphi|^2_g dV_g = \frac{1}{n-2}\int_{R_s}^\infty u^2\varphi_r^2 r^{n-1}dr = m.$$

That is, the equality case of the mass-capacity inequality is achieved by the Riemannian Schwarzschild manifold; this should be the extremal case for the inequality and it is our motivational starting point.

In view of the above example we now generalize the notion of Riemannian Schwarzschild metric to general euclidean domains, not necessarily the complement of a round ball.

Definition. Let $\Omega^c \subset \mathbb{R}^n$. The *harmonic* metric of Ω^c is the asymptotically flat, conformally flat metric $g_s = u_s^{\frac{4}{n-2}}\delta$, where $u_s > 0$ is a positive function on Ω^c which is uniquely determined by the following conditions:

- $\Delta_0 u_s = 0$ (thus $R_{g_s} = 0$),
- $u_s \to 1$ as $x \to \infty$,
- $(u_s)_\nu = -\frac{n-2}{2(n-1)} u_s H_0$ on $\Sigma = \partial\Omega$, or equivalently, $H_{g_s}(\Sigma) = 0$.

(Note that harmonic metrics are CF-manifolds whenever $\Sigma = \partial\Omega$ is Euclidean mean-convex.) We are now ready to state our main result.

Theorem 4. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain with boundary $\Sigma = \partial\Omega$, not necessarily connected. Let (M, g) be isometric to a conformally flat metric $g_{ij} = u^{\frac{4}{n-2}}\delta_{ij}$ on Ω^c which is asymptotically flat with ADM mass m. (Here $u > 0$ and $u \to 1$ towards infinity.) Assume further that (M, g) has non-negative scalar curvature $R_g \geq 0$. Then*

(I) If Σ is Euclidean mean-convex ($H_0 > 0$) and g-minimal ($H_g = 0$), then

$$C_0(\Sigma) < C_g(\Sigma) \leq C_0(\Sigma) + \frac{m}{2}.$$

Equality occurs in the second inequality if and only if g is a harmonic metric, i.e. $\Delta_0 u = 0$.

(II) (Euclidean estimate.) Assume $H_0 > 0$ on Σ. Then:

$$C_0(\Sigma) \leq \frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0.$$

Equality holds if and only if Σ is a round sphere.

(III) Let $\alpha = \min_\Sigma u$. Under the same assumptions on Σ as in (I), we have:

$$\frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0 \leq \frac{m}{\alpha}.$$

Equality holds if and only if g is the Riemannian Schwarzschild metric, for which we have $\alpha = 2$. (Note that by Lemma 5 below, $\alpha > 1$.)

(IV) Under the same assumptions on Σ as in (I)

$$C_0(\Sigma) \leq \frac{m}{2}.$$

Equality holds only for the Riemannian Schwarzschild manifold.

(V) Assume $H_0 > 0$ on Σ, and Σ is outer-minimizing with area A. Then:

$$\frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0 \geq \left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

Equality holds if and only if Σ is a round sphere.

4. Proof of Theorem 4

The first inequality of part (I) of Theorem 4 follows from the following lemma of [12], as we see below.

Lemma 5 ([12])**.** *Assume $u > 0$ and $\Delta_0 u \leq 0$ in $\Omega^c = \mathbb{R}^n \setminus \Omega$, with $\Sigma = \partial\Omega$ mean-convex for the euclidean metric ($H_0 > 0$) and minimal for the metric $g = u^{\frac{4}{n-2}}\delta$ ($H_g = 0$), where $u > 0, u \to 1$ at infinity. Then $u > 1$ on Ω^c.*

The key ingredients in the proof of this Lemma are the minimum principle for superharmonic functions and the transformation formula for mean curvature of a hypersurface under conformal deformations of the metric. This is given by the equation

$$H_g = u^{-\frac{2}{n-2}}\left(H_0 + \frac{2(n-1)}{n-2}\frac{u_\nu}{u}\right), \tag{3}$$

where ν is the euclidean-unit outward normal of Ω. (To check the constant multiplying u_ν/u, observe that the boundary is minimal for the Riemannian Schwarzschild metric).

Proof of Theorem 4. As remarked earlier, we may assume (M, g) is harmonically flat at infinity. Namely, we have that

$$\Delta_0 u = 0 \text{ for } r > R_0, \quad u = 1 + \frac{m}{2r^{n-2}} + O\left(\frac{1}{r^{n-1}}\right), \quad u_r = -\frac{(n-2)m}{2r^{n-1}} + O\left(\frac{1}{r^n}\right).$$

Proof of (I). To prove the first (strict) inequality of (I), we note that

$$\int_M |\nabla_g\varphi|_g^2 dV_g = \int_{\Omega^c} u^{-\frac{4}{n-2}}|\nabla_0\varphi|^2 u^{\frac{2n}{n-2}} dV_0 = \int_{\Omega^c} u^2|\nabla_0\varphi|^2 dV_0. \tag{4}$$

Since $u > 1$ on Ω^c from Lemma 5 above, we conclude that $C_0(\Sigma) \leq C_g(\Sigma)$. To show that equality is not possible in this inequality, note that both infima for the capacities are achieved, as discussed in Remark 2. Therefore, if $C_0(\Sigma) = C_g(\Sigma)$, there exists functions φ, ψ with equal boundary conditions so that $C_0(\Sigma) = \int_M |\nabla_0\psi|_0^2 dV_0 = C_g(\Sigma) = \int_M |\nabla_g\varphi|_g^2 dV_g$. Using equation (4) and the fact that $u > 1$ we get $\int_{\Omega^c} |\nabla_0\varphi|^2 dV_0 < \int_{\Omega^c} u^2|\nabla_0\varphi|^2 dV_0 = \int_M |\nabla_0\psi|_0^2 dV_0$, contradicting the fact that ψ achieves the infimum for the euclidean capacity. (Here we have used the fact that a non-constant harmonic function is not constant over sets of positive measure.)

For the second inequality in (I), let $v : \Omega^c \to (0,1)$ be the unique harmonic function ($\Delta_0 v = 0$) satisfying $v_{|\Sigma} = 0, v(x) \to 1$ as $x \to \infty$. Then with $\varphi = \frac{v}{u}$ we have $\varphi_{|\Sigma} = 0, \varphi \to 1$ at infinity. Thus

$$(n-2)\omega_{n-1}C_g(\Sigma) \leq \int_M |\nabla_g\varphi|_g^2 dV_g = \int_{\Omega^c} u^2|\nabla_0(\frac{v}{u})|^2 dV_0 := \mathcal{I} = \lim_{\rho\to\infty} \mathcal{I}_\rho,$$

where

$$\mathcal{I}_\rho := \int_{\mathbb{B}_\rho\setminus\Omega} u^2|\nabla_0(\frac{v}{u})|^2 dV_0 = \int_{\mathbb{B}_\rho\setminus\Omega}\left[|\nabla_0 v|^2 - \nabla_0(v^2)\cdot\frac{\nabla_0 u}{u} + v^2\frac{|\nabla_0 u|^2}{u^2}\right] dV_0.$$

Now

$$\int_{\mathbb{B}_\rho\setminus\Omega} \nabla_0(v^2)\cdot\frac{\nabla_0 u}{u} dV_0 = -\int_{\mathbb{B}_\rho\setminus\Omega} v^2 \mathrm{div}_0(\frac{\nabla_0 u}{u}) dV_0 + \int_{S_\rho} v^2\frac{u_r}{u} d\sigma_\rho^0$$

since $v_{|\Sigma} = 0$. Noting $\operatorname{div}_0(u^{-1}\nabla_0 u) = u^{-1}\Delta_0 u - u^{-2}|\nabla_0 u|^2$ and $\Delta_0 u \leq 0$, we have

$$\int_{\mathbb{B}_\rho\setminus\Omega} \nabla_0(v^2)\cdot\frac{\nabla_0 u}{u} dV_0 \geq \int_{\mathbb{B}_\rho\setminus\Omega} v^2\frac{|\nabla_0 u|^2}{u^2}dV_0 + \int_{S_\rho} v^2\frac{u_r}{u}d\sigma^0_\rho, \tag{5}$$

and hence we obtain

$$\mathcal{I}_\rho \leq \int_{\mathbb{B}_\rho\setminus\Omega}|\nabla_0 v|^2 dV_0 - \int_{S_\rho} v^2\frac{u_r}{u}d\sigma^0_\rho.$$

Taking limits as $\rho\to\infty$ and using the asymptotics of u we find

$$\mathcal{I} \leq (n-2)\omega_{n-1}C_0(\Sigma) + (n-2)\omega_{n-1}\frac{m}{2}.$$

From this it follows $C_g(\Sigma) \leq C_0(\Sigma) + \frac{m}{2}$, as claimed.

Rigidity. For the rigidity statement of (I) we first assume that all the above inequalities are equalities. From the equality attained in equation (5) it follows directly that u is harmonic. This proves one direction of the equivalence.

Assume, on the other hand, that u is harmonic. Let ψ be the function that achieves the infimum for the capacity $C_g(\Sigma)$. From Lemma 3 it follows that such function (i.e. the g-harmonic function which is exactly zero on Σ and goes to one at infinity) satisfies $\Delta_0(u\psi) = 0$. We immediately recognize that $u\psi$ must be equal to the function v from above since both are harmonic and equal on Σ and at infinity. Therefore, ψ equals $\varphi = v/u$ from above, and all the above inequalities become equalities. This proves (I). □

Proof of (II). Here we use a modification of the method described in [4]. First, we get an upper bound for $C_0(\Sigma)$ using test functions of the form $\varphi = f\circ\phi$, where $\phi \in C^1(\Omega^c, \mathbb{R}_+)$ is a (soon to be determined) proper function vanishing on $\Sigma = \Sigma_0$ whose level sets define a foliation $(\Sigma_t)_{t\geq 0}$ of Ω^c. As noted in [4], we have

$$(n-2)\omega_{n-1}C_0(\Sigma) \leq \inf\left\{\int_0^\infty (f')^2 w(t)dt \;:\; f(0)=0, f(\infty)=1\right\}, \tag{6}$$

where $w(t) = \int_{\Sigma_t}|\nabla_0\phi|d\sigma^0_t > 0$.

(We omit the subscript/superscript '0' for the remainder of the proof of (II).)

Moving away from the method of [4], we note that the one-dimensional variational problem (6) is easily solved.

Claim. Provided $w^{-1}\in L^1(0,\infty)$, the infimum of the right hand side of (6) equals $\mathbb{I}^{-1} = (\int_0^\infty \frac{1}{w(s)}ds)^{-1}$, and is attained by the function $f(t) = \frac{1}{\mathbb{I}}\int_0^t w^{-1}(s)ds$.

Proof. This follows from

$$1 = \int_0^\infty f'dt = \int_0^\infty f'w^{1/2}w^{-1/2}dt \leq \left(\int_0^\infty (f')^2 w(t)dt\right)^{1/2}\left(\int_0^\infty w^{-1}(t)dt\right)^{1/2}.$$

□

Remark 5. If $\Omega\subset\mathbb{R}^n$ is *convex*, it is natural to try to use the distance function $\phi = dist(\cdot,\Sigma)$ for the above process. In this case, the level sets of ϕ give a foliation of Ω^c by outer parallel hypersurfaces. We get $|\nabla\phi| \equiv 1$, so $w(t) = |\Sigma_t|$ is the Euclidean (n-1)-dimensional area. By a well-known formula

$$|\Sigma_t| = |\Sigma| + \sum_{j=0}^{n-2} \left(\int_\Sigma \sigma_j(\vec{k}) d\sigma \right) t^j + \omega_{n-1} t^{n-1},$$

where $\sigma_j(\vec{k})$ is the j-th elementary symmetric function of the principal curvatures $\vec{k} = (k_1, \dots, k_{n-1}),\ k_i > 0$ of Σ. Now since

$$\sigma_1(\vec{k}) = H \text{ and } \sigma_j(\vec{k}) \leq H^j \text{ for } j = 1, \dots, n-1,$$

we see that an estimate based on this foliation would involve the integrals $\int_\Sigma H^j d\sigma$. Since we are interested in estimating the capacity in terms of the ADM mass (especially in view of part (III)), we choose a different function to construct the foliation.

Consider the foliation $(\Sigma_t)_{t\geq 0}$ defined by the level sets of the function given by Huisken and Ilmanen's inverse mean curvature flow [7,8] in $\Omega^c \subset \mathbb{R}^n$. We recall the summary given in [4] (which holds in all dimensions):

Theorem 6 (Huisken-Ilmanen, [7,8])**.**

- *There exists a proper, locally Lipschitz function $\phi \geq 0$ on Ω^c, $\phi_{|\Sigma} = 0$. For $t > 0$, $\Sigma_t = \partial\{\phi \geq t\}$ and $\Sigma'_t = \partial\{\phi > t\}$ define increasing families of $C^{1,\alpha}$ hypersurfaces;*
- *The hypersurfaces Σ_t (resp.Σ'_t) minimize (resp. strictly minimize) area among surfaces homologous to Σ_t in $\{\phi \geq t\} \subset \Omega^c$. The hypersurface $\Sigma' = \partial\{\phi > 0\}$ strictly minimizes area among hypersurfaces homologous to Σ in Ω^c.*
- *For almost all $t > 0$, the weak mean curvature of Σ_t is defined and equals $|\nabla\phi|$, which is positive a.e. on Σ_t.*

From Theorem 6 and the Claim from above it follows that

$$(n-2)\omega_{n-1} C_0(\Sigma) \leq \left(\int_0^\infty w^{-1}(t) dt \right)^{-1}, \text{ where } w(t) := \int_{\Sigma_t} H d\sigma_t. \tag{7}$$

Lemma 7. *Consider the foliation $\{\Sigma_t\}$ given by IMCF in $\Omega^c \subset \mathbb{R}^n$ as above. Then*

$$\int_{\Sigma_t} H d\sigma \leq \left(\int_{\Sigma_0} H d\sigma \right) e^{\frac{n-2}{n-1} \cdot t}$$

for $t \geq 0$.

Remark 6. Note that equality holds in the above inequality for the foliation by IMCF outside a sphere, which is given by $\Sigma_t = \partial\mathbb{B}_{R(t)} \subset \mathbb{R}^n$, where $R(t) = e^{\frac{t}{n-1}}$.

Proof of Lemma 7. From [7] we have that, so long as the evolution remains smooth,

$$\frac{d}{dt}\left(\int_{\Sigma_t} H d\sigma_t \right) = \int_{\Sigma_t} \left(H - \frac{|A|^2}{H} \right) d\sigma_t \leq \frac{n-2}{n-1} \int_{\Sigma_t} H d\sigma_t, \tag{8}$$

where A denotes the second fundamental form, and the second inequality follows from

$$H - \frac{|A|^2}{H} - \frac{n-2}{n-1} H = \frac{1}{(n-1)H} (H^2 - (n-1)|A|^2) \leq 0. \tag{9}$$

(Note that equality occurs in this last inequality if and only if each connected component of Σ_t is a sphere.) It is easy to see that the inequalities extend through countably many jump times since the total mean curvature does not increase at the jump times. □

By straightforward integration, Lemma 7 implies:

$$\left(\int_0^\infty w^{-1}(t)dt\right)^{-1} \leq \frac{n-2}{n-1}\int_\Sigma H d\sigma.$$

Together with equation (7) this gives

$$C_0(\Sigma) \leq \frac{1}{(n-1)\omega_{n-1}}\int_\Sigma H d\sigma,$$

as claimed in part (II) of the main theorem.

Rigidity. From Remark 6 it follows that the inequality of part (II) is an equality whenever Σ is a round sphere.

On the other hand, if equality holds in part (II), it follows that

$$\int_{\Sigma_t} H d\sigma = \left(\int_{\Sigma_0} H d\sigma\right) e^{\frac{n-2}{n-1}\cdot t} \text{ for a.e. } t \geq 0,$$

and therefore:

$$H^2 = (n-1)|A|^2 \text{ on } \Sigma_t, \text{ for a.e. } t \geq 0.$$

This implies Σ_t is a disjoint union of round spheres, for a.e. $t \geq 0$. For a solution of inverse mean curvature flow in $\mathbb{R}^n$, this is only possible if Σ_t is, in fact, a single round sphere for every t. (See e.g. the Two Spheres Example 1.5 of [7].) This proves part (II). □

Proof of (III). From the transformation law for the mean curvature given by equation (3), together with the divergence theorem, it follows that

$$\begin{aligned}\int_{\mathbb{B}_\rho\setminus\Omega} \Delta_0 u dV_0 &= \int_{S_\rho} u_r d\sigma_\rho^0 - \int_\Sigma u_\nu d\sigma_0 \\ &= -m\omega_{n-1}\frac{n-2}{2} + O(\rho^{-1}) + \frac{n-2}{2(n-1)}\int_\Sigma H_0 u d\sigma_0.\end{aligned}$$

Taking the limit $\rho \to \infty$ we obtain

$$m = -\frac{2}{(n-2)\omega_{n-1}}\int_{\Omega^c} \Delta_0 u dV_0 + \frac{1}{(n-1)\omega_{n-1}}\int_\Sigma H_0 u d\sigma_0. \tag{10}$$

Since $\Delta_0 u \leq 0$ on Ω^c and $u \geq \alpha$ on Σ, this gives the inequality in (III), as well as the fact that equality in (III) implies $\Delta_0 u \equiv 0$ on Ω^c.

Rigidity. For the rigidity statement of (III) we only need to prove one direction since (clearly) for the Riemannian Schwarzschild manifold, the above inequalities are all equalities. Thus, we now assume that all the above inequalities are equalities. In particular, we have that

$$\int_\Sigma H_0 d\sigma_0 = (n-1)\omega_{n-1}\frac{m}{\alpha}. \tag{11}$$

Claim. u is harmonic on Ω^c, and is (the same) constant on (all components of) Σ.

Proof. Indeed, notice that from the remark below equation (10) it follows $\Delta_0 u = 0$. Furthermore, since the inequality in (III) is obtained from equation (10) by replacing u by its minimum on Σ, it follows that, in the case of equality in (III), u equals its minimum on Σ, i.e. $u|_\Sigma \equiv \min_\Sigma u = \alpha$. $\square$

Claim. Σ is a (single) round sphere.

Proof. From the previous claim $\Delta_0 u = 0$, so

$$0 = \int_{\Omega^c} u\Delta_0 u dV_0 = -\int_{\Omega^c} |\nabla_0 u|^2 dV_0 - \frac{m}{2}\omega_{n-1}(n-2) - \int_\Sigma u u_\nu d\sigma_0.$$

Also, from that claim $u_{|\Sigma} \equiv \alpha$, so we know u is the optimal function for $C_0^{(\alpha,1)}(\Sigma)$ (cf. Remark 2). Furthermore, using Remark 3 it follows that $\int_{\Omega^c} |\nabla_0 u|^2 dV_0 = (n-2)\omega_{n-1}(\alpha-1)^2 C_0(\Sigma)$. Combining this with the above equation we obtain

$$(n-2)\omega_{n-1}(\alpha-1)^2 C_0(\Sigma) = -\frac{m}{2}\omega_{n-1}(n-2) + \frac{n-2}{2(n-1)}\alpha^2 \int_\Sigma H_0 d\sigma_0.$$

We now use equation (11) to substitute the last term in the above equation. We get

$$(\alpha - 1)C_0(\Sigma) = \frac{m}{2}. \tag{12}$$

We now claim that $\alpha = 2$ and we are actually in the equality case of (II).

Indeed, equation (11) combined with the inequality in (II) implies $\alpha \leq 2$. On the other hand, consider the inequality $C_0(\Sigma) \leq m/2$ from part (IV) (which is independent of part (III)). Together with equation (12) this implies $\alpha \geq 2$. Hence $\alpha = 2$ and

$$C_0(\Sigma) = \frac{m}{2} = \frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0.$$

Thus we are in the equality case of part (II), and it follows from the already proven rigidity there that Σ is a round sphere. $\square$

Claim. g is the Riemannian Schwarzschild manifold, or equivalently, u is a function like the one in equation (2).

Proof. From the above claim we obtain that Σ is a sphere of radius $r_0 > 0$ centered at some p. Consider the function $v(x) := 1 + r_0^{n-2}|x-p|^{2-n}$. Then both u and v are harmonic, equal on Σ (i.e. $u|_\Sigma = v|_\Sigma \equiv 2$), and tend to 1 at infinity. By the maximum principle, $u = v$ on Ω^c. This concludes the proof of (III). $\square$

Proof of (IV). Given $\epsilon > 0$ arbitrary, let $f : [1,\infty) \to \mathbb{R}_+$ be a C^2 function satisfying the following conditions:

$f > 0, f' < 0, f'' > 0$ on $[1,\infty)$, $f(1) = 1, f'(1) = -1$ and $f(\alpha) = \epsilon$, where $\alpha = \min_\Sigma u > 1$.

Then $f \circ u \to 1$ at infinity, while $0 < (f \circ u)_{|\Sigma} \leq \epsilon$. It follows from Remark 3 that

$$(1-\epsilon)^2(n-2)\omega_{n-1}C_0(\Sigma) \leq \int_{\Omega^c} |\nabla_0(f\circ u)|^2 dV_0.$$

Thus, to prove part (IV) it suffices to show that the right hand side of the above inequality is bounded above by $(n-2)\omega_{n-1}(m/2)$.

To prove this, we note that since

$$\Delta_0(f \circ u) = (f'' \circ u)|\nabla_0 u|^2 + (f' \circ u)\Delta_0 u \geq 0, \tag{13}$$

we have

$$\int_{\mathbb{B}_\rho \setminus \Omega} |\nabla_0 (f \circ u)|^2 dV_0 \leq \int_{S_\rho} f \circ u (f \circ u)_r d\sigma_\rho^0 - \int_\Sigma f \circ u (f \circ u)_\nu d\sigma. \tag{14}$$

We see that the second term of the right hand side of equation (14) (i.e. the integral over Σ) may be dropped. Indeed, using the boundary condition on u we get

$$\int_\Sigma f \circ u (f \circ u)_\nu d\sigma = \int_\Sigma f \circ u (f' \circ u) u_\nu d\sigma = -\frac{2(n-2)}{n-1}\int_\Sigma f \circ u (f' \circ u) u H_0 d\sigma \geq 0.$$

For the first term of the right hand side of equation (14) we use the asymptotics of u_r to see that, in the limit $\rho \to \infty$, the integral over S_ρ satisfies

$$\lim_{\rho \to \infty} \int_{S_\rho} f \circ u (f \circ u)_r d\sigma_\rho^0 = -\frac{m}{2}(n-2)\omega_{n-1} f(1) f'(1) = \frac{m}{2}(n-2)\omega_{n-1},$$

concluding the proof of the inequality in part (IV).

Rigidity. For the rigidity statement of (IV) we only need to prove one direction, since direct calculation shows the Riemannian Schwarzschild manifold satisfies equality in (IV). Thus, we now assume that $C_0(\Sigma) = m/2$.

Claim. $\alpha \leq 2$.

Proof. This follows directly from parts (II) and (III), which can be combined to form the double inequality

$$\frac{m}{2} \leq \frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0 \leq \frac{m}{\alpha}. \tag{15}$$

□

Claim. $\alpha = 2$.

Proof. To prove this, we use some of the calculations in the proof of the *inequality* part of (IV) from above. It follows from equation (13) and the one before it that

$$\begin{aligned}(n-2)\omega_{n-1}\left[(1-\epsilon)^2 C_0(\Sigma) - \frac{m}{2}\right] \leq &- \int_{\Omega^c} (f_\epsilon \circ u)(f''_\epsilon \circ u)|\nabla_0 u|^2 dV_0 \\ &- \int_{\Omega_\alpha^-} (f_\epsilon \circ u)(f'_\epsilon \circ u)\Delta_0 u dV_0 \\ &+ \frac{2(n-2)}{n-1}\int_\Sigma (f_\epsilon \circ u)(f'_\epsilon \circ u) u H_0 d\sigma_0,\end{aligned} \tag{16}$$

where f_ϵ is the function denoted by f in the proof of (IV).

We now prepare to take the limit as $\epsilon \searrow 0_+$ in inequality (16).

We may assume the family (f_ϵ) is pointwise decreasing as ϵ decreases to zero, bounded below by zero. We have $f_\epsilon \to g$ in $C^2[1,\alpha] \cap Lip_{loc}[1,\infty)$, where g satisfies:

$$g > 0, g' < 0, g'' \geq 0 \text{ on } [1,\alpha), g(\alpha) = 0, g \equiv 0 \text{ on } [\alpha,\infty).$$

Consider the sets:

$$\Omega_\alpha^+ := \{x \in \Omega^c; u(x) \geq \alpha\}, \quad \Omega_\alpha^- := \{x \in \Omega^c; u(x) \leq \alpha\}.$$

Note that $\Sigma \subset \Omega_\alpha^+$ and Ω_α^- contains a neighborhood of the end $\mathcal{E}$ (since $\alpha > 1$). In the limit $\epsilon \searrow 0_+$

$$\begin{aligned}
&(f_\epsilon \circ u)(f_\epsilon' \circ u) \to (g \circ u)(g' \circ u) \text{ in } \Omega^c, \text{ with } (g \circ u)_{|\Sigma} = 0,\\
&(f_\epsilon \circ u)(f_\epsilon'' \circ u) \to (g \circ u)(g'' \circ u) \text{ in } \Omega_\alpha^-,\\
&(f_\epsilon \circ u)(f_\epsilon'' \circ u) \to 0 \text{ in } \Omega_\alpha^+.
\end{aligned}$$

We now take the limit as $\epsilon \searrow 0$ in inequality (16). Since $C_0(\Sigma) = m/2$, we get that

$$\begin{aligned}
0 \leq &- \int_{\Omega_\alpha^-} (g \circ u)(g'' \circ u)|\nabla_0 u|^2 dV_0 - \int_{\Omega_\alpha^-} (g \circ u)(g' \circ u)\Delta_0 u dV_0\\
&+ \frac{2(n-2)}{n-1} \int_\Sigma (g \circ u)(g' \circ u) u H_0 d\sigma_0.
\end{aligned}$$

The third term in the right hand side above vanishes since $(g \circ u)_{|\Sigma} = 0$. Using $g > 0$ and $g'' \geq 0$, together with $g' \leq 0$ and $\Delta_0 u \leq 0$, we get a sign for the first two terms in the right hand side above. From this it follows that

$$(g \circ u)(g'' \circ u)|\nabla_0 u|^2 = 0 \text{ and } (g \circ u)(g' \circ u)\Delta_0 u = 0, \text{ both a.e. in } \Omega_\alpha^-. \tag{17}$$

We are interested in showing that $g'' \circ u = 0$ a.e. in Ω_α^-. For this, we use the first equality of equation (17). Note that $g \circ u > 0$ a.e. in Ω_α^- since $g > 0$ in $[1,\alpha)$ and u is harmonic. Also, $|\nabla_0 u|^2 > 0$ a.e in Ω_α^-. Hence, $g'' \circ u = 0$ a.e. in Ω_α^-, as desired. We claim this implies $\alpha = 2$.

Indeed if $\alpha < 2$, necessarily $g'' > 0$ on some open interval $I \subset [1,\alpha]$ (for $g'' \equiv 0$ in $[1,\alpha]$ would force $g(t) = 2 - t$ for $t \in [1,\alpha]$, hence $\alpha = 2$). But then $u^{-1}(I) \subset \Omega_\alpha^-$ is a non-empty open set on which $g'' \circ u > 0$, a contradiction. This proves the claim. □

We have shown that $\alpha = 2$. From equation (15) we see that we are in the equality case of part (III). By the rigidity statement, it follows that g is the Riemannian Schwarzschild metric. This ends the proof of (IV). □

Proof of (V). Recall that a hypersurface $\Sigma = \partial\Omega$ is called **outer-minimizing** if whenever Ω' is a domain with $\Omega' \supset \Omega$ then $|\partial\Omega'| \geq |\Sigma|$. (An example of such a hypersurface is given by the boundary of a collection of sufficiently far-apart convex bodies in $\mathbb{R}^n$.) Let us denote by $|\Sigma_t|$ the area of the evolving hypersurface Σ_t moving by IMCF with initial condition $\Sigma_0 \equiv \Sigma$. Then, by Lemma 1.4 of [7], one has $|\Sigma_t| = e^t|\Sigma|$ for all $t \geq 0$, provided Σ is outer-minimizing.

Now, from Lemma 7 and the fact that $e^{(\frac{n-2}{n-1})t} = (|\Sigma_t|/|\Sigma|)^{\frac{n-2}{n-1}}$, we have that the function

$$f(t) := |\Sigma_t|^{-\frac{n-2}{n-1}} \int_{\Sigma_t} H d\sigma_t$$

is non-increasing along IMCF in $\mathbb{R}^n$. By a known property of Euclidean IMCF, for t large enough Σ_t is arbitrarily close to a round sphere, and hence $f(t) \to$

$(n-1)\omega_{n-1}^{1/(n-1)}$ as $t \to \infty$. This proves the inequality in (V), since $f(0) = |\Sigma|^{-(n-2)/(n-1)} \int_\Sigma H d\sigma$.

Rigidity. From Remark 6 it follows that the inequality of part (V) is an equality whenever Σ is a round sphere. On the other hand, if the inequality in (V) were an equality, we have $f(\infty) = f(0)$, so $f(t) \equiv f(0)$ for all t since f is non-increasing. This implies $\int_{\Sigma_t} H d\sigma_t = ce^{t(n-2)/(n-1)}$, and inequality (8) becomes an equality. Thus, we have reduced rigidity here to the case of rigidity of part (II). □

5. Applications of the Main Theorem

Proof of part (a) of Theorem 1. The inequality $C_g(\Sigma) \leq m$ follows immediately combining parts (I) and (IV) of Theorem 5.

Rigidity. From the calculations in the Model Case, $C_g(\Sigma) = m$ for Riemannian Schwarzschild. On the other hand, if $C_g(\Sigma) = m$ it follows from parts (I) and (IV) of Theorem 4 that $C_0(\Sigma) = m/2$. Rigidity of (IV) gives that g is Riemannian Schwarzschild. □

Proof of part (b) of Theorem 1. As observed in [12], by spherical decreasing rearrangement the Euclidean capacity of $\partial\Omega$ is bounded from below by the capacity of a ball with the same volume as Ω. Namely, the ball of radius $R = (V/\beta_n)^{1/n}, \beta_n = vol_0(\mathbb{B}^n), V_0 = vol_0(\Omega)$. In other words,

$$C_0(\Sigma) \geq \left(\frac{V_0}{\beta_n}\right)^{\frac{n-2}{n}}. \tag{18}$$

On the other hand, part (IV) of Theorem 4 gives that $m \geq 2C_0(\Sigma)$. Together with (18) this gives

$$m \geq 2\left(\frac{V_0}{\beta_n}\right)^{\frac{n-2}{n}},$$

which is the claim of part (b) of Theorem 1.

Rigidity. Whenever equality holds above (clearly does for Riemannian Schwarzschild since $m = 2R_s^{n-2}$), we must have $C_0(\Sigma) = m/2$ by part (IV) of Theorem 4. The rigidity statement there implies g is Riemannian Schwarzschild. □

Proof of Theorem 2. This is just parts (II) and (V) of Theorem 4. □

Remark 7. Combining theorem 1(b) and theorem 2(a), we find:

$$\frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0 \geq C_0(\Sigma) \geq \left(\frac{V_0}{\beta_n}\right)^{\frac{n-2}{n}}.$$

The resulting inequality between total mean curvature and volume, while weaker than theorem 2(b) (via the isoperimetric inequality in $\mathbb{R}^n$), holds without the requirement that Σ be outer-minimizing.

Remark 8. The proof of the Riemannian Penrose inequality in [1] involves the construction of a conformal flow of asymptotically flat Riemannian metrics $(g(t))_{t\geq 0}$ in the conformal class of the initial metric $g(0)$. It is crucial for the argument in [1] that the ADM mass $m(t)$ of $g(t)$ be non-increasing in t. The proof of this is based on the relation ([1], section 7):

$$\frac{d}{dt}m(t)_{|t=t_0} = C_{g_{t_0}}(\Sigma(t_0)) - m(t_0)$$

(using the normalization in the present paper for the capacity, and one-sided derivatives at the "jump times"). Given this relation, the fact that $m(t)$ is non-increasing follows from the mass-capacity inequality obtained here for conformally flat metrics (independently of the positive mass theorem, or PMT), while in [1] (for more general metrics, in dimension 3) it is obtained applying the reflection argument of [5] and the PMT. (In fact, this is apparently the only place in [1] where the PMT is needed.) Thus our result of part (a) may be regarded as evidence that the Riemannian Penrose inequality for conformally flat metrics in all dimensions can be obtained from arguments of classical linear elliptic theory, as conjectured by Bray and Iga in [2].

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