FOR HAUSDORFF SPACES, H-CLOSED = D-PSEUDOCOMPACT FOR ALL ULTRAFILTERS D

PAOLO LIPPARINI

Abstract. We prove that a topological space is $H(i)$ if and only if it is D-pseudocompact, for every ultrafilter D.

Locally, our result asserts that if X is a weakly initially λ-compact topological space, and $2^\mu \leq \lambda$, then X is D-pseudocompact, for every ultrafilter D over any set of cardinality $\leq \mu$. As a consequence, if $2^\mu \leq \lambda$, then the product of any family of weakly initially λ-compact spaces is weakly initially μ-compact.

Throughout this note λ and μ are infinite cardinals. No separation axiom is assumed, if not otherwise specified. By a product of topological spaces we shall always mean the Tychonoff product.

A topological space is said to be *weakly initially λ-compact* if and only if every open cover of cardinality at most λ has a finite subset with dense union. This notion has been introduced by Z. Frolík under a different name and studied by various authors. See [L, Remark 3] for references. Notice that, for regular spaces, weakly initial ω-compactness is equivalent to pseudocompactness.

If D is an ultrafilter over some set I, a topological space X is said to be *D-pseudocompact* [GS, GF] if and only if every I-indexed sequence of nonempty open sets of X has some D-limit point, where x is called a *D-limit point* of the sequence $(O_i)_{i\in I}$ if and only if, for every neighborhood U of x in X, $\{i \in I \mid U \cap O_i \neq \emptyset\} \in D$.

Theorem 1. *If X is a weakly initially λ-compact topological space, and $2^\mu \leq \lambda$, then X is D-pseudocompact, for every ultrafilter D over any set of cardinality $\leq \mu$.*

Proof. Suppose by contradiction that X is weakly initially λ-compact, D is an ultrafilter over I, $2^{|I|} \leq \lambda$, and X is not D-pseudocompact. Thus, there is a sequence $(O_i)_{i\in I}$ of nonempty open sets of X which

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has no D-limit point in X. This means that, for every $x \in X$, there is an open neighborhood U_x of x such that $\{i \in I \mid U_x \cap O_i \neq \emptyset\} \notin D$, that is, $\{i \in I \mid U_x \cap O_i = \emptyset\} \in D$, since D is an ultrafilter. For each $x \in X$, choose some U_x as above, and let $Z_x = \{i \in I \mid U_x \cap O_i = \emptyset\}$. Thus, $Z_x \in D$.

For each $Z \in D$, let $V_Z = \bigcup\{U_x \mid x \text{ is such that } Z_x = Z\}$. Notice that if $i \in Z \in D$, then $V_Z \cap O_i = \emptyset$. Notice also that $(V_Z)_{Z\in D}$ is an open cover of X. Since $|D| = 2^{|I|} \leq \lambda$, then, by weakly initial λ-compactness, there is a finite number $Z_1, \dots, Z_n$ of elements of D such that $V_{Z_1} \cup \dots \cup V_{Z_n}$ is dense in X. Since D is a filter, $Z = Z_1 \cap \dots \cap Z_n \in D$, hence $Z_1 \cap \dots \cap Z_n \neq \emptyset$. Choose $i \in Z_1 \cap \dots \cap Z_n$. Then $O_i \cap V_{Z_1} = \emptyset$, $\dots$, $O_i \cap V_{Z_n} = \emptyset$, hence $O_i \cap (V_{Z_1} \cup \dots \cup V_{Z_n}) = \emptyset$, contradicting the conclusion that $V_{Z_1} \cup \dots \cup V_{Z_n}$ is dense in X, since, by assumption, O_i is nonempty. □

Corollary 2. *If $2^\mu \leq \lambda$, then the product of any family of weakly initially λ-compact spaces is weakly initially μ-compact.*

Proof. Choose some regular ultrafilter D over μ. Recall that an ultrafilter over μ is *regular* if and only if there is a family of μ elements of D such that the intersection of any infinite subset of the family is empty. As a consequence of the Axiom of Choice (actually, the Prime Ideal Theorem suffices), for every infinite μ there is a regular ultrafilter over μ.

It is easy to see that if X is D-pseudocompact, for some regular ultrafilter D over μ, then X is weakly initially μ-compact. See, e. g., [L, Corollary 15].

Given any family of weakly initially λ-compact spaces, then, by Theorem 1, each member of the family is D-pseudocompact. Since D-pseudocompactness is productive [GS], their product is D-pseudocompact, hence weakly initially μ-compact, because of the choice of D, and by the preceding paragraph. □

A topological space X is $H(i)$ if and only if every open filter base on X has nonvoid adherence. Equivalently, a topological space is $H(i)$ if and only if every open cover has a finite subset with dense union. A Hausdorff space is *H-closed* (or *Hausdorff-closed*, or *absolutely closed*) if and only if it is closed in every Hausdorff space in which it is embedded. It is well known that a Hausdorff topological space is H-closed if and only if it is $H(i)$. A regular Hausdorff space is H-closed if and only if it is compact. See, e. g., [SS] for references.

Theorem 3. *For every topological space X, the following conditions are equivalent.*

(1) *X is $H(i)$.*
(2) *X is weakly initially λ-compact, for every infinite cardinal λ.*
(3) *X is D-pseudocompact, for every ultrafilter D.*
(4) *For every infinite cardinal λ, there exists some regular ultrafilter D over λ such that X is D-pseudocompact.*

If X is Hausdorff (respectively, Hausdorff and regular) then the preceding conditions are also equivalent to, respectively:

(5) *X is H-closed.*
(6) *X is compact.*

Proof. (1) and (2) are trivially equivalent, because of the above mentioned characterization of $H(i)$ spaces.

(2) $\Rightarrow$ (3) is immediate from Theorem 1.

(3) $\Rightarrow$ (4) follows from the fact that, as we mentioned in the proof of Corollary 2, for every infinite cardinal λ, there does exist some regular ultrafilter over λ.

(4) $\Rightarrow$ (1) follows from [L, Corollary 15], as in the proof of Corollary 2.

The equivalences of (1) and (5), and of (1) and (6), under the respective assumptions, follow from the remarks before the statement of the theorem. □

As a consequence of Theorem 3, we get another proof of some classical results.

Corollary 4. *Any product of a family of $H(i)$ spaces is an $H(i)$ space. Any product of a family of H-closed Hausdorff spaces is H-closed. Any product of a family of compact spaces is compact.*

Proof. By Theorem 3, and the mentioned result by Ginsburd and Saks [GS] that D-pseudocompactness is productive. □

In conclusion, a few remarks are in order. The situation described in this note is almost entirely similar to the case dealing with initial λ-compactness and D-compactness. Indeed, the proof of Theorem 1 can be easily modified in order to show directly that if $2^\mu \leq \lambda$, then every initially λ-compact topological space is D-compact, for every ultrafilter over any cardinal $\leq \mu$. This result, however, is already an immediate consequence of implications (8) and (5) in [S, Diagram 3.6]. Since D-compactness, too, is productive, we get that if $2^\mu \leq \lambda$, then any product of initially λ-compact spaces is initially μ-compact, the result analogue to Corollary 2. The above arguments furnish also a proof of the well known result that a space is compact if and only if it is D-compact, for every ultrafilter D, a theorem which, in turn, has the

Tychonoff theorem that every product of compact spaces is compact as an immediate consequence. This is entirely parallel to Theorem 3 and Corollary 4.

However, a subtle difference exists between the two cases. A sufficient condition for a topological space X to be initially λ-compact is that, for every λ' with $\omega \leq \lambda' \leq \lambda$, there exists some ultrafilter D uniform over λ' such that X is D-compact, while the parallel statement fails, in general, for weakly initial λ-compactness and D-pseudocompactness. Indeed, under some set theoretical hypothesis, [GF, Example 1.9] constructed a space X which is D-pseudocompact, for some ultrafilter uniform D over ω_1, hence necessarily D'-pseudocompact, for some ultrafilter D' uniform over ω, but X is not weakly initially ω_1-compact, actually, not even ω_1-pseudocompact. Cf. also [L, Remark 30].

The above counterexample shows that, in our arguments, and, in particular, in Condition (4) of Theorem 3 we do need the notion of a regular ultrafilter, while, in the corresponding theory for initial compactness, (a sufficient number of) uniform ultrafilters are enough. See, as an example, [S, Theorem 5.13].

Apart from the above comment, which mentions well known results, we do not know which results proved in this note are already known. To the best of our knowledge, they do not appear in the literature.

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Dipartimento di Matematica, Viale dei Decreti Attuattivi Scientifici, II Università di Roma (Tor Vergata), I-00133 ROME ITALY

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