The 2D Euler equation on singular domains

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Abstract

We establish the existence of global weak solutions of the 2D incompressible Euler equation, for a large class of non-smooth open sets. These open sets are the complements (in a simply connected domain) of a finite number of connected compact sets with positive capacity. Existence of weak solutions with L^p vorticity is deduced from an approximation argument, that relates to the so-called Γ-convergence of domains. Our results complete those obtained for convex domains in [15], or for domains with asymptotically small holes [4, 11]. Connection is made to the recent papers [6, 7] on the Euler equation in the exterior of a Jordan arc.

1 Introduction

Our concern in this paper is the existence theory for the 2D incompressible Euler flow: for Ω an open subset of $\mathbb{R}^2$, we consider the equations

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = 0, & t>0, x\in\Omega \\ \operatorname{div} u = 0, & t>0, x\in\Omega \end{cases} \tag{1.1}$$

endowed with an initial condition and an impermeability condition at the boundary $\partial\Omega$:

$$u|_{t=0} = u^0, \quad u\cdot n|_{\partial\Omega} = 0. \tag{1.2}$$

As usual, $u(t,x) = (u_1(t,x_1,x_2), u_2(t,x_1,x_2))$ and $p = p(t,x_1,x_2)$ denote the velocity and pressure fields, and the vorticity

$$\operatorname{curl} u \ := \ \partial_1 u_2 - \partial_2 u_1$$

plays a crucial role in their dynamics.

The well-posedness of system (1.1)-(1.2) has been of course the matter of many works, starting from the seminal paper of Wolibner on smooth data [16]. In the case where the vorticity is only assumed to be bounded, existence and uniqueness of a weak solution was established by Yudovich in [17]. We quote that the well-posedness result of Yudovich applies to smooth bounded domains, and to unbounded ones under further decay assumptions. As regards smooth exterior domains, one can also mention the articles of Kikuchi [5]. Since the work of Yudovich, the theory of weak solutions has been considerably improved, accounting for vorticities that are only in $L^1\cap L^p$, or that are positive Radon measures: we refer to the textbook [12] for extensive discussion and bibliography.

A common point in all above studies is that $\partial\Omega$ is at least $C^{1,1}$. Roughly, the reason is the following: due to the non-local character of the Euler equation, these works rely on global in space estimates of u in terms of $\operatorname{curl} u$. These estimates *up to the boundary* involve Biot and Savart type kernels, corresponding

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to operators such as $\nabla\Delta^{-1}$. Unfortunately, such operators are known to behave badly in general non-smooth domains. This explains why well-posedness results are dedicated to regular domains, with a few exceptions.

Among those exceptions, one can mention the work [15] of M. Taylor related to convex domains Ω. Indeed, it is well known that if Ω is convex, the solution v of the Dirichlet problem

$$\Delta v = f \ \text{ in } \ \Omega, \quad v|_{\partial\Omega} = 0$$

belongs to $H^2(\Omega)$ when the source term f belongs to $L^2(\Omega)$, no matter the regularity of the domain. Pondering on this remark, Taylor was able to prove in [15] the *existence of global weak solutions in bounded convex domains*. Nevertheless, this interesting result still leaves aside many situations of practical interest, notably flows around irregular obstacles.

This kind of situations has been partly studied in several recent papers, notably by the second author: [6, 7]. Put together, these papers provide an asymptotic analysis of the Euler flow around smooth obstacles, when these obstacles shrink to points (see [4, 11]) or to Jordan arcs (see [6, 7]). In all cases, a modified Euler flow in the whole plane is obtained at the limit: it is governed by the usual transport equation of vorticity, but the Biot and Savart law includes an additional singular measure, supported by the points or the arcs. We will provide more details on such results in due course. Let us point out that they yield the existence of Yudovich like solutions of the Euler equation in some singular domains, that is in the complement of lower dimensional smooth objects. Let us also stress that theses results rely on various analytic techniques, notably conformal mapping. *Our goal here is to establish in a simpler way the existence of weak solutions for a large class of singular domains.*

The first part of the paper is devoted to a large class of bounded open sets Ω. We just assume that Ω has a finite number of holes and that these holes have a positive capacity. They can be written as

$$\Omega := \widetilde{\Omega} \setminus \left(\cup_{i=1}^{k} \mathcal{C}^i\right), \quad k \in \mathbb{N} \tag{1.3}$$

with the following assumptions

(H1) (connectedness) $\widetilde{\Omega}$ is a bounded simply connected domain, $\mathcal{C}^1, \dots, \mathcal{C}^k$ are disjoint connected compact subsets of $\widetilde{\Omega}$.

(H2) (capacity) For all $i = 1..k$, $\text{cap}(\mathcal{C}^i) > 0$, where cap denotes the Sobolev H^1 capacity.

Reminders on the notion of capacity are provided in Section 2. In particular, our assumptions allow to handle flows around obstacles of positive Lebesgue measure, as well as flows around Jordan arcs or curves. They do not cover the case of point obstacles, which have zero capacity. Let us insist that no regularity is assumed on Ω: exotic geometries, such as the Koch snowflake, can be considered.

Within this setting, it is possible to establish the existence of global weak solutions of the Euler equation with L^p vorticity. More precisely, we consider initial data satisfying

$$u^0 \in L^2(\Omega), \quad \text{curl}\, u^0 \in L^p(\Omega), \quad \text{div}\ u^0 = 0, \quad u^0 \cdot n|_{\partial\Omega} = 0, \tag{1.4}$$

for some $p \in]1, \infty]$. Note that, due to the irregularity of Ω, the condition $u^0 \cdot n|_{\partial\Omega} = 0$ has to be understood in a weak sense: for any $\varphi \in C^1_c(\mathbb{R}^2)$,

$$\int_\Omega u^0 \cdot \nabla\varphi = -\int_\Omega \text{div}\ u^0\, \varphi = 0. \tag{1.5}$$

Let us stress that this set of initial data is large: we will show later that for any function $\omega^0 \in L^p(\Omega)$, there exists u^0 verifying (1.4) and $\text{curl}\, u^0 = \omega^0$.

Similarly to (1.5), the weak form of the divergence free and tangency conditions on the Euler solution u will read:

$$\forall \varphi \in \mathcal{D}\left([0,+\infty); C^1_c(\mathbb{R}^2)\right), \quad \int_{\mathbb{R}^+} \int_\Omega u \cdot \nabla \varphi = 0. \tag{1.6}$$

Finally, the weak form of the momentum equation on u will read:

$$\text{for all } \varphi \in \mathcal{D}\left([0,+\infty[\times \Omega\right) \text{ with } \operatorname{div} \varphi = 0, \quad \int_0^\infty \int_\Omega \left(u \cdot \partial_t \varphi + (u \otimes u) : \nabla \varphi\right) = \int_\Omega u^0 \cdot \varphi(0,\cdot). \tag{1.7}$$

Our first main theorem is

Theorem 1. *Assume that* Ω *is of type* (1.3)*, with (H1)-(H2). Let* $p \in (1,\infty]$ *and* u^0 *as in* (1.4)*. Then there exists*

$$u \in L^\infty(\mathbb{R}^+; L^2(\Omega)), \quad \textit{with } \operatorname{curl} u \in L^\infty(\mathbb{R}^+; L^p(\Omega))$$

which is a global weak solution of (1.1)-(1.2) *in the sense of* (1.6) *and* (1.7)*.*

In a few words, our existence theorem will follow from a compactness argument, performed on smooth solutions u_n of the Euler equation in smooth approximate domains Ω_n. By *approximate*, we mean close to Ω in the Hausdorff topology. These approximate domains, to be built in Section 2, read

$$\Omega_n := \widetilde{\Omega}_n \setminus \left(\cup_{i=1}^k \overline{O^i_n}\right)$$

for some smooth Jordan domains $\widetilde{\Omega}_n$ and O^i_n. A keypoint is the so-called Γ-convergence of Ω_n to Ω. All necessary prerequisites on Hausdorff or Γ-convergence will be given in Section 2. The compactness argument will be given in Section 3 ($p = \infty$) and Section 5 (finite p).

The analysis of the first part at hand, we then turn to a class of exterior domains Ω. We assume that Ω is the exterior of a bounded hole with positive capacity. It reads

$$\Omega := \mathbb{R}^2 \setminus \mathcal{C} \tag{1.8}$$

with

(H1') (connectedness) $\mathcal{C}$ is a connected compact set.

(H2') (capacity) $\operatorname{cap}(\mathcal{C}) > 0$.

Let us point out that to work with square integrable velocities in exterior domains is too restrictive. Therefore, we relax the condition (1.4) on the initial data into

$$u^0 \in L^2_{\text{loc}}(\overline{\Omega}), \quad u^0 \to 0 \text{ as } |x| \to +\infty, \quad \operatorname{curl} u^0 \in L^p(\Omega), \quad \operatorname{div} u^0 = 0, \quad u^0 \cdot n|_{\partial\Omega} = 0. \tag{1.9}$$

We make the additional assumption that

$$\operatorname{curl} u^0 \text{ is supported in a compact subset of } \mathbb{R}^2 \tag{1.10}$$

We prove in Sections 4 and 5 the following result:

Theorem 2. *Assume that* Ω *is of type* (1.8)*, with (H1')-(H2'). Let* $p \in (2,\infty]$ *and* u^0 *satisfying* (1.9)-(1.10)*. Then, there exists*

$$u \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^2_{\text{loc}}(\overline{\Omega})), \quad \textit{with } \operatorname{curl} u \in L^\infty(\mathbb{R}^+; L^1 \cap L^p(\Omega))$$

which is a global weak solution of (1.1)-(1.2) *in the sense of* (1.6) *and* (1.7)*.*

Again, the weak solution u is obtained from the compactness of a sequence of smooth solutions u_n in the approximate domains $\Omega_n := \mathbb{R}^2 \setminus \overline{O_n}$. Our theorem improves the result in [6, 7] in two ways. First, we treat more shapes than just C^2 Jordan arcs. Second, the convergence of Ω_n to Ω is expressed through the Hausdorff distance, which is more general and simple than the conditions in [7]. Therein, one needs stringent convergence properties of the biholomorphisms that map Ω_n to the set $\{|z| > 1\}$. In particular, to obtain the uniform convergence of the first derivatives requires the convergence of the tangent angles of ∂O_n. We refer to [7] for detailed statements.

We point out that the limit dynamics in Theorem 2 is expressed differently than in [6]. Indeed, in this article, extensions $\tilde{u}_n$ of u_n to the whole plane are considered, resulting in a modified Euler system in the whole plane at the limit. This system is expressed in vorticity form, and reads

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad \omega := \operatorname{curl} u - g_\omega \delta_{\mathcal{C}}, \quad t > 0, \, x \in \mathbb{R}^2, \tag{1.11}$$

with an additional Dirac mass along the curve. The equivalence between (1.11) and the standard formulation (1.7) of our theorem will be discussed in Section 6. In particular, it is proved in [6] that the velocity blows up near the end-points like the inverse of the square root of the distance, which belongs to L^p_{loc} for $p < 4$. Here, we will obtain some uniform estimates of the velocity in L^2_{loc} (see (4.14)-(4.16)) which are in agreement with the former estimates.

Our global existence results will be proved through several steps. The special case $p = \infty$ will be treated with full details in Section 3 (Theorem 1) and Section 4 (Theorem 2). The extension to finite p will be sketched in Section 5.

Let us finally insist that even for Yudovich type solutions ($p = \infty$), we only deal with global existence, not uniqueness. The uniqueness problem in singular domains is a hard issue, that has been so far only resolved in special cases: see [8].

2 Reminders

This section collects some geometric material needed for the rest of our study. The first paragraph gives some basic knowledge of Hausdorff convergence. In the second paragraph, given some domain Ω satisfying (1.3) and (H1), or (1.8) and (H1'), we construct a "nice" sequence of open sets (Ω_n) that converges to Ω in the Hausdorff sense. Later on, we will use compactness properties for solutions of the Euler equations in these approximate domains Ω_n. In the third paragraph, we remind some basic facts about the Sobolev capacity, to be used later on with assumptions (H2) or (H2'). The notion of Γ-convergence of open sets, which plays a crucial role in the proof of our theorems, is discussed in the fourth paragraph. Finally, we discuss the so-called kernel convergence of Caratheodory, to be used in our treatment of exterior domains. Most of the material in this section is taken from the books [3, 13], where the reader can find more details and proofs.

2.1 Hausdorff distance

We first introduce the Hausdorff distance for compact sets. Let $\mathcal{K}$ the set of all non-empty compact sets of $\mathbb{R}^N$, $N \geq 1$. For $K_1, K_2 \in \mathcal{K}$, we define

$$d_H(K_1, K_2) := \max\left(\rho(K_1, K_2), \rho(K_2, K_1)\right), \quad \rho(K, K') := \sup_{x \in K} d(x, K').$$

It is an easy exercise to show that d_H defines a distance on $\mathcal{K}$. Sequences that converge with respect to this distance are said to converge in the Hausdorff sense. One has the following basic properties

Proposition 1. *1. A decreasing sequence of non-empty compact sets converges in the Hausdorff sense to its intersection.*

2. *An increasing sequence of non-empty compact sets converges in the Hausdorff sense to the closure of its union.*

3. *Inclusion is stable for convergence in the Hausdorff sense.*

4. *The Hausdorff convergence preserves connectedness. More generally, if $(K_n)_{n\in\mathbb{N}}$ converges to K, and K_n has at most p connected components, K has at most p connected components.*

A remarkable feature of the Hausdorff topology is given by the following

Proposition 2. *Any bounded sequence of $(\mathcal{K}, d_H)$ has a convergent subsequence.*

From the Hausdorff topology on $\mathcal{K}$, one can define a Hausdorff topology on *confined* open sets, that is on all open sets included in some big given compact. Thus, let B some compact domain in $\mathbb{R}^N$, $N \geq 1$, and $\mathcal{O}_B$ the set of all open sets included in B. The Hausdorff distance on $\mathcal{O}_B$ is defined by:

$$d_H(\Omega_1, \Omega_2) := d_H(B \setminus \Omega_1, B \setminus \Omega_2)$$

the r.h.s refering to the Hausdorff distance for compact sets. Let us note that this distance does not really depend on B: that is, for $B \subset B'$ two compact sets, and Ω_1, Ω_2 in $\mathcal{O}_B$,

$$d_H(B' \setminus \Omega_1, B' \setminus \Omega_2) = d_H(B \setminus \Omega_1, B \setminus \Omega_2).$$

Proposition 3. 1. *An increasing sequence of (confined) open sets converges in the Hausdorff sense to its union.*

2. *A decreasing sequence of (confined) open sets converges in the Hausdorff sense to the interior of its intersection.*

3. *Inclusion is stable for convergence in the Hausdorff sense.*

4. *Finite intersection is stable for convergence in the Hausdorff sense*

5. *Let (Ω_n) a sequence that converges to Ω in the Hausdorff sense. Let $x \in \partial\Omega$. There exists a sequence (x_n) with $x_n \in \partial\Omega_n$ that converges to x.*

6. *Let $(\Omega_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{O}_B$. There exists an open set $\Omega \in \mathcal{O}_B$ and a subsequence $(\Omega_{n_k})_{k\in\mathbb{N}}$ that converges to Ω in the Hausdorff sense*

Let us note that the Hausdorff convergence of open sets, contrary to the one of compact sets, does not preserve connectedness. Let us finally point out the following result, to be used later on:

Proposition 4. *If $(\Omega_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense to Ω and K is a compact subset of Ω, then there exists n_0 such that $\Omega_n \supset K$ for $n \geq n_0$.*

2.2 Approximation of domains verifying (H1) or (H1')

The aim of this paragraph is to construct nice open sets Ω_n, that approximate in the Hausdorff topology the open sets Ω described in the introduction. We state

Proposition 5. 1. *Let Ω of type (1.3), satisfying (H1). Then, Ω is the Hausdorff limit of a sequence $\Omega_n := \widetilde{\Omega}_n \setminus \left(\cup_{i=1}^k \overline{O_n^i}\right)$ where $\widetilde{\Omega}_n$ and the O_n^i's are smooth Jordan domains, and such that $\widetilde{\Omega}_n$, resp. $\overline{O_n^i}$, converges in the Hausdorff sense to $\widetilde{\Omega}$, resp. $\mathcal{C}^i$.*

2. *Let Ω of type* (1.8), *satisfying (H1'). Then, Ω is the Hausdorff limit of a sequence*

$$\Omega_n := \mathbb{R}^2 \setminus \overline{O_n}$$

where O_n is a smooth Jordan domain, whose closure converges in the Hausdorff sense to $\mathcal{C}$.

Sketch of proof: Let U be a bounded simply connected domain. By the Riemann mapping theorem, there exists a unique biholomorphism $\mathcal{T} : \{|z| < 1\} \mapsto \widetilde{\Omega}$ satisfying $\mathcal{T}'(0) > 0$. The sets $U_n := \mathcal{T}(\{|z| < 1 - 1/n\})$ are smooth Jordan domains, with (U_n), resp. $\overline{U_n}$ converging respectively to U and $\overline{U}$. In particular, applying this argument with $U = \tilde{\Omega}$ yields the sequence $(\widetilde{\Omega}_n)$ from the lemma. To conclude the proof, it remains to show that any connected compact set $\mathcal{C}$ can be approximated in the Hausdorff topology by the closure of a bounded simply connected domain. Clearly, $\mathcal{C}$ can be approximated by the closure of a non-disjoint and finite union of open disks $O = \cup D_i$ (the fact that the union is not disjoint comes from the connectedness of $\mathcal{C}$). Then, to approximate O by a simply connected domain, one makes slits in the disks, connecting the gaps left by the disks either to each other or to the outside. Details are left to the reader. □

Besides the approximate domains Ω_n, we need for later purposes *to introduce Jordan curves "separating" the holes $\mathcal{C}^i$'s in* (1.3). One must take into account that one compact set $\mathcal{C}^i$ can enclose some of the other $\mathcal{C}^j$'s. The reader can think of the following example:

$$\widetilde{\Omega} = B(0,3),\ \mathcal{C}^1 = \partial B(0,1),\ \mathcal{C}^2 = \overline{B(0,1/2)},\ \mathcal{C}^3 = \partial B((0,2),1/2).$$

In particular, one cannot find a closed curve enclosing $\mathcal{C}^1$ and not $\mathcal{C}^2$. We define separating curves as follows. For any closed curve J, let $C_{ub}(J)$ the unbounded connected component of $\mathbb{R}^2 \setminus J$. Let

$$E_i := \Big\{ j = 1..k,\ j \neq i,\ \mathcal{C}^j \subset (\text{bounded connected components of } \mathbb{R}^2 \setminus \mathcal{C}^i) \Big\}. \tag{2.1}$$

(for the above example, $E_1 = \{2\}$ and $E_2 = E_3 = \emptyset$). We set

Definition 1. *Let $J^1, \dots, J^k \subset \Omega$ some disjoint smooth closed curves. $J^1, \dots, J^k$ are* separating curves *if they satisfy : for all $i = 1 \dots k$*

- *J^i encloses $\mathcal{C}^i$, i.e. $\mathcal{C}^i \subset \mathbb{R}^2 \setminus C_{ub}(J^i)$.*
- *J^i does not enclose $\mathcal{C}^j$ for all $j \notin E_i$, i.e. $\mathcal{C}^j \subset C_{ub}(J^i), \forall j \notin E_i$.*

2.3 Capacity

Let $E \subset \mathbb{R}^N,\ N \geq 1$. The capacity of E (with respect to the Sobolev space $H^1(\mathbb{R}^N)$) is defined by

$$\text{cap}(E) := \inf\{\|v\|^2_{H^1(\mathbb{R}^N)},\ v \geq 1 \text{ a.e. in a neighborhood of } E\},$$

with the convention that $\text{cap}(E) = +\infty$ when the set at the r.h.s. is empty. The capacity is not a measure, but has similar good properties:

Proposition 6. *1. $A \subset B \Rightarrow \text{cap}(A) \leq \text{cap}(B)$.*

2. *Let $(K_n)_{n\in\mathbb{N}}$ a decreasing sequence of compact sets, with $K = \cap K_n$. Then, $\text{cap}(K) = \lim \text{cap}(K_n)$.*

3. *Let $(E_n)_{n\in\mathbb{N}}$ an increasing sequence of sets, with $E = \cup E_n$. Then, $\text{cap}(E) = \lim \text{cap}(E_n)$.*

4. (Strong subadditivity) For all sets A and B, one has

$$\text{cap}(A \cup B) + \text{cap}(A \cap B) \;\leq\; \text{cap}(A) + \text{cap}(B).$$

If D is a bounded open set of $\mathbb{R}^N$, one can also define a capacity relatively to D: for $E \subset D$,

$$\text{cap}_D(E) \;:=\; \inf \left\{ \|\nabla v\|^2_{L^2(D)},\; v \in H^1_0(D),\; v \geq 1 \text{ a.e. in a neighborhood of } E \right\},$$

with the same convention as before. It is clear from this definition and the Poincaré inequality that $\text{cap}(E) \;\leq\; C\,\text{cap}_D(E)$.

For nice sets E in $\mathbb{R}^N$, the capacity of E can be thought very roughly as some $n-1$ dimensional Hausdorff measure of its boundary. More precisely:

Proposition 7. *1. For all compact set K included in a bounded open set D, $\text{cap}(K) = \text{cap}(\partial K)$.*

2. If $E \subset \mathbb{R}^N$ is contained in a manifold of dimension $N-2$, then $\text{cap}(E) = 0$.

3. If $E \subset \mathbb{R}^N$ contains a piece of some smooth hypersurface (manifold of dimension N-1), then $\text{cap}(E) > 0$.

The last result concerns $H^1_0(\Omega)$. When Ω is a smooth open set, $H^1_0(\Omega)$ can be defined as the set of function in $H^1(\mathbb{R}^2)$ which are equal to zero almost everywhere in $\mathbb{R}^2 \setminus \Omega$. But this result does not hold for general open sets Ω. To generalize such a characterization, the notion of capacity is appropriate.

Proposition 8. *Let D and Ω be open sets such that $\Omega \subset D$. Then*

$$\left(u \in H^1_0(\Omega)\right) \iff \left(u \in H^1_0(D) \text{ and } u = 0 \text{ quasi everywhere in } D \setminus \Omega\right),$$

which means that $u = 0$ except on a set with zero capacity.

2.4 Γ-convergence of open sets

Let D be a bounded open set. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets included in D. One says that $(\Omega_n)_{n\in\mathbb{N}}$ Γ-converges to $\Omega \subset D$ if for any $f \in H^{-1}(D)$, the sequence of solutions $u_n \in H^1_0(\Omega_n)$ of

$$-\Delta u_n = f \;\text{ in }\; \Omega_n, \quad u_n|_{\partial\Omega_n} = 0.$$

converges in $H^1_0(D)$ to the solution $u \in H^1_0(\Omega)$ of

$$-\Delta u = f \;\text{ in }\; \Omega, \quad u|_{\partial\Omega} = 0.$$

In this definition, $H^1_0(\Omega)$ and $H^1_0(\Omega_n)$ are seen as subsets of $H^1_0(D)$, through extension by zero. In a dual way, $H^{-1}(D)$ is seen as a subset of $H^{-1}(\Omega_n)$ and $H^{-1}(\Omega)$. As for the Hausdorff convergence of open sets, the definition of Γ-convergence does not depend on the choice of the confining set D.

The notion of Γ-convergence is extensively discussed in [3]. The basic example of Γ-convergence is given by increasing sequences:

Proposition 9. *If $(\Omega_n)_{n\in\mathbb{N}}$ is an increasing sequence in D, it Γ-converges to $\Omega = \cup\Omega_n$. More generally, if $(\Omega_n)_{n\in\mathbb{N}}$ is included in Ω and converges to Ω in the Hausdorff sense, then it Γ-converges to Ω.*

In general, Hausdorff converging sequences are not Γ-converging. We refer to [3] for counterexamples, with domains Ω_n that have more and more holes as n goes to infinity. This kind of counterexamples, reminiscent of homogenization problems, is the only one in dimension 2, as proved by Sverak:

Proposition 10. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets in $\mathbb{R}^2$, included in D. Assume that the number of connected components of $D \setminus \Omega_n$ is bounded uniformly in n. If $(\Omega_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense to Ω, it Γ-converges to Ω.*

This result will be a crucial ingredient of the next sections.

One can characterize the Γ-convergence in terms of the Mosco-convergence of $H^1_0(\Omega_n)$ to $H^1_0(\Omega)$. Namely:

Proposition 11. *$(\Omega_n)_{n\in\mathbb{N}}$ Γ-converges to Ω if and only if the following two conditions are satisfied:*

1. *For all $u \in H^1_0(\Omega)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ with u_n in $H^1_0(\Omega_n)$ that converges to u.*
2. *For any sequence $(u_n)_{n\in\mathbb{N}}$ with u_n in $H^1_0(\Omega_n)$, weakly converging to u in $H^1_0(D)$, $u \in H^1_0(\Omega)$.*

One can also characterize Γ-convergence with capacity, see [3, Proposition 3.5.5 page 114].

2.5 Convergence of biholomorphisms in exterior domains

Through the identification of $\mathbb{R}^2$ with $\mathbb{C}$, complex analysis is a great tool for the study of two-dimensional ideal flows in exterior domains. The proof of Theorem 2 will require some results on conformal mapping that we now explain.

Let Ω of type (1.8), satisfying (H1')-(H2'). We denote $D := \{|z| < 1\}$ the open unit disk, $\Omega_n := \mathbb{R}^2 \setminus \overline{O_n}$ the approximate exterior domain given by Proposition 5, and $\Delta := \{|z| > 1\}$ the exterior of the closed unit disk. From the Riemann mapping theorem, it is easily seen that there is a unique biholomorphism

$$\mathcal{T}_n : \Omega_n \mapsto \Delta, \quad \text{with } \mathcal{T}_n(\infty) = \infty, \quad \mathcal{T}_n'(\infty) > 0.$$

We remind that the last two conditions mean

$$\mathcal{T}_n(z) \sim \lambda_n z, \quad |z| \sim +\infty, \quad \text{for some } \lambda_n > 0.$$

Like in [4, 6, 7], a key point will be to control $\mathcal{T}_n$ when $\overline{O_n}$ tends to $\mathcal{C}$. Therefore, we will rely on the notion of kernel convergence introduced by Caratheodory in 1912, see [13, p28] (we remind that the word *domain* refers to a connected open set):

Definition 2. *Let (F_n) be a sequence of domains, with $0 \in F_n$ for all n. Its kernel F (with respect to 0) is the set consisting of 0 together with all points $w \in \mathbb{C}$ that satisfy:* there exists a domain H including 0 and w such that $H \subset F_n$ for all n large enough.

If F is the kernel of any subsequence of (F_n), we say that (F_n) converges to F in the kernel sense.

This type of geometric convergence is related to the famous Caratheodory theorem, see [13, Theorem 1.8, p29]:

Proposition 12. *Let (f_n) be a sequence of biholomorphisms from D to $F_n := f_n(D)$, with $f_n(0) = 0$, $f_n'(0) > 0$. Then, f_n converges locally uniformly in D if and only if (F_n) converges to its kernel F and if $F \neq \mathbb{C}$. Moreover, the limit function maps D onto F.*

From the Caratheodory theorem, it is possible to deduce the following property, which will be crucial in our proof of Theorem 2:

Proposition 13. *Let Π be the unbounded connected component of Ω. There is a unique biholomorphism $\mathcal{T}$ from Π to Δ, satisfying $\mathcal{T}(\infty) = \infty$, $\mathcal{T}'(\infty) > 0$. Moreover, one has the following convergence properties:*

i) *$\mathcal{T}_n^{-1}$ converges uniformly locally to $\mathcal{T}^{-1}$ in Δ.*

ii) *$\mathcal{T}_n$ (resp. $\mathcal{T}_n'$) converges uniformly locally to $\mathcal{T}$ (resp. to $\mathcal{T}'$) in Π.*

iii) *$|\mathcal{T}_n|$ converges uniformly locally to 1 in $\Omega \setminus \Pi$.*

Proof of the proposition: Let us first point out that, because of Hausdorff convergence and Proposition 4, any compact of Ω is included in Ω_n for n large enough. Thus, the local convergence properties stated in ii) and iii) make sense.

Up to a change of coordinates, we can always assume that $0 \in \partial\Omega \subset \mathcal{C}$. By Proposition 3, there exists $x_n \in \partial\Omega_n$ converging to 0. Then, if we introduce the domains

$$F_n := \frac{1}{(\Omega_n - x_n) \cup \{\infty\}} := \left\{\frac{1}{z},\ z + x_n \in \Omega_n\right\} \cup \{0\},$$

and

$$F := \frac{1}{\Pi \cup \{\infty\}} := \left\{\frac{1}{z},\ z \in \Pi\right\} \cup \{0\},$$

it follows easily from (H1'), Proposition 3 and Proposition 4 that F_n converges to F in the kernel sense. Note that by the choice of (x_n), the F_n's do not include ∞.

Hence, by the Caratheodory theorem, the sequence of biholomorphisms (f_n) defined by

$$f_n : D \mapsto f_n(D) = F_n, \quad f_n(z) := \frac{1}{\mathcal{T}_n^{-1}(1/z) - x_n}$$

converges uniformly locally in D to some function f from D onto F. By the Weierstrass convergence theorem, f is holomorphic over D. Moreover, by a standard application of the Rouché formula, as f_n is one-to-one for all n, so is f. Going on with standard arguments, f is the unique biholomorphism that maps D to F and that satisfies $f(0) = 0$, $f'(0) > 0$. Back to $\mathcal{T}_n^{-1}$, this yields i) with $\mathcal{T}^{-1}(z) := \dfrac{1}{f(1/z)}$. Actually, one has clearly uniform convergence of $\mathcal{T}_n^{-1}$ to $\mathcal{T}^{-1}$ in $\Delta_\delta := \{|z| \geq 1 + \delta\}$ for all $\delta > 0$. Then, by the Weierstrass theorem, the sequence of derivatives $(\mathcal{T}_n^{-1})'$ converges locally uniformly to $(\mathcal{T}^{-1})'$.

As regards ii), let $z_0 \in \Pi$, and $J_n := \mathcal{T}_n^{-1}(\{z', |z' - \mathcal{T}(z_0)| = \delta\})$. By i), for $\delta > 0$ small enough and n large enough, J_n is a closed curve that encloses z_0 and is contained in Π. For all z in a small enough neighborhood of z_0, we can then write the Cauchy formula:

$$\mathcal{T}_n(z) = \frac{1}{2i\pi} \int_{J_n} \frac{\mathcal{T}_n(\xi)}{\xi - z} d\xi = \frac{1}{2i\pi} \int_{\{|\xi' - \mathcal{T}(z_0)| = \delta\}} \frac{\xi'}{\mathcal{T}_n^{-1}(\xi') - z} (\mathcal{T}_n^{-1})'(\xi')\, d\xi'$$

where the last equality comes from the change of variable $\xi = \mathcal{T}_n^{-1}(\xi')$. Thanks to i), we may let n go to infinity to obtain the convergence of $\mathcal{T}_n$ to $\mathcal{T}$ uniformly in a neighborhood of z_0. Again, the convergence of derivatives follows from the Weierstrass theorem. This ends the proof of ii).

To obtain iii), we argue by contradiction. We assume *a contrario* that there exists a $\delta > 0$ and a sequence z_n located in a given closed ball B of $\Omega \setminus \Pi$ such that $|\mathcal{T}_n(z_n)| \geq 1 + \delta$. Up to extract a subsequence, we can assume that $z_n \to z \in B$. By the uniform convergence of $\mathcal{T}_n^{-1}$ in Δ_δ (see above) we have $z \in \mathcal{T}^{-1}(\Delta_\delta) \subset \Pi$. Thus, we reach a contradiction, which proves iii).

□

3 Theorem 1 for $p = \infty$

This section is devoted to the proof of Theorem 1, in the case $p = \infty$. Our starting point is the approximation of Ω by smooth domains

$$\Omega_n := \widetilde{\Omega}_n \setminus \left(\cup_{i=1}^k \overline{O_n^i} \right),$$

given in Proposition 5, 1). We also need to approximate the initial data u^0, to generate some strong Euler solution in Ω_n. We proceed as follows. Let $\omega^0 := \operatorname{curl} u^0$. By truncation and convolution, there exists some sequence $\omega_k^0 \in C_c^\infty(\Omega)$ such that

$$\omega_k^0 \to \omega^0 \quad \text{weakly in } L^p(\Omega), \quad \|\omega_k^0\|_{L^p} \le \|\omega^0\|_{L^p}, \quad \forall p \in [1, \infty].$$

As $(\Omega_n)_{n\in\mathbb{N}}$ converges to Ω in the Hausdorff sense, it follows from Proposition 4 that $\omega_k^0 \in C_c^\infty(\Omega_{n_k})$ for n_k large enough. Hence, up to extract a subsequence from $(\Omega_n)_{n\in\mathbb{N}}$ one can assume

$$\omega_n^0 \in C_c^\infty(\Omega_n), \quad \forall n \in \mathbb{N}.$$

To build up a velocity field u_n^0 from ω_n^0, we still need to specify the circulation around each obstacle. Let $J^1, \dots, J^k$ be separating curves, as in Definition 1. Up to consider large enough indices n, we can always assume that

$$\partial\widetilde{\Omega}_n \subset C_{ub}(J^i) \text{ for all } i = 1..k, \qquad \overline{O_n^i} \subset C_{ub}(J^i)^c \bigcap_{\{j,\ i \notin E_j\}} C_{ub}(J^j) \bigcap_{\{j,\ i \in E_j\}} C_{ub}(J^j)^c.$$

By standard results related to the Hodge-De Rham theorem, there exists a unique field $u_n^0 \in C_c^\infty(\overline{\Omega_n})$ satisfying

$$\operatorname{curl} u_n^0 = \omega_n^0, \quad \operatorname{div} u_n^0 = 0, \quad u_n^0 \cdot n|_{\partial\Omega_n} = 0, \quad \int_{J^i} u_n^0 \cdot \tau ds = \int_{J^i} u^0 \cdot \tau ds,$$

where τ denotes the unit tangent vector rotating counterclockwise. Let us remind that u^0 satisfies (1.4) with $p = \infty$, so that it belongs to $W^{1,q}_{\text{loc}}$ for all finite q, and so that the integrals at the r.h.s. are well-defined.

We take (u_n^0) as our sequence of initial data. We shall postpone the convergence of u_n^0 to u^0 to the end of the section. We consider for all n the unique smooth solution u_n of the Euler equation in Ω_n, such that

$$u_n \cdot n|_{\partial\Omega_n} = 0, \quad u_n|_{t=0} = u_n^0.$$

Again, from classical results related to the Hodge-De Rham theorem, the divergence-free smooth fields u_n satisfy in Ω_n

$$u_n(t,x) = \nabla^\perp \psi_n^0(t,x) + \sum_{i=1}^k \alpha_n^i(t) \nabla^\perp \psi_n^i(x) \tag{3.1}$$

where ψ_n^0 satisfies the Dirichlet problem

$$\Delta \psi_n^0 = \omega_n := \operatorname{curl} u_n \text{ in } \Omega_n, \quad \psi_n^0|_{\partial\Omega_n} = 0 \tag{3.2}$$

whereas ψ_n^i, $i = 1 \dots k$ are harmonic functions satisfying

$$\Delta \psi_n^i = 0 \text{ in } \Omega_n, \quad \frac{\partial \psi_n^i}{\partial \tau}|_{\partial\Omega_n} = 0, \quad \int_{\partial O_n^j} \frac{\partial \psi_n^i}{\partial n} = -\delta_{ij}, \quad \psi_n^i|_{\partial\widetilde{\Omega}_n} = 0, \tag{3.3}$$

where δ_{ij} is the Kronecker symbol and n denotes the unit vector pointing outside Ω_n. Note that α_n^i, $i = 1 \dots k$ only depends on time (the formula will be given in Proposition 14).

We refer to [5] and [11] for all details. The key point in proving Theorem 1 is to obtain some compactness on u_n through the study of the ψ_n^i's.

3.1 Study of the harmonic part

We first focus on the harmonic part of u_n, that is the sum at the r.h.s. of (3.1). Note that the harmonic functions $\psi_n^i,\ i = 1\dots k$ are defined up to an additive constant. We fix this constant by imposing

$$\psi_n^i = 0 \ \text{ on } \partial\widetilde{\Omega}_n.$$

We then introduce the auxiliary harmonic functions $\phi_n^i,\ i = 1\dots k$, that satisfy

$$\Delta\phi_n^i = 0, \quad \phi_n^i|_{\partial\widetilde{\Omega}_n} = 0, \quad \phi_n^i|_{\partial O_n^j} = \delta_{ij}, \quad j = 1\dots k. \tag{3.4}$$

Clearly, one can decompose each ψ_n^i on the ϕ_n^j's:

$$\psi_n^i \ = \ \sum_{j=1}^{k} c_n^{i,j}\,\phi_n^j. \tag{3.5}$$

Our ambition in this section is to prove the convergence of the ϕ_n^j's, the $c_n^{i,j}$'s and the α_n^i's as n goes to infinity.

For all $i = 1\dots k$ and $\varepsilon > 0$, we denote $\mathcal{C}^{i,\varepsilon} \ := \ \{x,\ d(x, \mathcal{C}^i) \le \varepsilon\}$ the ε-neighborhood of $\mathcal{C}^i$. Let $\chi^{i,\varepsilon} \in C_c^\infty(\mathbb{R}^2)$ smooth functions satisfying

$$\chi^{i,\varepsilon} = 1 \ \text{ on } \mathcal{C}^{i,\varepsilon}, \quad \chi^{i,\varepsilon} = 0 \ \text{ on } \mathbb{R}^2 \setminus \mathcal{C}^{i,2\varepsilon}.$$

By Assumptions (H1)-(H2), there exists $\varepsilon > 0$ and $n_0 = n_0(\varepsilon)$ such that

$$\chi^{i,\varepsilon} = 1 \ \text{ on } \overline{O_n^i}, \quad \chi^{i,\varepsilon} = 0 \ \text{ on } \overline{O_n^j}, \quad j \neq i, \quad \chi^{i,\varepsilon} = 0 \ \text{ on } \partial\widetilde{\Omega}_n, \quad \text{for all } \ n \ge n_0.$$

For brevity, we drop the upperscript ε. We notice that the function $\Phi_n^i \ := \ \phi_n^i - \chi^i$ satisfies

$$\Delta\Phi_n^i = -\Delta\chi^i \quad \text{in } \ \Omega_n, \quad \Phi_n^i|_{\partial\Omega_n} = 0.$$

Let D some big open ball containing all the Ω_n's. We can use Proposition 10: as $(\Omega_n)_{n\in\mathbb{N}}$ converges to Ω in the Hausdorff sense and the complement in D of Ω_n has at most $k+1$ connected components for all n, $(\Omega_n)_{n\in\mathbb{N}}$ Γ-converges to Ω. We deduce that Φ_n^i converges in $H_0^1(D)$ to the solution $\Phi^i \in H_0^1(\Omega)$ of

$$\Delta\Phi^i = -\Delta\chi^i \quad \text{in } \ \Omega, \quad \Phi^i|_{\partial\Omega} = 0.$$

Setting $\phi^i \ := \ \Phi^i + \chi^i$, we have for $i = 1\dots k$ the convergence of ϕ_n^i to ϕ^i strongly in $H_0^1(D)$.

Let us now turn to the convergence of the constants $c_n^{i,j}$. We take the normal derivative at both sides of (3.5) and integrate along ∂O_n^m, for $m \in \{1, \dots k\}$. We obtain thanks to (3.3) and (3.4):

$$-\delta_{im} \ = \ \sum_{j=1}^{k} c_n^{i,j} \int_{\partial O_n^m} \frac{\partial\phi_n^j}{\partial n} \ = \ \sum_{j=1}^{k} c_n^{i,j} \int_{\Omega_n} \nabla\phi_n^j \cdot \nabla\phi_n^m.$$

Introducing the $k \times k$ identity matrix Id, this last line reads:

$$-Id \ = \ C_n\,P_n, \quad \text{ with } \ C_n = \left(c_n^{i,j}\right)_{1\le i,j\le k}, \quad P_n = \left(\int_{\Omega_n} \nabla\phi_n^i \cdot \nabla\phi_n^j\right)_{1\le i,j\le k}.$$

Our goal is to show the convergence of C_n: it is therefore enough to prove the convergence of P_n to an invertible matrix P. But from the previous step, that is the convergence of ϕ_n^i to ϕ^i in $H_0^1(D)$, we know that P_n converges to

$$P := \left(\int_D \nabla\phi^i \cdot \nabla\phi^j\right)_{1\le i,j\le k}.$$

The matrix P is selfadjoint and nonnegative: namely, for any vector $\lambda \in \mathbb{R}^k$,

$$P\lambda \cdot \lambda \;=\; \int_D |\nabla \sum_{i=1}^k \lambda_i \, \phi^i|^2.$$

Thus, to prove the invertibility of P, it is enough to show that the ϕ^i's are linearly independant. Assume *a contrario* that

$$\sum \lambda_i \, \phi^i = 0 \text{ almost everywhere, for some non-zero vector } \lambda.$$

Up to reindex the functions, one can assume that $\lambda_1 \neq 0$. We remind that the functions $\Phi^i := \phi^i - \chi^i$ belong to $H^1_0(\Omega)$ (see above). Thus, there exists a sequence of functions $\tilde{\Phi}^i_n$ in $C^\infty_c(\Omega)$ converging to Φ^i in $H^1_0(\Omega)$, $i = 1...k$. We set $\tilde{\phi}^i_n \;:=\; \tilde{\Phi}^i_n + \chi^i$, and introduce

$$v_n \;:=\; \frac{1}{\lambda_1} \left(\sum_{i=1}^k \lambda_i \, \tilde{\phi}^i_n \right) \;\to\; 0 \quad \text{ in } \; H^1_0(D).$$

Clearly, $v_n = 1$ on a neighborhood of $\mathcal{C}^1$. It follows that

$$\int_D |\nabla v_n|^2 \;\geq\; \text{cap}_D(\mathcal{C}^1),$$

and letting n go to infinity leads to $\text{cap}_D(\mathcal{C}^1) = 0$. This contradicts Assumption (H3).

Eventually, we obtain that P is invertible, which yields a uniform bound on the $c^{i,j}_n$'s, and their convergence (up to subsequences) to some limit constants $c^{i,j}$. From the above lines and from relation (3.5), we deduce that $(\psi^i_n)_{n \in \mathbb{N}}$ converges (up to a subsequence) to $\psi^i \;:=\; \sum_{j=1}^{j} c^{i,j} \, \phi^i$ in $H^1_0(D)$, for all $i = 1, \ldots, k$.

To completely control the harmonic part of the velocity u_n, it remains to show convergence of the time dependent functions α^i_n, $i = 1 \ldots k$ in (3.1). We shall use the following proposition, to be found in [11]:

Proposition 14. *For all* $i = 1 \ldots k$, $\alpha^i_n \;=\; \int_{\Omega_n} \phi^i_n \, \omega_n \, dx \;+\; \int_{\partial O^i_n} u_n \cdot \tau \, ds.$

By Kelvin's theorem, the circulation of u_n on each ∂O^i_n is constant in time so that

$$\alpha^i_n \;=\; \int_{\Omega_n} \phi^i_n \, \omega_n \, dx \;+\; \int_{\partial O^i_n} u^0_n \cdot \tau \, ds.$$

With the notations of Subsection 2.2, *we introduce for all* $i = 1...k$ *the region* A^i_n *included between* J^i *and* ∂O^i_n:

$$A^i_n := \Omega_n \bigcap C_{ub}(J^i)^c \bigcap_{j \in E_i} C_{ub}(J^j).$$

We also introduce: $A^i := \Omega \bigcap C_{ub}(J^i)^c \bigcap_{j \in E_i} C_{ub}(J^j)$. Then we compute

$$\int_{J^i} u^0_n \cdot \tau \, ds = \int_{A^i_n} \omega^0_n \, dx + \int_{\partial O^i_n} u^0_n \cdot \tau \, ds + \sum_{j \in E_i} \int_{J^j} u^0_n \cdot \tau \, ds.$$

It follows that

$$\alpha^i_n \;=\; \int_{\Omega_n} \phi^i_n \, \omega_n \, dx + \int_{J^i} u^0 \cdot \tau \, ds - \int_{A^i_n} \omega^0_n \, dx - \sum_{j \in E_i} \int_{J^j} u^0 \cdot \tau \, ds. \tag{3.6}$$

Moreover, we remind that the vorticity ω_n obeys the transport equation

$$\partial_t \omega_n + u_n \cdot \nabla \omega_n = 0$$

so that the L^p norms are conserved:

$$\|\omega_n(t,\cdot)\|_{L^p(\Omega_n)} = \|\omega_n^0\|_{L^p(\Omega_n)} \le \|\omega^0\|_{L^p(\Omega)}, \;\; 1 \le p \le \infty. \tag{3.7}$$

We now extend ω_n by 0 outside Ω_n for all n, so that the sequence $(\omega_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^+ \times D)$. Up to an extraction, we deduce that

$$\omega_n \to \omega \;\text{ weakly * in }\; L^\infty(\mathbb{R}^+ \times D). \tag{3.8}$$

One has easily that

$$\omega = 0 \quad \text{outside } \overline{\Omega}. \tag{3.9}$$

From this convergence and (3.6), we infer that α_n^i converges weakly* in $L^\infty(\mathbb{R}^+)$ to

$$\begin{aligned} \alpha^i \;&:=\; \int_D \phi^i \,\omega\, dx \,+ \int_{J^i} u^0 \cdot \tau\, ds - \int_D 1_{A^i} \omega^0\, dx - \sum_{j\in E_i} \int_{J^j} u^0 \cdot \tau\, ds \\ &= \int_\Omega \phi^i \,\omega\, dx \,+ \int_{J^i} u^0 \cdot \tau\, ds - \int_{A^i} \omega^0\, dx - \sum_{j\in E_i} \int_{J^j} u^0 \cdot \tau\, ds. \end{aligned}$$

Unfortunately, we cannot establish rightaway strong convergence of (α_n^i). We need some uniform $L^\infty L^2$ bounds on u_n, to be obtained in the section below.

3.2 Study of the rotational part

A simple energy estimate on (3.2) yields

$$\|\nabla \psi_n^0(t,\cdot)\|^2_{L^2(\Omega_n)} \;\le\; \|\omega_n(t,\cdot)\|_{L^2(\Omega_n)} \|\psi_n^0(t,\cdot)\|_{L^2(\Omega_n)}, \quad \forall t, n.$$

Extending ψ_n^0 by zero outside Ω_n, we can see it as an element of $H_0^1(D)$. By applying the Poincaré inequality on D, we end up with

$$\|\psi_n^0(t,\cdot)\|_{H_0^1(D)} \;=\; \|\psi_n^0(t,\cdot)\|_{H_0^1(\Omega_n)} \;\le\; C\,\|\omega_n(t,\cdot)\|_{L^2(\Omega_n)} \;\le\; C'$$

uniformly in t, n, in particular for $t = 0$. Combining this bound on $\psi_n^0(0,\cdot)$ with the estimates on ψ_n^i and $\alpha_n^i(0)$, we obtain that u_n^0 is uniformly bounded in L^2. Then, the conservation of energy implies that

$$\|u_n(t)\|_{L^2(\Omega_n)} \;=\; \|u_n^0\|_{L^2(\Omega_n)} \;\le\; C, \quad \forall t,$$

that is a uniform $L^\infty L^2$ estimate on u_n.

On one hand, this estimate implies the strong convergence of α_n^i (and completes the analysis of the harmonic part). Indeed, we compute

$$(\alpha_n^i)' \;=\; \int_{\Omega_n} \phi_n^i \,\partial_t \omega_n\, dx \;=\; -\int_{\Omega_n} \phi_n^i \,\text{div}\,(u_n \omega_n)\, dx \;=\; \int_{\Omega_n} \nabla \phi_n^i \cdot u_n \omega_n\, dx.$$

Using the uniform L^2 bounds on $\nabla \phi_n^i$ and u_n, we infer that α_n^i is uniformly bounded in $W^{1,\infty}(\mathbb{R}^+)$ which means that the converges holds strongly in $C^0_{\text{loc}}(\mathbb{R}^+)$.

On the other hand, this estimate allows a control of the time derivatives of ψ^0_n. Indeed, we observe that $\partial_t \psi^0_n$ satisfies

$$\Delta\left(\partial_t \psi^0_n\right) \;=\; \partial_t \omega_n \;=\; -\mathrm{div}\;(u_n \omega_n) \;\text{ in }\; \Omega_n, \quad \partial_t \psi^0_n|_{\partial\Omega_n} = 0.$$

Using the uniform $L^\infty L^2$ and L^∞ bounds on u_n and ω_n respectively, we get similarly

$$\|\partial_t \psi^0_n(t,\cdot)\|_{H^1_0(D)} \;\le\; C, \quad \forall t, n.$$

From these bounds and standard compactness lemma [14], there exists $\psi^0 \in W^{1,\infty}(\mathbb{R}^+; H^1_0(D))$ such that up to a subsequence:

$$\psi^0_n \to \psi^0 \;\text{ weakly* in }\; W^{1,\infty}(\mathbb{R}^+; H^1_0(D)) \text{ and strongly in } C^0(0,T; L^2(D)), \quad \forall T > 0.$$

From the weak convergence of ψ^0_n and ω_n, we infer that

$$\Delta\psi^0(t,\cdot) = \omega(t,\cdot) \text{ in } \mathcal{D}'(\Omega), \text{ for almost every } t \tag{3.10}$$

using again that any compact subset of Ω is included in Ω_n for n large enough.

As Ω_n Γ-converges to Ω, we can use Proposition 11: $\psi^0_n(t,\cdot)$ has for every t a subsequence that converges weakly in $H^1_0(D)$ to a limit in $H^1_0(\Omega)$. Thus, for every t, $\psi^0(t,\cdot)$ belongs to $H^1_0(\Omega)$.

Finally, let us prove the strong convergence of ψ^0_n to ψ^0 in $L^2(0,T; H^1_0(D))$ for all $T > 0$. Therefore, we go back to the equation (3.2). We compute:

$$\int_0^T \int_D |\nabla\psi^0_n|^2 = \int_0^T \int_{\Omega_n} |\nabla\psi^0_n|^2 = -\int_0^T \int_{\Omega_n} \omega_n\, \psi^0_n = -\int_0^T \int_D \omega_n\, \psi^0_n \to -\int_0^T \int_D \omega\, \psi^0$$

As we know from the previous paragraph that $\psi^0(t,\cdot)$ belongs to $H^1_0(\Omega)$ for every t, we can perform an energy estimate on (3.10) as well. We get

$$\int_0^T \int_D |\nabla\psi^0|^2 = \int_0^T \int_\Omega |\nabla\psi^0|^2 = -\int_0^T \int_\Omega \omega\, \psi^0 = -\int_0^T \int_D \omega\, \psi^0$$

Hence,

$$\int_0^T \int_D |\nabla\psi^0_n|^2 \;\to\; \int_0^T \int_D |\nabla\psi^0|^2$$

which together with the weak convergence in $W^{1,\infty}(0,T; H^1_0(D))$ yields the strong convergence of ψ^0_n to ψ^0 in $L^2(0,T; H^1_0(D))$ for all $T > 0$.

3.3 Conclusion of the proof

We can now conclude the proof of Theorem 1. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of Euler solutions in Ω_n, associated to the initial data u^0_n. Each field u_n has the Hodge decomposition (3.1). Through obvious extension of ψ^m_n, $m = 0\ldots k$, it can be seen as an element of $L^\infty(\mathbb{R}^+; L^2(D))$. By the results of the previous subsections, it *converges strongly in* $L^2((0,T)\times D)$ *and weakly* in* $L^\infty(\mathbb{R}^+; L^2(D))$, $T > 0$, to the field

$$u(t,x) \;=\; \nabla^\perp\psi^0(t,x) \,+\, \sum_{i=1}^k \alpha^i(t)\nabla^\perp\psi^i(x).$$

Note that ψ^0 belongs to $L^\infty(\mathbb{R}^+; H^1_0(\Omega))$ whereas for $i = 1,\ldots,k$, α^i belongs to $C^0(\mathbb{R}^+)$ and ψ^i belongs to $H^1_0(\widetilde{\Omega})$. Moreover, by construction, one has $\mathrm{curl}\, u \;=\; \Delta\psi^0 \;=\; \omega \in L^\infty(\mathbb{R}^+\times\Omega)$ as well as the divergence-free and tangency conditions, *cf* (1.6).

An important remark is that all the reasoning we have made so far also applies to the initial data (without the difficulties linked to time dependence). In particular, it can be seen that the sequence (u_n^0) converges strongly in $L^2(\Omega)$ (up to a subsequence). Moreover, its limit $\tilde{u}^0$ has a Hodge decomposition,

$$\tilde{u}^0(x) \;=\; \nabla^\perp \psi^{0,0}(x) \;+\; \sum_{i=1}^k \alpha^{0,i} \nabla^\perp \psi^i(x).$$

with $\psi^{0,0} \in H^1_0(\Omega)$, $\Delta \psi^{0,0} = \omega^0$ and $\alpha^{0,i} := \int_\Omega \phi^i \, \omega^0 \, dx \, + \int_{J^i} u^0 \cdot \tau \, ds - \int_{A^i} \omega^0 \, dx - \sum_{j \in E_i} \int_{J^j} u^0 \cdot \tau \, ds$. In particular, it satisfies

$$\mathrm{curl}\, \tilde{u}^0 \;=\; \omega^0, \quad \int_{J^i} \tilde{u}^0 \cdot \tau = \int_{J^i} u^0 \cdot \tau, \quad i = 1 \dots k,$$

as well as the divergence-free and tangency conditions (1.5). It follows that the difference $\tilde{u}^0 - u^0$ is curl-free, divergence free, with zero circulation around each $\mathcal{C}^i$ and a tangency condition. By a slight modification of the argument used in [4, Proposition 2.1], it follows that $\tilde{u}^0 - u^0 = 0$. In particular, u_n^0 converges to u^0 strongly in L^2. As a byproduct, we obtain the existence of a Hodge decomposition for data u^0 satisfying (1.4) in the irregular domain Ω.

Finally, let $\varphi \in \mathcal{D}\left([0, +\infty[\times \Omega\right)$, with div $\varphi = 0$. For n large enough, the support of φ is included in Ω_n so that:

$$\int_0^\infty \int_\Omega \left(u_n \cdot \partial_t \varphi + (u_n \otimes u_n) : \nabla \varphi \right) = \int_\Omega u_n^0 \cdot \varphi(0, \cdot)$$

By the strong L^2 convergence of u_n to u, and of u_n^0 to u^0, it follows that u satisfies the weak form of the Euler equation (1.7).

4 Theorem 2 for $p = \infty$

Similarly to the previous section, we first introduce a sequence of domains $\Omega_n := \mathbb{R}^2 \setminus \overline{O_n}$, see Proposition 5, 2). We introduce a sequence (ω_n^0) such that $\omega_n^0 \in C_c^\infty(\Omega_n) \cap C_c^\infty(\Omega)$ for all n,

$$\omega_0^n \to \omega^0 := \mathrm{curl}\, u^0 \quad \text{ weakly in } L^p(\Omega) \quad \text{with } \; \|\omega_n^0\|_{L^p} \leq \|\omega^0\|_{L^p} \text{ for all } \; p \in [1, \infty],$$

and such that for $\rho_0 > 0$ large enough, $\omega_n^0 = 0$ outside $B(0, \rho_0)$ for all n (see(1.9)-(1.10)).

To build up an appropriate initial velocity u_n^0 in $\Omega_n := \mathbb{R}^2 \setminus \overline{O_n}$ from the vorticity, we need to specify the value of the circulation somewhere. Let J be a smooth closed Jordan curve in Ω such that $\mathcal{C}$ is included in the bounded component of $\mathbb{R}^2 \setminus J$. For any n, we consider as an initial velocity u_n^0 the unique vector field in Ω_n which verifies

$$\mathrm{div}\; u_n^0 = 0, \quad \mathrm{curl}\, u_n^0 = \omega_n^0, \quad u_n^0|_{\partial O_n} \cdot n = 0, \quad \int_J u_n^0 \cdot \tau \, ds = \int_J u^0 \cdot \tau \, ds \;\text{ and }\; \lim_{|x| \to \infty} u_n^0(x) = 0.$$

Note that the quantity $\int_J u^0 \cdot \tau \, ds$ is well-defined because u^0 belongs to $W^{1,q}_{\mathrm{loc}}(\Omega)$ for all finite q. As Ω_n is smooth, u_n^0 generates a unique global strong solution of the Euler equation (see e.g. [5]). The transport equation governing the vorticity implies that the L^p norms are conserved:

$$\|\omega_n(t, \cdot)\|_{L^p(\Omega_n)} = \|\omega_n^0\|_{L^p} \leq \|\omega^0\|_{L^p}, \;\; 1 \leq p \leq +\infty. \tag{4.1}$$

As in the previous part, the Hodge-decomposition will be useful to obtain estimates on the velocity:

$$u_n(t, x) \;=\; \nabla^\perp \psi_n^0(t, x) \;+\; \alpha_n(t) \nabla^\perp \psi_n(x) \tag{4.2}$$

where ψ_n^0 satisfies for any t the Dirichlet problem

$$\Delta\psi_n^0 = \omega_n \text{ in } \Omega_n, \quad \psi_n^0|_{\partial\Omega_n} = 0, \quad \psi_n^0(x) = \mathcal{O}\big(\frac{1}{|x|}\big) \text{ as } x\to\infty, \tag{4.3}$$

whereas ψ_n is the harmonic function satisfying

$$\Delta\psi_n = 0 \text{ in } \Omega_n, \quad \frac{\partial\psi_n}{\partial\tau}|_{\partial\Omega_n} = 0, \quad \int_{\partial O_n}\frac{\partial\psi_n}{\partial n} = -1, \quad \psi_n(x) = \mathcal{O}\big(\ln|x|\big) \text{ as } x\to\infty. \tag{4.4}$$

The function $\alpha_n(t)$ is the sum of the circulation of the velocity around O_n and the mass of the vorticity $\int_{\Omega_n}\omega_n(t,\cdot)$. By the Kelvin's circulation theorem and the transport nature of the vorticity equation, we infer that these two quantities are conserved, hence

$$\alpha_n(t) \equiv \alpha_n = \int_{\partial O_n} u_n^0\cdot\tau\,ds + \int_{\Omega_n}\omega_n^0 = \int_{\partial J} u_n^0\cdot\tau\,ds - \int_{A_n}\omega_n^0 + \int_{\Omega_n}\omega_n^0 \tag{4.5}$$

where $A_n := \Omega_n\setminus C_{ub}(J)$ (see Subsection 2.2), hence

$$\alpha_n \to \alpha := \int_J u^0\cdot\tau\,ds + \int_{C_{ub}(J)}\omega^0. \tag{4.6}$$

4.1 Poincaré inequality in exterior domains

Thanks to the properties in (4.3), we integrate by parts to obtain:

$$\|\nabla\psi_n^0\|_{L^2(\Omega_n)}^2 \le \|\omega_n\|_{L^2(\Omega_n)}\|\psi_n^0\|_{L^2(\Omega_n\cap\operatorname{supp}\omega_n)}. \tag{4.7}$$

In the case of a bounded domain, we used the Poincaré inequality on a domain D containing all the Ω_n's. The idea here is to establish a similar inequality, thanks to the Γ-convergence of $\overline{O_n}$ to $\mathcal{C}$ with cap $\mathcal{C}>0$.

Lemma 1. *Let ρ be a positive such that $\Pi^c\subset B(0,\rho)$. As Ω is of type* (1.8)*, with (H1')-(H2'), then there exists $C_\rho>0$ and N_ρ, depending only on ρ, such that*

$$\|\varphi\|_{L^2(\Omega_n\cap B(0,\rho))} \le C_\rho\|\nabla\varphi\|_{L^2(\Omega_n\cap B(0,\rho))},\ \forall\varphi\in C_c^\infty(\Omega_n),\ \forall n\ge N_\rho.$$

Proof. Let us assume that the conclusion is false, which means that for any $k\in\mathbb{N}$, if we choose $C_\rho=k$ and $N_\rho = max(k,n_{k-1})$, then there exist $n_k\ge N_\rho$ and $\varphi_k\in C_c^\infty(\Omega_{n_k})$ such that

$$\|\varphi_k\|_{L^2(\Omega_{n_k}\cap B(0,\rho))} > k\|\nabla\varphi_k\|_{L^2(\Omega_{n_k}\cap B(0,\rho))}.$$

Dividing φ_k by $\|\varphi_k\|_{L^2(\Omega_{n_k}\cap B(0,\rho))}$, we can consider that $\|\varphi_k\|_{L^2(\Omega_{n_k}\cap B(0,\rho))}=1$, which implies that $\|\nabla\varphi_k\|_{L^2(\Omega_{n_k}\cap B(0,\rho))}$ tends to zero as k tends to infinity. Therefore, extracting a subsequence if necessary, we have that

$$\varphi_k \rightharpoonup \varphi \text{ weakly in } H^1(B(0,\rho)) \text{ and } \varphi_k\to\varphi \text{ strongly in } L^2(B(0,\rho)).$$

It follows that φ is a non zero contant, because $\|\varphi\|_{L^2(B(0,\rho))}=1$ and $\nabla\varphi_k\rightharpoonup 0$ weakly in $L^2(B(0,\rho))$.

We introduce a cutoff function χ which is equal to zero in $B(0,\rho)^c$, and equal to 1 in some neighborhoods of O_n's and $\mathcal{C}$. Then,

$$\chi\varphi_k \text{ belongs to } H_0^1(B(0,\rho)\setminus\overline{O_n}), \tag{4.8}$$

and

$$\chi\varphi_k \rightharpoonup \chi\varphi \text{ weakly in } H_0^1(B(0,\rho)). \tag{4.9}$$

However, the sequence $(B(0,\rho)\setminus\overline{O_n})$ converges to $B(0,\rho)\setminus\mathcal{C}$ in the Hausdorff sense and as $\overline{O_n}$ is connected for all n, Proposition 10 implies:

$$B(0,\rho)\setminus\overline{O_n}\ \Gamma\text{-converges to } B(0,\rho)\setminus\mathcal{C}.$$

Combining (4.8), (4.9) and Proposition 11, we obtain that $\chi\varphi$ belongs to $H^1_0(B(0,\rho)\setminus\mathcal{C})$.

Next, Proposition 8 implies that $\chi\varphi=0$ quasi everywhere in $\mathcal{C}$. This is in contradiction with $\mathrm{cap}(\mathcal{C})>0$ and the fact that $\chi\varphi$ is equal to a non zero constant in $\mathcal{C}$. The conclusion of the proof follows. ☐

We want to apply the previous lemma to (4.7), but we remark that an important issue is to control the size of the support of ω_n independently of n. As ω_n is transported by u_n, we will prove that the velocity is uniformly bounded far from the domains O_n.

4.2 Uniform estimates of the velocity far from the boundaries.

The advantage of working outside one simply connected domain is the explicit formula of ψ^0_n and ψ_n in terms of biholomorphisms. We shall use the notations and results of Subsection 2.5. With such notations, we have

$$\nabla^\perp\psi^0_n(t,x)=\frac{1}{2\pi}D\mathcal{T}_n^T(x)\int_{\Omega_n}\Big(\frac{\mathcal{T}_n(x)-\mathcal{T}_n(y)}{|\mathcal{T}_n(x)-\mathcal{T}_n(y)|^2}-\frac{\mathcal{T}_n(x)-\mathcal{T}_n(y)^*}{|\mathcal{T}_n(x)-\mathcal{T}_n(y)^*|^2}\Big)^\perp\omega_n(t,y)\,dy \tag{4.10}$$

and

$$\nabla^\perp\psi_n(t,x)=\frac{1}{2\pi}D\mathcal{T}_n^T(x)\frac{\mathcal{T}_n(x)^\perp}{|\mathcal{T}_n|^2} \tag{4.11}$$

with the notation $z^*=\frac{z}{|z|^2}$ (see e.g. [4, 7] for an introduction to the Biot-Savart law in exterior domains).

Lemma 2. *Let R_0 large enough so that $\Pi^c\subset B(0,R_0)$. Then, there exists $C_0=C(\|\omega^0\|_{L^1},\|\omega^0\|_{L^\infty},R_0)$ such that*

$$f_n(t,x):=\frac{1}{2\pi}D\mathcal{T}^T(x)\int_{\Pi}\Big(\frac{\mathcal{T}(x)-\mathcal{T}(y)}{|\mathcal{T}(x)-\mathcal{T}(y)|^2}-\frac{\mathcal{T}(x)-\mathcal{T}(y)^*}{|\mathcal{T}(x)-\mathcal{T}(y)^*|^2}\Big)^\perp\omega_n(t,y)\,dy \tag{4.12}$$

verifies

$$\|f_n(t,x)\|_{L^\infty(\mathbb{R}^+\times B(0,R_0)^c)}\le C_0,\ \forall n.$$

Moreover, for any compact K outside $\overline{B(0,R_0)}$, there exists N_K such that

$$\Big\|\nabla^\perp\psi^0_n(t,x)\Big\|_{L^\infty(\mathbb{R}^+\times K)}\le 2C_0+1,\ \forall n\ge N_K.$$

Proof. Let $\tilde R_0<R_0$ such that $\Pi^c\subset B(0,\tilde R_0)$. We decompose the integral (4.12) into three parts:

$$\begin{aligned}
f_n(t,x)&=\frac{1}{2\pi}D\mathcal{T}^T(x)\Big[\int_{\Pi\cap B(0,\tilde R_0)}\frac{(\mathcal{T}(x)-\mathcal{T}(y))^\perp}{|\mathcal{T}(x)-\mathcal{T}(y)|^2}\omega_n(t,y)\,dy\\
&\quad+\int_{B(0,\tilde R_0)^c}\frac{(\mathcal{T}(x)-\mathcal{T}(y))^\perp}{|\mathcal{T}(x)-\mathcal{T}(y)|^2}\omega_n(t,y)\,dy-\int_{\Pi}\frac{(\mathcal{T}(x)-\mathcal{T}(y)^*)^\perp}{|\mathcal{T}(x)-\mathcal{T}(y)^*|^2}\omega_n(t,y)\,dy\Big]\\
&=\frac{1}{2\pi}D\mathcal{T}^T(x)(\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3).
\end{aligned}$$

By the definition of $\mathcal{T}$ (see Proposition 13), there exists some $(\beta,\tilde\beta)\in\mathbb{R}^+_*\times\mathbb{C}$ such that:

$$\mathcal{T}(z)=\beta z+\tilde\beta+\mathcal{O}(\frac{1}{z})\text{ as } z\to\infty.$$

Then there exists C_1 such that $\|D\mathcal{T}\|_{L^\infty(B(0,R_0)^c)} \leq C_1$. If the boundary $\partial\Pi$ is rough, we recall that such an inequality does not hold in $L^\infty(\Pi)$ (see for instance [10]). This remark underlines the importance of R_0.

As $\mathcal{T}$ is continuous and one-to-one, there exists $\delta > 0$ such that

$$\text{dist}\Big(\mathcal{T}(\partial B(0,\tilde{R}_0)); \mathcal{T}(\partial B(0,R_0))\Big) \geq \delta.$$

Then $|\mathcal{T}(x) - \mathcal{T}(y)| \geq \delta$ for any $x \in B(0,R_0)^c$ and $y \in \Pi \cap B(0,\tilde{R}_0)$. Hence, for any $x \in B(0,R_0)^c$, we have

$$|\mathcal{I}_1| \leq \frac{1}{\delta}\int_{\Pi\cap B(0,\tilde{R}_0)} |\omega_n(t,y)|\,dy \leq \frac{\|\omega^0\|_{L^1}}{\delta},$$

where we have used (4.1).

As $|\mathcal{T}(y)^*| \leq 1 \leq |\mathcal{T}(y)|$, we also have $|\mathcal{T}(x) - \mathcal{T}(y)^*| \geq \delta$ for any $x \in B(0,R_0)^c$. Therefore, we obtain

$$|\mathcal{I}_3| \leq \frac{1}{\delta}\int_{\Pi} |\omega_n(t,y)|\,dy \leq \frac{\|\omega^0\|_{L^1}}{\delta},$$

Concerning the last part $\mathcal{I}_2$, we introduce $z = \mathcal{T}(x)$ and

$$g(t,\eta) := \omega_n(t,\mathcal{T}^{-1}(\eta))|\det D\mathcal{T}^{-1}(\eta)|\mathbf{1}_{\mathcal{T}(B(0,\tilde{R}_0)^c)}(\eta).$$

Changing variables $\eta = \mathcal{T}(y)$, we compute

$$\mathcal{I}_2 = \int_{\mathbb{R}^2} \frac{(z-\eta)^\perp}{|z-\eta|^2} g(t,\eta)\,d\eta.$$

Changing variables back, we obtain that

$$\|g(t,\cdot)\|_{L^1(\mathbb{R}^2)} = \|\omega_n(t,\cdot)\|_{L^1(B(0,\tilde{R}_0)^c)} \leq \|\omega^0\|_{L^1}.$$

Using the behavior at infinity of $\mathcal{T}^{-1}$, we infer that there exists C_2 such that

$$|\det D\mathcal{T}^{-1}(\eta)| \leq C_2,\ \forall \eta \in \mathcal{T}(B(0,\tilde{R}_0)^c),$$

hence

$$\|g(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \leq C_2\|\omega_n(t,\cdot)\|_{L^\infty} \leq C_2\|\omega^0\|_{L^\infty}.$$

This last argument explains why we split the integral into several parts: we cannot prove that $\det D\mathcal{T}^{-1}$ is bounded up to the boundary, in particular when its boundary is rough. Using a classical estimate for the Biot-Savart kernel in $\mathbb{R}^2$, we write

$$|\mathcal{I}_2| \leq C_3\|g(t,\cdot)\|_{L^1(\mathbb{R}^2)}^{1/2}\|g(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}^{1/2} \leq C_3\sqrt{C_2}\|\omega^0\|_{L^1}^{1/2}\|\omega^0\|_{L^\infty}^{1/2},$$

where C_3 is a universal constant.

Putting $C_0 := \frac{C_1}{2\pi}\Big(\frac{2\|\omega^0\|_{L^1}}{\delta} + C_3\sqrt{C_2}\|\omega^0\|_{L^1}^{1/2}\|\omega^0\|_{L^\infty}^{1/2}\Big)$, we have established the first inequality:

$$\|f_n(t,x)\|_{L^\infty(\mathbb{R}^+\times B(0,R_0)^c)} \leq C_0,\ \forall n.$$

We treat now $\nabla^\perp\psi_n^0$. Let K be a compact set in $B(0,R_0)^c$. Let $\tilde{K}$ a compact set satisfying

$$K \subset \tilde{K} \subset B(0,\tilde{R}_0)^c, \quad \text{and } \text{dist}\Big(\mathcal{T}(\partial K); \mathcal{T}(\partial\tilde{K})\Big) \geq \delta.$$

One can take for instance $\tilde{K} = \{\tilde{R}_0 \leq |z| \leq R_1\}$ for R_1 large enough. Again, we decompose the integral (4.10) into three parts:

$$\begin{aligned}\nabla^\perp \psi_n^0(t,x) &= \frac{1}{2\pi} D\mathcal{T}_n^T(x)\Big[\int_{\Omega_n\setminus\tilde{K}} \frac{(\mathcal{T}_n(x)-\mathcal{T}_n(y))^\perp}{|\mathcal{T}_n(x)-\mathcal{T}_n(y)|^2}\omega_n(t,y)\,dy \\ &\quad + \int_{\tilde{K}} \frac{(\mathcal{T}_n(x)-\mathcal{T}_n(y))^\perp}{|\mathcal{T}_n(x)-\mathcal{T}_n(y)|^2}\omega_n(t,y)\,dy - \int_{\Omega_n} \frac{(\mathcal{T}_n(x)-\mathcal{T}_n(y)^*)^\perp}{|\mathcal{T}_n(x)-\mathcal{T}_n(y)^*|^2}\omega_n(t,y)\,dy\Big] \\ &= \frac{1}{2\pi} D\mathcal{T}_n^T(x)(\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3).\end{aligned}$$

By the uniform convergence of $D\mathcal{T}_n$ to $D\mathcal{T}$ in K (see Proposition 13), for any $\varepsilon_1 > 0$ there exists N_1 such that

$$\|D\mathcal{T}_n\|_{L^\infty(K)} \leq C_1 + \varepsilon_1,\ \forall n \geq N_1.$$

By the uniform convergence of $\mathcal{T}_n$ to $\mathcal{T}$ in $\tilde{K}$, there exists $N_2 > 0$ such that

$$\mathrm{dist}\Big(\mathcal{T}_n(\partial\tilde{K}); \mathcal{T}_n(\partial K)\Big) \geq \delta/2,\ \forall n \geq N_2.$$

Then $|\mathcal{T}_n(x) - \mathcal{T}_n(y)| \geq \delta/2$ for any $x \in K$ and $y \in \Omega_n \setminus \tilde{K}$. Hence, for any $x \in K$, we have

$$|\mathcal{J}_1| \leq \frac{2}{\delta}\int_{\Omega_n\setminus\tilde{K}} |\omega_n(t,y)|\,dy \leq \frac{2\|\omega^0\|_{L^1}}{\delta},\ \forall n \geq N_2.$$

As $|\mathcal{T}_n(y)^*| \leq 1 \leq |\mathcal{T}_n(y)|$, we also have $|\mathcal{T}_n(x) - \mathcal{T}_n(y)^*| \geq \delta/2$ for any $x \in K$. Therefore, we obtain

$$|\mathcal{J}_3| \leq \frac{2}{\delta}\int_{\Omega_n} |\omega_n(t,y)|\,dy \ \leq\ \frac{2\|\omega^0\|_{L^1}}{\delta},$$

Concerning the last part $\mathcal{J}_2$, we introduce $z = \mathcal{T}_n(x)$ and

$$g_n(t,\eta) := \omega_n(t, \mathcal{T}_n^{-1}(\eta))|\det D\mathcal{T}_n^{-1}(\eta)|\mathbf{1}_{\mathcal{T}_n(\tilde{K})}(\eta).$$

Changing variables $\eta = \mathcal{T}_n(y)$, we compute

$$\mathcal{J}_2 = \int_{\mathbb{R}^2} \frac{(z-\eta)^\perp}{|z-\eta|^2} g_n(t,\eta)\,d\eta.$$

Changing variables back, we obtain that

$$\|g_n(t,\cdot)\|_{L^1(\mathbb{R}^2)} = \|\omega_n(t,\cdot)\|_{L^1(\tilde{K})} \leq \|\omega^0\|_{L^1}.$$

Using the uniform convergence of $D\mathcal{T}_n^{-1}$ to $D\mathcal{T}^{-1}$ in a compact big enough (such that $\mathcal{T}_n(\tilde{K}) \subset D$), for any $\varepsilon_3 > 0$ there exists N_3 such that

$$|\det D\mathcal{T}_n^{-1}(\eta)| \leq C_2 + \varepsilon_3,\ \forall \eta \in \mathcal{T}_n(\tilde{K}),\ \forall n \geq N_3$$

hence

$$\|g_n(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \leq (C_2+\varepsilon_3)\|\omega_n(t,\cdot)\|_{L^\infty} \ \leq\ (C_2+\varepsilon_3)\|\omega^0\|_{L^\infty}.$$

Finnaly, we use the classical estimate for the Biot-Savart kernel in $\mathbb{R}^2$:

$$|\mathcal{J}_2| \leq C_3\|g_n(t,\cdot)\|_{L^1(\mathbb{R}^2)}^{1/2}\|g_n(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}^{1/2} \leq C_3\sqrt{C_2+\varepsilon_3}\|\omega^0\|_{L^1}^{1/2}\|\omega^0\|_{L^\infty}^{1/2}.$$

Choosing well ε_1 and ε_3, we find $N_K = \max(N_1, N_2, N_3)$ such that

$$\Big\|\nabla^\perp\psi_n^0(t,x)\Big\|_{L^\infty(\mathbb{R}^+\times K)} \leq 2C_0 + 1,\ \forall n \geq N_K,$$

which ends the proof. □

The reason why we divide the proof in two parts is to obtain a constant C_0 independent of the compact set K. Although N_K depends on K, the independence of C_0 with respect to K will be crucial for the uniform estimate of the vorticity support. The harmonic part is easier to estimate.

Lemma 3. *Let R_0 a positive number such that $\Pi^c \subset B(0,R_0)$. Then, there exists $C_0 = C(R_0)$ such that*

$$\psi(x) := \frac{1}{2\pi}\ln|\mathcal{T}(x)| \tag{4.13}$$

verifies

$$\left\|\nabla^\perp\psi(x)\right\|_{L^\infty(B(0,R_0)^c)} \leq C_0.$$

Moreover, for any compact K outside $\overline{B(0,R_0)}$, there exists N_K such that

$$\left\|\nabla^\perp\psi_n(x)\right\|_{L^\infty(K)} \leq 2C_0+1,\ \forall n \geq N_K.$$

Proof. The first part comes from the behavior of $\mathcal{T}$ at infinity:

$$\mathcal{T}(z) = \beta z + \tilde{\beta} + \mathcal{O}(\frac{1}{z}) \text{ as } z \to \infty.$$

The second point is a direct consequence of the uniform convergence of $\mathcal{T}_n$ in K (see Proposition 13). □

4.3 Support of the vorticity and H^1 estimates

Let ρ_0 such that $\cup_n \operatorname{supp}\omega_n^0 \cup \operatorname{supp}\omega^0 \cup \Pi^c \subset B(0,\rho_0)$.

$$C_0 = C_0(\|\omega^0\|_{L^1}, \|\omega^0\|_{L^\infty}, \rho_0)$$

the constant of Lemmata 2 and 3. Let $C := (2C_0+1)(2+|\alpha|)$, where α was defined in (4.6). We fix a time $T>0$ and we introduce

$$K_T := \overline{B(0,\rho_0+CT)} \setminus B(0,\rho_0).$$

Together with (4.2)-(4.6), Lemmata 2 and 3 provide some N_T such that

$$\|u_n\|_{L^\infty(\mathbb{R}^+\times K_T)} \leq C,\ \forall n \geq N_T.$$

As ω_n is transported by u_n, we can conclude that

$$\operatorname{supp}\omega_n(t,\cdot) \subset B(0,\rho_0+Ct),\ \forall t\in[0,T],\ \forall n \geq N_T.$$

Finally we can complete the estimate of $\|\nabla\psi_n^0\|_{L^2(\Omega_n)}$. Let $\rho_T := \rho_0 + CT$, then Lemma 1 implies that there exist C_{ρ_T} and N_{ρ_T} such that

$$\|\psi_n^0\|_{L^2(\Omega_n\cap\operatorname{supp}\omega_n)} \leq C_{\rho_T}\|\nabla\psi_n^0\|_{L^2(\Omega_n\cap B(0,\rho_T))},\ \forall t\in[0,T],\ \forall n\geq\max(N_{\rho_T},N_T).$$

Combining with (4.7) and (4.1), we obtain: $\forall t\in[0,T],\ \forall n\geq\max(N_{\rho_T},N_T)$:

$$\|\nabla\psi_n^0\|_{L^2(\Omega_n)} \leq C_{\rho_T}\|\omega_n\|_{L^2(\Omega_n)} = C_{\rho_T}\|\omega^0\|_{L^2}. \tag{4.14}$$

Using again Lemma 1 on any compact K of $\mathbb{R}^2$, we conclude that there exist C_K and N_K such that

$$\|\psi_n^0\|_{L^2(\Omega_n\cap K)} \leq C_K\|\nabla\psi_n^0\|_{L^2(\Omega_n)} \leq C_K C_{\rho_T}\|\omega^0\|_{L^2},\ \forall t\in[0,T],\ \forall n\geq\max(N_{\rho_T},N_T,N_K), \tag{4.15}$$

where C_K depends on the diameter of K. We recall that ψ_n^0 is not square integrable at infinity (see (4.3)), but (4.15) will be sufficient to obtain local convergence.

We end this subsection with a L^2_{loc} estimate of $\nabla^\perp \psi_n$ up to the boundary. Let $R_0 > 0$ and χ be a cutoff function equal to 1 in $B(0,R_0)$ and to 0 outside $B(0,R_0+1)$. Then, $\chi\psi_n$ verifies

$$\Delta(\chi\psi_n) \;=\; \tilde{\omega}_n := 2\nabla\chi\cdot\nabla\psi_n + \psi_n\Delta\chi \;\text{ in }\; \Omega_n\cap B(0,R_0+1), \quad \chi\psi_n \;=\; 0 \;\text{ on }\; \partial\Omega_n \cup \partial B(0,R_0+1).$$

Note that the connectedness of $\partial\Omega_n$ allows to impose a Dirichlet condition on ψ_n. This Dirichlet condition can also be read on the formula $\psi_n = \frac{1}{2\pi}\ln|\mathcal{T}_n(x)|$, as $\mathcal{T}_n$ maps $\partial\Omega_n$ to $\partial B(0,1)$. Therefore, by a classical energy estimate and Poincaré inequality applied in $B(0,R_0+1)$, we obtain that:

$$\|\nabla(\chi\psi_n)\|^2_{L^2(\Omega_n)} \le \|\tilde{\omega}_n\|_{L^2(\Omega_n)}\|\chi\psi_n\|_{L^2(\Omega_n\cap B(0,R_0+1))} \le C_{R_0}\|\tilde{\omega}_n\|_{L^2(\Omega_n)}\|\nabla(\chi\psi_n)\|_{L^2(\Omega_n)}.$$

Using that ψ_n and $\nabla\psi_n$ converge uniformly to ψ and $\nabla\psi$ in $B(0,R_0+1)\setminus B(0,R_0)$ (see Proposition 13), we get that $\|\tilde{\omega}_n\|_{L^2(\Omega_n)}$ is uniformly bounded. This yields the existence of a constant C, depending only on R_0, such that

$$\|\nabla\psi_n\|_{L^2(\Omega_n\cap B(0,R_0))} \le C, \;\forall n. \tag{4.16}$$

4.4 Conclusion of the proof

The proof follows the one for bounded domains, taking care of integrability at infinity. We fix $T > 0$ and a compact K in Ω. We denote

$$D := K \cup B(0,\rho_T),$$

with ρ_T defined in the previous subsection.

a) Compactness of the rotational part.
We deduce from (4.14) and (4.15) that

$$\|\psi_n^0(t,\cdot)\|_{H^1(D)} \;\le\; C_{T,K} \quad \forall t\in[0,T],\; \forall n \ge N_K.$$

As regards time derivatives, we observe that $\partial_t\psi_n^0$ satisfies

$$\Delta\left(\partial_t\psi_n^0\right) \;=\; \partial_t\omega_n \;=\; -\text{div}\;(u_n\omega_n) \;\text{ in }\; \Omega_n, \quad \partial_t\psi_n^0|_{\partial\Omega_n} = 0.$$

Using the uniform $L^\infty([0,T],L^2(B(0,\rho_T)))$ bound of u_n (see (4.14) and (4.16)) and the L^∞ bounds on ω_n, we get

$$\|\partial_t\psi_n^0(t,\cdot)\|_{H^1(D)} \;\le\; C, \quad \forall t\in[0,T],\; \forall n\ge N_K.$$

From these bounds and standard compactness lemma, there exists $\psi^0 \in W^{1,\infty}(0,T;H^1(D))$ such that up to a subsequence:

$$\psi_n^0 \to \psi^0 \;\text{ weakly* in }\; W^{1,\infty}(0,T;H^1(D)) \text{ and strongly in } C^0(0,T;L^2(D)).$$

We now extend ω_n by 0 outside Ω_n for all n, so that the sequence $(\omega_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^+;L^1\cap L^\infty(\mathbb{R}^2))$. Up to another extraction, we deduce that

$$\omega_n \to \omega \;\text{ weakly * in }\; L^\infty(\mathbb{R}^+;L^1\cap L^\infty(\mathbb{R}^2)). \tag{4.17}$$

From the weak convergence of ψ_n^0 and ω_n, we infer that

$$\Delta\psi^0(t,\cdot) = \omega(t,\cdot) \text{ in } \mathcal{D}'(\Omega\cap D), \text{ for almost every } t \tag{4.18}$$

using again that any compact subset of Ω is included in Ω_n for n large enough.

Now, we use Proposition 10: as $(\Omega_n\cap B(0,\rho_T))_{n\in\mathbb{N}}$ converges to $\Omega\cap B(0,\rho_T)$ in the Hausdorff sense and as the complement in a large closed ball B of $\Omega_n\cap B(0,\rho_T)$ has at most 2 connected components for all n, $(\Omega_n\cap B(0,\rho_T))_{n\in\mathbb{N}}$ Γ-converges to $\Omega\cap B(0,\rho_T)$. Let χ be a cutoff function equal to 1 in a neighborhood of the O_n's, and to 0 outside $B(0,\rho_T)$. By Proposition 11, $\chi\psi_n^0(t,\cdot)$ has for every t a subsequence that converges weakly in $H_0^1(B(0,\rho_T))$ to a limit in $H_0^1(\Omega\cap B(0,\rho_T))$. Thus, for every $t\in[0,T]$, $\chi\psi^0(t,\cdot)$ belongs to $H_0^1(\Omega\cap B(0,\rho_T))$.

Finally, let us prove the strong convergence of ψ_n^0 to ψ^0 in $L^2(0,T;H^1(D))$ for all $T>0$. Therefore, we go back to the equation (4.3). As $|\nabla\psi_n^0||\psi_n^0| = \mathcal{O}(1/|x|^3)$, we can integrate by part:

$$\int_0^T\int_{\mathbb{R}^2}|\nabla\psi_n^0|^2 = \int_0^T\int_{\Omega_n}|\nabla\psi_n^0|^2 = -\int_0^T\int_{\Omega_n}\omega_n\,\psi_n^0 = -\int_0^T\int_D\omega_n\,\psi_n^0 \to -\int_0^T\int_D\omega\,\psi^0$$

As we know from the previous paragraph that $\chi\psi^0(t,\cdot)$ belongs to $H_0^1(\Omega\cap B(0,\rho_T))$ for every t, we can perform an energy estimate on (4.18) as well. We get

$$\int_0^T\int_{\mathbb{R}^2}|\nabla\psi^0|^2 = \int_0^T\int_{\Omega}|\nabla\psi^0|^2 = -\int_0^T\int_{\Omega}\omega\,\psi^0 = -\int_0^T\int_D\omega\,\psi^0$$

Hence,

$$\int_0^T\int_{\mathbb{R}^2}|\nabla\psi_n^0|^2 \to \int_0^T\int_{\mathbb{R}^2}|\nabla\psi^0|^2$$

which together with the weak convergence in $W^{1,\infty}(0,T;H^1(D))$ yields the strong convergence of ψ_n^0 to ψ^0 in $L^2(0,T;H^1(K))$.

b) Compactness of the harmonic part.

By the convergence results on $(\mathcal{T}_n)$ (see Proposition 13), we obtain directly that $\psi_n = \frac{1}{2\pi}\ln|\mathcal{T}_n(x)|$ converges uniformly to ψ, resp. to 0, in any compact subset K of Π (the unbounded connected component of Ω), resp. of $\Omega\setminus\Pi$ (the bounded connected components of Ω). As $\psi_n-\psi$ is harmonic, local uniform convergence implies $H^1_{\rm loc}$ convergence by the mean-value theorem.

c) Limit equation.

We can now conclude the proof of Theorem 2. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of Euler solutions in Ω_n, associated to the initial data u_0^n. Each field u_n has the Hodge decomposition (4.2). By diagonal extraction, it *converges strongly in* $L^2_{\rm loc}(\mathbb{R}^+\times\overline{\Omega})$ *and weakly* in* $L^\infty_{\rm loc}(\mathbb{R}^+;L^2_{\rm loc}(\overline{\Omega}))$ to the field

$$u(t,x) \;=\; \begin{cases}\nabla^\perp\psi^0(t,x) \;+\; \alpha\nabla^\perp\psi(x), & \text{if } x\in\Pi,\\ \nabla^\perp\psi^0(t,x), & \text{if } x\in\Omega\setminus\Pi.\end{cases}$$

Note that $\nabla^\perp\psi^0$ belongs to $L^\infty_{\rm loc}(\mathbb{R}^+;L^2(\Omega))$ (see (4.14)) whereas $\nabla^\perp\psi$ is only locally square integrable. It follows that $u\in L^\infty_{\rm loc}(\mathbb{R}^+;L^2_{\rm loc}(\overline{\Omega}))$. From this explicit form, we deduce that u is divergence free, tangent to the boundary, with a conserved circulation along the closed curve J. Moreover, inside Ω, one has

$$\operatorname{curl} u \;=\; \Delta\psi^0 \;=\; \omega\in L^\infty(\mathbb{R}^+;L^1\cap L^\infty(\Omega)).$$

The uniform estimate of the support of ω_n means that ω is also compactly supported.

Finally, let $\varphi\in\mathcal{D}\left([0,+\infty[,\mathcal{V}(\Omega)\right)$. For n large enough, the support of φ is included in Ω_n so that:

$$\int_0^\infty\int_\Omega\left(u_n\cdot\partial_t\varphi + (u_n\otimes u_n):\nabla\varphi\right) = \int_\Omega u_n^0\cdot\varphi(0,\cdot)$$

By the strong $L^2_{\rm loc}$ convergence of u_n to u, and also the strong $L^2_{\rm loc}$ convergence of u_n^0 to u^0 (see the previous section for details), it follows that u satisfies the weak form of the Euler equation (1.7).

Let us emphasize that this convergence in $L^2_{\rm loc}$ does not hold for the situation studied in [4] (one small obstacle shrinking to a point), or in [11] (bounded domain with several holes, one of them shrinking to a point). In such situations, the limit velocity is the sum of a smooth part and a harmonic part $x^\perp/|x|^2$, so it does not even belong to $L^2_{\rm loc}$.

5 Initial vorticity in L^p

We complete in this section the proof of Theorems 1 and 2, by passing from L^∞ vorticity to L^p vorticity.

5.1 Theorem 1 for $p > 1$

Let $p > 1$, u^0 satisfying (1.4). Let $\omega^0 := \operatorname{curl} u^0$. We introduce a sequence of smooth functions $(\omega_n^0)_{n\in\mathbb{N}}$ such that $\omega_n^0 \to \omega^0$ strongly in $L^p(\Omega)$. We remind that we have established in Section 4 that there is for each n and each real k-uplet $c^1, \dots, c^k$ a unique $u_n^0 \in L^2(\Omega)$ satisfying

$$\operatorname{curl} u_n^0 = \omega_n^0, \quad \int_{J^i} u_n^0 \cdot \tau = c^i, \quad \forall i = 1, \dots, k$$

together with the divergence-free and tangency conditions. We choose here $c^i := \int_{J^i} u^0 \cdot \tau$, which is well-defined, as u_0 belongs to $W^{1,p}_{\text{loc}}(\Omega)$. We then denote by u_n a weak solution constructed in Section 4. We denote $\omega_n := \operatorname{curl} u_n$.

Approaching u_n as in Section 4 by a sequence of smooth solutions $u_{n,N}$ of Euler in Ω_N, we notice by (3.7) that:

$$\|\omega_n\|_{L^\infty(L^p(\Omega))} \le \liminf_{N\to\infty} \|\omega_{n,N}\|_{L^\infty(L^p(\Omega_N))} \le \|\omega_n^0\|_{L^p(\Omega)} \le C_p. \tag{5.1}$$

Then we have, up to a subsequence, the weak $*$ convergence of ω_n to some ω in $L^\infty(\mathbb{R}^+; L^p(\Omega))$.

Moreover, we proved that the velocity can be written as

$$u_n(t,x) \;=\; \nabla^\perp \psi_n^0(t,x) \;+\; \sum_{i=1}^k \alpha_n^i(t) \nabla^\perp \psi^i(x)$$

where

$$\psi_n^0 \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)) \text{ and } \Delta\psi_n^0(t,\cdot) = \omega_n(t,\cdot) \text{ in } \mathcal{D}'(\Omega), \text{ for almost every } t;$$

$$\psi^i \;=\; \sum_{j=1}^{j} c^{i,j}\, \phi^i \text{ for all } i = 1, \dots, k;$$

$$\alpha_n^i \;=\; \int_\Omega \phi^i\, \omega_n\, dx \;+ \int_{J^i} u^0 \cdot \tau\, ds - \int_{A^i} \omega_n^0\, dx - \sum_{j\in E_i} \int_{J^j} u^0 \cdot \tau\, ds, \tag{5.2}$$

with

$$\phi^i \in H_0^1(\tilde\Omega),\ \Delta\phi^i = 0 \quad \text{in } \Omega, \quad \phi^i|_{\partial\mathcal{C}^j} = \delta_{ij} \text{ in a weak sense, see Section 4,}$$

and

$$C = \left(c^{i,j}\right)_{1\le i,j\le N} = -\left(\int_\Omega \nabla\phi^i \cdot \nabla\phi^j\right)^{-1}_{1\le i,j\le N}.$$

Note that ϕ^i, ψ^i and C do not depend on k.

By the energy estimate, we obtain that $\|\nabla\psi_n^0(t,\cdot)\|^2_{L^2(\Omega)} \le \|\omega_n(t,\cdot)\|_{H^{-1}(\Omega)} \|\psi_n^0(t,\cdot)\|_{H^1(\Omega)}$, which implies by (5.1) and the Poincaré inequality on a big ball D that ψ_n^0 is bounded in $L^\infty(\mathbb{R}^+; H_0^1(\Omega))$. Also by (5.1), the sequences $(\alpha_n^i)_{n\in\mathbb{N}}$ are bounded in $L^\infty(\mathbb{R}^+)$.

Therefore, we can write $u_n = \nabla^\perp \psi_n$ with ψ_n bounded in $L^\infty(\mathbb{R}^+; H^1(\Omega))$. By this estimate, we extract a subsequence such that $u_n \to u$ weakly* in $L^\infty(\mathbb{R}^+; L^2(\Omega))$, which implies that u verifies the divergence-free and tangency conditions (1.6).

The last step consists in obtaining strong compactness of (u_n) in $C^0\left([0,T]; L^2_{\rm loc}(\Omega)\right)$, as it allows to pass to the limit in the momentum equation (1.7). Thanks to (5.1), the sequences $(\mathrm{curl}\,(\chi u_n))$ and $(\mathrm{div}\,(\chi u_n))$ are bounded in $L^\infty\left(\mathbb{R}^+, L^q(\mathbb{R}^2)\right)$, $q := \min(2,p)$, for any $\chi \in C^\infty_c(\Omega)$. It follows that χu_n is bounded in $L^\infty\left(\mathbb{R}^+, W^{1,q}(\mathbb{R}^2)\right)$. In other words (u_n) is bounded in $L^\infty\left(\mathbb{R}^+; W^{1,q}(\Omega')\right)$ for any $\Omega' \Subset \Omega$. Moreover, the time derivative

$$\partial_t u_n = -\mathrm{div}\,(u_n \otimes u_n) - \nabla p_n$$

is bounded uniformly in the space $H^{-2}_\sigma(\Omega')$, that is the dual space of

$$H^2_\sigma(\Omega') := \{\varphi = \lim \varphi_n \ \text{ in } \ H^2(\Omega'), \quad \varphi_n \in C^\infty_c(\Omega'), \quad \mathrm{div}\,\varphi_n = 0\}$$

From Aubin-Lions compactness lemma, see [14], (u_n) is strongly compact in $C^0([0,T]; L^2(\Omega'))$, for all $t > 0$, all $\Omega' \Subset \Omega$. This ends the proof.

5.2 Theorem 2 for $p > 2$

To go from $p = \infty$ to $p > 2$, one follows the lines of the bounded case. Let us remark that for solutions in Theorem 2, we have that $\alpha_n(t) = \alpha_n = \int_{C_{ub}(J)} \omega^0_n + \int_J u^0 \cdot \tau$ which tends easily to $\alpha := \int_{C_{ub}(J)} \omega^0 + \int_J u^0 \cdot \tau$.

Then, it is sufficient to prove the convergence of the rotational part. In order to obtain an estimate in L^2 of $\nabla \psi^0_n$ independent of n (see (4.14)), we need to control uniformly the size of the support of ω_n. Therefore, we want to prove Lemma 2 with $\|\omega_n\|_{L^p}$ instead of $\|\omega_n\|_{L^\infty}$. Such an extension is possible for $p > 2$ (see e.g. the technics used in [7, Lemma 3.5]). More precisely, we assume $p > 2$ in the unbounded domain in order to obtain an L^∞ estimate of the velocity far from the boundaries, and a uniform control of the size of the support of ω_n.

We conclude by the same compactness argument as above.

6 Final remarks

6.1 Domain continuity for Euler

Theorems 1 and 2 yield existence of global weak solutions in singular domains. However, their proof yields more, namely some domain continuity for the Euler equation. It shows that solutions of Euler in

$$\Omega_n := \widetilde{\Omega}_n \setminus \left(\cup_{i=1}^k \overline{O^i_n}\right), \quad \text{resp. } \Omega_n := \mathbb{R}^2 \setminus \overline{O_n}$$

converge to solutions of Euler in

$$\Omega := \widetilde{\Omega} \setminus \left(\cup_{i=1}^k \mathcal{C}^i\right), \quad \text{resp. } \Omega := \mathbb{R}^2 \setminus \mathcal{C}.$$

We discuss here some consequences of this convergence result.

Rugosity. A typical problem in rugosity theory is the following: let Ω be a smooth domain with a rough wall $y = 0$. Let Ω_ε be obtained from Ω by a boundary perturbation of the form $y = \varepsilon^\alpha \cos(x/\varepsilon)$ ($\alpha > 0$ fixed). What is the asymptotic behaviour of the flow in Ω_ε as $\varepsilon \to 0$? In the case of viscous flows, it has been shown that there is a drastic effect of the rugosity at the limit, see [2, 1]. In the opposite direction, *one can deduce* from our analysis that such effect does not hold for ideal incompressible flows: the solution u_ε of the Euler equations on Ω_ε converges to the solution u of the Euler equations on Ω.

Trapping of a flow. The complements of the domains Ω_n and Ω that we consider have the same number of connected components. Thus, the domain continuity that we show does not extend to the fusion of two obstacles as in Figure 1. In such a case, we do not pretend that u_n solution in Ω_n (see Figure 1) converges to u solution in Ω. Actually, we guess that it does not hold because of the Kelvin's circulation theorem.

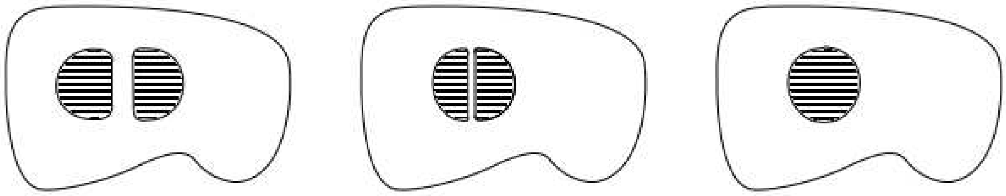

Figure 1: fusion of two obstacles.

However, an example that we can include in our analysis is an obstacle $\overline{O_n}$ which closes on itself (see Figure 2). In this picture, although Ω_n as a unique connected component, Ω has several connected components. Here, $\mathcal{C}$ is a Jordan curve, and it is an example of a compact set obtain as a Hausdorff limit, but not as a decreasing sequence of smooth simply connected domains. In such a setting, the present work still shows that u_n solution in Ω_n (see Figure 2) converges to u solution in Ω.

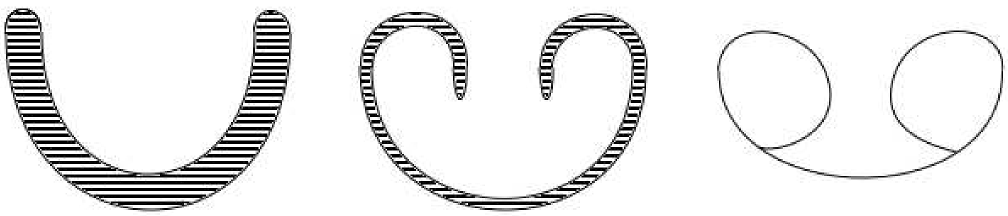

Figure 2: $\overline{O_n}$ tends to $\mathcal{C}$ in the Hausdorff sense.

6.2 The case of the Jordan arc

In this subsection, we pay special attention to the case of a smooth Jordan arc $\mathcal{C}$. We shall denote 0_1 and 0_2 the endpoints of the arc. As mentioned earlier, this geometry has already been investigated by the second author in [6]. In this article, the existence of Yudovitch type solutions is established through an approximation by regular domains Ω_ε. The corresponding regular solutions u_ε and their curl ω_ε are then truncated smoothly over a size ε around the obstacle. The resulting truncations $\tilde{u}_\varepsilon$ and $\tilde{\omega}_\varepsilon$, defined over the whole of $\mathbb{R}^2$, are shown to converge in appropriate topologies to the solutions $\tilde{u}$, $\tilde{\omega}$ of the system

$$\begin{cases} \operatorname{div} \tilde{u} = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ \partial_t \tilde{\omega} + \tilde{u} \cdot \nabla \tilde{\omega} = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ \tilde{\omega} \ := \ \operatorname{curl} \tilde{u} \ - \ g_{\tilde{\omega}} \delta_{\mathcal{C}}, \quad t > 0,\ x \in \mathbb{R}^2, \end{cases} \tag{6.1}$$

(plus a circulation condition). This is an Euler like equation, modified by a Dirac mass along the arc. The density function $g_{\tilde{\omega}}$ is given explicitly in terms of $\tilde{\omega}$ and $\mathcal{C}$. Moreover, it is shown that it is equal to the jump of the tangential component of the velocity across the arc. We refer to [6] for all necessary details. Our point in this section is to link this formulation in the whole space to the classical formulation of the Euler equation in Ω, see (1.7)-(1.5).

More precisely, let u be the solution of (1.7)-(1.5) built in Section 4, and $\omega := \operatorname{curl} u$. We want to show that extending u and ω by 0 yields a solution of (6.1) in $\mathbb{R}^2$. Therefore, we first notice that these extensions (still defined by u and ω) satisfy

$$u \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^2_{\text{loc}}(\mathbb{R}^2)), \quad \omega \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)).$$

It follows easily from the estimates (4.14)-(4.16), and (3.7)-(3.8). Then, we remark that $u = \nabla^\perp \psi^0 + \alpha \nabla^\perp \psi$ is clearly divergence free over the whole of $\mathbb{R}^2$.

We now turn to the transport equation for the vorticity. Taking $\varphi = \nabla^\perp \psi$ in (1.7), with some ψ compactly supported in $]0, +\infty[\times\Omega$, we first obtain

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad \text{ in }]0, +\infty[\times\Omega \tag{6.2}$$

in the distributional sense. Let now

$$\varphi \in \mathcal{D}\left([0, +\infty[\times(\mathbb{R}^2 \setminus \{0_1, 0_2\})\right)$$

be a scalar test function. We want to prove that

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \partial_t \varphi \, \omega + \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \nabla \varphi \cdot (\omega u) = \int_{\mathbb{R}^2} \varphi(0, \cdot) \, \omega^0, \tag{6.3}$$

meaning that the transport equation is satisfied over $\mathbb{R}^2 \setminus \{0_1, 0_2\}$. We introduce a curvilinear coordinate $s \in]0, S[$ and a transverse coordinate $r \in [-R, R]$, so that in a neighborhood of $\mathcal{C} \setminus \{0_1, 0_2\}$, one has $x = J(s) + r\, n(s)$, n a normal vector field. In view of (6.2), we can assume with no loss of generality that φ is compactly supported in this neighborhood. We then consider a truncation function that reads

$$\varphi_\varepsilon(t, x) = \varphi(t, x)(1 - \chi(r/\varepsilon))$$

where $\chi \in C^\infty_c(\mathbb{R})$, $\chi = 1$ near 0. One can use φ_ε as a test function in (6.2). Hence, to prove (6.3), it remains to prove that

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \partial_t \varphi \, \chi_\varepsilon \, \omega + \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \nabla(\varphi \, \chi_\varepsilon) \cdot (\omega u) - \int_{\mathbb{R}^2} \varphi(0, \cdot) \, \chi_\varepsilon \, \omega^0 \to 0, \quad \text{ as } \varepsilon \to 0, \quad \chi_\varepsilon(x) := \chi(r/\varepsilon).$$

The only difficult term is

$$I_\varepsilon := \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} (\varphi \, \nabla \chi_\varepsilon) \cdot (\omega u).$$

We remind that the streamfunction $\eta = \psi^0 + \alpha \psi$ associated to u satisfies $\Delta \eta = \omega$ in Ω, and that it is constant at $\mathcal{C}$ by the impermeability condition. As ω is bounded, it follows from elliptic regularity that η has $W^{2,p}$ regularity for all finite p on each side of $\mathcal{C}$, *away from the endpoints* $0_1, 0_2$. In particular, one has

$$\|u\|_{W^{1,p}(K_\varepsilon)} \leq C_p \tag{6.4}$$

over the support K_ε of $\varphi \nabla \chi_\varepsilon$. Denoting $u_n(x) := u(x) \cdot n(s)$ the "normal" component of u, one has

$$\begin{aligned} |I_\varepsilon| &\leq C \int_{K_\varepsilon} \frac{1}{\varepsilon} |\chi'(r/\varepsilon)| \, |u_n(x)| \, dx \leq C \int_{K_\varepsilon} \frac{r}{\varepsilon} |\chi'(r/\varepsilon)| \, \frac{|u_n(x)|}{r} \, dx \\ &\leq C \sup_\theta \left(\theta \, |\chi'(\theta)|\right) \int_{K_\varepsilon} \frac{|u_n(x)|}{r} \, dx \leq C' \sqrt{\int_{K_\varepsilon} dx} \sqrt{\int_{K_\varepsilon} |\nabla u_n(x)|^2 \, dx} \end{aligned}$$

where the last bound comes from the Hardy inequality, applied on each side of $\mathcal{C}$ to u_n (which vanishes at $\mathcal{C}$ by the impermeability condition). It follows from (6.4) that I_ε vanishes to zero with ε, as expected.

Thus, to establish the transport equation for the vorticity on the whole plane, it remains to handle the neighborhood of the endpoints $0_1, 0_2$, say 0_1. This time, we introduce the truncation

$$\chi_\varepsilon(x) := \chi\left(\frac{x - 0_1}{\varepsilon}\right) \quad \text{with } \chi \in C_c^\infty(\mathbb{R}^2),\ \chi = 1 \text{ near } 0.$$

As before, one is left with showing that

$$I_\varepsilon \ := \ \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} (\varphi \, \nabla \chi_\varepsilon) \cdot (\omega u)$$

goes to zero with ε. But this time, as $\nabla \chi_\varepsilon$ is uniformly bounded in L^2, one has the simple inequality

$$|I_\varepsilon| \ \le \ C \, \|\nabla \chi_\varepsilon\| \, \|\omega u\|_{L^2(K_\varepsilon)} \ \le \ C' \|\omega u\|_{L^2(K_\varepsilon)}$$

where K_ε is the support of χ_ε. The r.h.s. goes to zero by Lebesgue dominated convergence theorem, and yields the result.

Eventually, we have to establish the third line of (6.1), which expresses ω in terms of u and a Dirac mass along the arc. Again, we notice that the streamfunction η has $W^{2,p}$ regularity for all finite p on each side of the arc, away from its endpoints. This implies that the velocity u has a trace from each side of the arc, denoted by $u_\pm$. These traces belong to $W^{1-1/p,p}_{\text{loc}}(int(\mathcal{C}))$ for any finite p. By the impermeability condition, only the tangential component of these traces is non-zero. Let now $\varphi \in C_c^\infty(\mathbb{R}^2 \setminus \{0_1, 0_2\})$ a scalar test function. Testing this function with the relation $\omega = \operatorname{curl} u$ (that clearly holds in the strong sense in $\mathbb{R}^2 \setminus \mathcal{C}$), and integrating by parts on each side of the arc, we end up with

$$\int_{\mathbb{R}^2} \omega \, \varphi \ = \ - \int_{\mathbb{R}^2} u \cdot \nabla^\perp \phi \ + \ \int_{\mathcal{C}} [u_\tau] \varphi,$$

almost surely in t, where $[u_\tau]$ *denotes the jump of the tangential component*: if n is the normal going from side $+$ to side $-$, $[u_\tau] \ := \ (u_+ - u_-) \cdot n^\perp$. The last equation can be written

$$\omega \ = \ \operatorname{curl} u \ - \ g_\omega \delta_{\mathcal{C}} \text{ in } \mathbb{R}^2 \setminus \{0_1, 0_2\}$$

in the sense of distributions, where $g_\omega(s) := [u_\tau](s)$ (s the curvilinear coordinate).

The last step is to go from $\mathbb{R}^2 \setminus \{0_1, 0_2\}$ to $\mathbb{R}^2$. Therefore, we introduce again truncation functions near the endpoints: say

$$\chi_\varepsilon(x) := \chi\left(\frac{x - 0_1}{\varepsilon}\right) \quad \text{with } \chi \in C_c^\infty(\mathbb{R}^2),\ \chi = 1 \text{ near } 0.$$

As before, the point is to show that

$$\int_{\mathbb{R}^2} \omega \, \varphi \chi_\varepsilon, \qquad \int_{\mathbb{R}}^2 u \cdot \nabla^\perp \phi \chi_\varepsilon, \quad \text{and} \ \int_{\mathcal{C}} [u_\tau] \varphi \chi_\varepsilon$$

all go to zero with ε. The only annoying quantity is the third one: it requires a control on the jump function $[u_\tau]$ *up to the endpoint* 0_1. Therefore, we use results related to elliptic equations in polygons, see [10]. Indeed, up to a smooth change of variable, the Laplace equation for η in $\mathbb{R}^2 \setminus \mathcal{C}$ turns into a divergence form elliptic equation in the exterior of a slit. In particular, it follows from the results in [10] that $u = \nabla^\perp \eta$ decomposes into $u_1 + u_2$, where u_1 behaves like $1/|x - 0_i|^{1/2}$ near the endpoint 0_i, and u_2 has $W^{1,p}_{\text{loc}}(\mathbb{R}^2 \setminus \mathcal{C})$ regularity for all $p < 2$. This allows to conclude that $\int_{\mathcal{C}} [u_\tau] \varphi \chi_\varepsilon$ goes to zero with ε. This concludes the proof.

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