Quaternion Gravi-Electromagnetism

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Abstract

Defining the generalized charge, potential, current and generalized fields as complex quantities where real and imaginary parts represent gravitation and electromagnetism respectively, corresponding field equation, equation of motion and other quantum equations are derived in manifestly covariant manner. It has been shown that the field equations are invariant under Lorentz as well as duality transformations. It has been shown that the quaternionic formulation presented here remains invariant under quaternion transformations.

Key Words: Quaternion, dyons, gravito-dyons, gravi-electromagnetism.

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1 Introduction

The idea of magnetic monopole was put forward by Dirac [1] in order to maintain the symmetry between electric and magnetic fields in Maxwell's equations. The analogy between linear gravitational and electromagnetic fields leads the asymmetry in Einstein's linear equation of gravity and suggests the existence of gravitational analogue [2] of magnetic monopole [3]. Like magnetic field, Cattani [4] introduced a new field (called Heavisidian field) depending upon the velocities of gravitational charges (masses) [2] and derived the covariant equations (like Maxwell's equation) of linear gravitational fields. On the other hand, some authors [5] described the existence of gravi-magnetic and gravi-electric fields instead of gravito-Heavisidian fields [6] associated with gravito-dyons (particle carrying simultaneously gravitational and Heavisidian charges). Avoiding the use of arbitrary string variables, Bisht et al [7] has developed manifestly covariant theory of gravito-dyons in terms of two four-potentials and maintained the structural symmetry between generalized electromagnetic field of dyons (particle carrying simultaneous existence of electric and magnetic charges) [8] and those of generalized gravito-Heavisidian fields of gravito-dyons. Corresponding field equation and equation of motion for unified fields of dyons and gravito-dyons are obtained by Rajput [6]. Quaternion formulation for generalized electromagnetic fields of dyons and generalized gravito-Heavisidian fields of gravito-dyons has also been developed [9] in compact and consistent manner. Accordingly, a consistent theory for the dynamics of four charges (masses) (namely electric, magnetic, gravitational, Heavisidian) have also been developed[10] in simple, compact and consistent manner. Considering an invariant Lagrangian density and its quaternionic representation, the consistent field equations for the dynamics of four charges have already been derived [11] and it has been shown that the present reformulation reproduces the dynamics of individual charges (masses) in the absence of other charge (masses) as well as the generalized theory of dyons (gravito - dyons) in the absence gravito - dyons (dyons). Keeping in view the recent potential importance of monopoles and the fact that the formalism necessary to develop them has been clumsy and manifestly non-covariant as well as the recent interest in linear gravity, in this paper, we have undertaken the study of gravitation and electromagnetism together by defining the complex four-potential, the real part of which represents gravitation and the imaginary part describes electromagnetism. Defining the generalized charge, potential, current and generalized fields as complex quantities where real and imaginary parts represent the constituents of gravitation and electromagnetism respectively, the generalized field equation and equation of motion are obtained in consistent and manifestly covariant manner. The suitable Lagrangian density for generalized fields of gravitation and electromagnetic charges has been described to yield the consistent form of corresponding field equation, equation of motion and continuity equation in manifestly covariant way. The electric, magnetic, gravitational and Heavisidian fields are discussed and Maxwell like equations for linear gravity and electromagnetism are obtained consistently in compact notation. It has been shown that the field equations are invariant under quaternion Lorentz transformations and duality transformations as well. The present theory reduces the the theory of linear gravity (or electromagnetism) in the absence of electromagnetism (gravitation) or vice versa.

2 Generalized Gravi-electromagnetism

Let us define the complex four-potential $\{\mathsf{V}_\mu\}$ associated with gravi-electromagnetic field as

$$\{\mathsf{V}_\mu\} = \{\mathrm{B}_\mu\} - i\ \{\mathrm{A}_\mu\} \qquad (i = \sqrt{-1}) \tag{1}$$

where $\{\mathrm{B}_\mu\}$ and $\{\mathrm{A}_\mu\}$ are respectively described as gravitational and electromagnetic four-potentials for linear gravitational and electromagnetic fields. $\{\mathrm{B}_\mu\}$ and $\{\mathrm{A}_\mu\}$ are now defined as

$$\{\mathrm{B}_\mu\} = \left\{\emptyset,\, \overrightarrow{B}\right\} \quad and \quad \{\mathrm{A}_\mu\} = \left\{\phi,\, \overrightarrow{A}\right\}. \tag{2}$$

Here $\emptyset$ and ϕ are respectively the temporal components of gravitational and electromagnetic four-potentials, whereas $\overrightarrow{B}$ and $\overrightarrow{A}$ are the spatial components of the respective four potentials. Thus, the generalized gravi-electromagnetic field tensor can be expressed as

$$\mathsf{G}_{\mu\nu} = \mathsf{F}_{\mu\nu} + i\,\mathsf{f}_{\mu\nu} \tag{3}$$

where the real part (i.e. $\mathsf{F}_{\mu\nu}$) is associated with linear gravitation and the imaginary part (i.e.$\mathsf{f}_{\mu\nu}$) is described for electromagnetism. The gravitational $\mathsf{F}_{\mu\nu}$ and electromagnetic $\mathsf{f}_{\mu\nu}$ field tensors are defined as

$$\begin{aligned} \mathsf{F}_{\mu\nu} &= \mathrm{B}_{\mu,\nu} - \mathrm{B}_{\nu,\mu}; \\ \mathsf{f}_{\mu\nu} &= \mathrm{A}_{\mu,\nu} - \mathrm{A}_{\nu,\mu}. \end{aligned} \tag{4}$$

From equation (3), we get

$$\mathsf{G}^{\dagger}_{\mu\nu} = \mathsf{G}^{\dagger}_{\mu\nu}. \tag{5}$$

Let us define the gravitational charge g_1 and electromagnetic charge $g_2$as

$$g_1 = \sqrt{k\,m} \quad and \quad g_2 = e \tag{6}$$

where k is gravitational constant (for brevity we take $k = 1$) and e is electronic charge. As such, we may write the generalized gravi-electromagnetic charge as complex quantity with gravitational and electronic charges as its real and imaginary constituents i.e.

$$q = g_1 + i\,g_2. \tag{7}$$

The generalized current is described as the product of generalized charge q and four-velocity $\{v_\mu\}$. So, the generalized current is expressed as complex quantity as

$$\{\mathsf{J}_\mu\} = q\,\{v_\mu\} = \{s_\mu\} + i\,\{\,j_\mu\} \tag{8}$$

where $\{s_\mu\}_\mu$and $\{\,j_\mu\}$are respectively the gravitational and electromagnetic four-currents which can be obtained from their respective field tensorial equations as

$$\begin{aligned} \mathsf{F}_{\mu\nu,\nu} &= s_\mu; \\ \mathsf{f}_{\mu\nu,\nu} &= j_\mu. \end{aligned} \tag{9}$$

So, the generalized field equation for gravi-electromagnetism is described as

$$\mathsf{G}_{\mu\nu,\nu} = \mathsf{J}_\mu \tag{10}$$

The Lagrangian density can now be expressed in terms of complex field tensor, four- potential and four-current as

$$\begin{aligned}\mathcal{L} =& -\frac{1}{4}\mathsf{G}^{\dagger}_{\mu\nu}\mathsf{G}_{\mu\nu}+\mathsf{V}^{\dagger}_{\mu}\mathsf{J}_{\mu} \\ =& -\frac{1}{4}\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}-\frac{1}{4}\mathsf{f}_{\mu\nu}\mathsf{f}^{\mu\nu}+\mathsf{A}_{\mu}j^{\mu}+\mathsf{B}_{\mu}s^{\mu}.\end{aligned} \tag{11}$$

This Lagrangian density yields the field equation (9) followed by the equation of continuity

$$\partial_{\mu}\mathsf{j}^{\mu} = \mathsf{j}_{\mu,\mu} = 0. \tag{12}$$

Hence, the Lorentz force equation of motion is described in the following form as

$$m\frac{d^2x_{\mu}}{d\tau^2} = Real(q\,\mathsf{G}_{\mu\nu})v^{\nu} = (g_1\mathsf{F}_{\mu\nu}+g_2\mathsf{f}_{\mu\nu})\,v^{\nu}. \tag{13}$$

Accordingly, the energy momentum tensor may be expressed as

$$\mathsf{T}_{\mu\nu} = \mathsf{G}^{\sigma}_{\mu}\mathsf{G}_{\nu\sigma}+\frac{1}{4}g_{\mu\nu}\mathsf{G}_{\alpha\beta}\mathsf{G}^{\alpha\beta} \tag{14}$$

the components of which are described as

$$\mathsf{T}_{00} = \frac{1}{2}\left[E^2+M^2\right]+\frac{1}{2}\left[G^2+H^2\right] \tag{15}$$

and

$$\mathsf{T}_{0a} = [E_a+M_a]+[G_a+H_a] \quad (\forall a=1,2,3) \tag{16}$$

where E_a and M_a denote the components of electric and magnetic fields while G_a and H_a describe corresponding components of gravitational and Heavisidian fields.

3 Quaternion Formalism for Gravi-electromagnetism

In order to write the quaternion formalism for gravi-electromagnetism discussed above, let us start with quaternion preliminaries.

3.1 Quaternion Preliminaries

The algebra $\mathbb{H}$ of quaternion is a four-dimensional algebra over the field of real numbers $\mathbb{R}$ and a quaternion ϕ is expressed in terms of its four base elements as

$$\phi = \phi_{\mu}e_{\mu} = \phi_0+e_1\phi_1+e_2\phi_2+e_3\phi_3(\mu=0,1,2,3), \tag{17}$$

where ϕ_0,ϕ_1,ϕ_2,ϕ_3 are the real quartets of a quaternion and e_0, e_1, e_2, e_3 are called quaternion units and satisfies the following relations,

$$\begin{aligned} e_0^2 = e_0 = & \; 1 \; , e_j^2 = -e_0, \\ e_0 e_i = e_i e_0 = & \; e_i (i = 1, 2, 3) \; , \\ e_i e_j = & \; -\delta_{ij} + \varepsilon_{ijk} e_k (\forall \, i, j, k = 1, 2, 3), \end{aligned} \tag{18}$$

where δ_{ij} is the delta symbol and ε_{ijk} is the Levi Civita three index symbol having value ($\varepsilon_{ijk} = +1$) for cyclic permutation , ($\varepsilon_{ijk} = -1$) for anti cyclic permutation and ($\varepsilon_{ijk} = 0$) for any two repeated indices. Addition and multiplication are defined by the usual distribution law $(e_j e_k) e_l = e_j (e_k e_l)$ along with the multiplication rules given by equation (18). $\mathbb{H}$ is an associative but non commutative algebra. If $\phi_0, \phi_1, \phi_2, \phi_3$ are taken as complex quantities, the quaternion is said to be a bi- quaternion. Alternatively, a quaternion is defined as a two dimensional algebra over the field of complex numbers $\mathbb{C}$. We thus have $\phi = \upsilon + e_2 \omega (\upsilon, \omega \in \mathbb{C})$ and $\upsilon = \phi_0 + e_1 \phi_1$, $\omega = \phi_2 - e_1 \phi_3$ with the basic multiplication law changes to $\upsilon e_2 = -e_2 \bar{\upsilon}$.The quaternion conjugate $\overline{\phi}$ is defined as

$$\overline{\phi} = \phi_\mu \bar{e_\mu} = \phi_0 - e_1 \phi_1 - e_2 \phi_2 - e_3 \phi_3. \tag{19}$$

In practice ϕ is often represented as a 2×2 matrix $\phi = \phi_0 - i \, \vec{\sigma} \cdot \vec{\phi}$ where $e_0 = I, \; e_j = -i \, \sigma_j (j = 1, 2, 3)$ and $\sigma_j$are the usual Pauli spin matrices. Then $\overline{\phi} = \sigma_2 \phi^T \sigma_2$ with ϕ^T the trans pose of ϕ. The real part of the quaternion ϕ_0 is also defined as

$$Re \, \phi = \frac{1}{2} (\overline{\phi} + \phi) \; , \tag{20}$$

where Re denotes the real part and if $Re \, \phi = 0$ then we have $\phi = -\overline{\phi}$ and imaginary ϕ is known as pure quaternion written as

$$\phi = e_1 \phi_1 + e_2 \phi_2 + e_3 \phi_3 \; . \tag{21}$$

The norm of a quaternion is expressed as $N(\phi) = \phi \overline{\phi} = \overline{\phi} \phi = \sum_{j=0}^{3} \phi_j^2$which is non negative i.e.

$$N(\phi) = |\phi| = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 = Det.(\phi) \geq 0. \tag{22}$$

Since there exists the norm of a quaternion, we have a division i.e. every ϕ has an inverse of a quaternion and is described as

$$\phi^{-1} = \frac{\overline{\phi}}{|\phi|} \; . \tag{23}$$

While the quaternion conjugation satisfies the following property

$$\overline{\phi_1 \phi_2} = \overline{\phi_2} \, \overline{\phi_1} \; . \tag{24}$$

The norm of the quaternion (22) is positive definite and enjoys the composition law

$$N(\phi_1 \phi_2) = N(\phi_1) N(\phi_2) \; . \tag{25}$$

Quaternion (17) is also written as $\phi = (\phi_0, \vec{\phi})$ where $\vec{\phi} = e_1\phi_1 + e_2\phi_2 + e_3\phi_3$ is its vector part and ϕ_0 is its scalar part. So, the sum and product of two quaternions are described as

$$\begin{aligned}(\alpha_0, \vec{\alpha}) + (\beta_0, \vec{\beta}) &= (\alpha_0 + \beta_0,\, \vec{\alpha} + \vec{\beta})\ , \\ (\alpha_0, \vec{\alpha}) \cdot (\beta_0, \vec{\beta}) &= (\alpha_0\beta_0 - \overrightarrow{\alpha} \cdot \overrightarrow{\beta}\,, \alpha_0 \overrightarrow{\beta} + \beta_0 \overrightarrow{\alpha} + \overrightarrow{\alpha} \times \overrightarrow{\beta})\ . \end{aligned} \tag{26}$$

Quaternion elements are non-Abelian in nature and thus represent a non commutative division ring.

3.2 Quaternion Gravi-electromagnetic Fields

Let us use the following definitions of electric, magnetic, gravitational and Heavisidian fields [6, 7] as

$$\begin{aligned}\overrightarrow{E} &= -\frac{\partial \overrightarrow{A}}{\partial t} - grad\,\phi; \\ \overrightarrow{M} &= \overrightarrow{\nabla} \times \overrightarrow{A} = curl\,\overrightarrow{A}; \\ \overrightarrow{G} &= \frac{\partial \overrightarrow{B}}{\partial t} + grad\,\emptyset; \\ \overrightarrow{H} &= \overrightarrow{\nabla} \times \overrightarrow{B} = curl\,\overrightarrow{B}; \end{aligned} \tag{27}$$

where we use the system of natural units $c = \hbar = 1$ throughout the text along with the gravitational and other constants are being taken to be unity. As such, we may express the complex four-potential $\{\mathsf{V}_\mu\}$ in terms of compact quaternion notation as

$$\mathbb{V} = \mathsf{V}_\mu e_\mu = V_0 + \sigma_1 V_1 + \sigma_2 V_2 + \sigma_3 V_3 \quad (\mu = 0, 1, 2, 3) \tag{28}$$

where σ_1, σ_2 and σ_3 are the Pauli spin matrices which are related to quaternion units as $e_k = -\,i\,\sigma_k$ $(\forall k = 1, 2, 3)$.The quaternion differential operator $\mathbb{D}$ may now be expressed as

$$\mathbb{D} = \partial_0 + \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3 \quad (\mu = 0, 1, 2, 3). \tag{29}$$

Operating the differential operator (29) to the quaternionic four-potential (28), we get [12],

$$\mathbb{D}\mathbb{V} = \Psi \tag{30}$$

where

$$\Psi = \Psi_0 + \sigma_1\Psi_1 + \sigma_2\Psi_2 + \sigma_3\Psi_3 \tag{31}$$

with

$$\Psi_0 = \partial_0 V_0 + \overrightarrow{\nabla} \cdot \overrightarrow{V} = 0 \tag{32}$$

due to Lorentz gauge condition and

$$\Psi_a = \partial_0 V_a + \partial_a V_0 + i\left(\overrightarrow{\nabla} \times \overrightarrow{V}\right)_a \quad (a = 1, 2, 3). \tag{33}$$

Using equations (33) and (27), we get

$$\begin{aligned} \overrightarrow{\Psi} &= \partial_0 \overrightarrow{V} + \overrightarrow{\nabla} V_0 + i\left(\overrightarrow{\nabla} \times \overrightarrow{V}\right) \\ &= \left(\overrightarrow{G} - \overrightarrow{M}\right) + i\left(\overrightarrow{H} - \overrightarrow{E}\right). \end{aligned} \tag{34}$$

Thus the generalized electromagnetic field vector $\overrightarrow{\Psi}$ is also expressed as complex quantity like the generalized potential and current. So, the gravi-electromagnetic field equations (i.e the generalized Maxwell's Dirac equations) may now be expressed [12] as

$$\begin{aligned} \overrightarrow{\nabla} \cdot \overrightarrow{\Psi} &= -\mathsf{J}_0; \\ \overrightarrow{\nabla} \times \overrightarrow{\Psi} &= i\overrightarrow{J} - i\frac{\partial \overrightarrow{\Psi}}{\partial t} \end{aligned} \tag{35}$$

Accordingly, we may write the conjugate (i.e. $\overline{\mathbb{D}}$) of quaternion differential operator $\mathbb{D}$ (29) as

$$\overline{\mathbb{D}} = \partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3 \quad (\mu = 0, 1, 2, 3). \tag{36}$$

Operating (36) to (34) and using equations (18), we get

$$\overline{\mathbb{D}}\, \Psi = \mathbb{J} \tag{37}$$

where

$$\mathbb{J} = J_0 + \sigma_1 J_1 + \sigma_2 J_2 + \sigma_3 J_3 \tag{38}$$

describes the quaternionic form of generalized current (35) for gravi-electromagnetism. Accordingly, operating (29) to (38) and using equations (18), we get

$$\mathbb{D}\, \mathbb{J} = \mathbb{S} \tag{39}$$

where

$$\mathbb{S} = S_0 + \sigma_1 S_1 + \sigma_2 S_2 + \sigma_3 S_3 \tag{40}$$

with

$$S_0 = \partial_0 J_0 + \overrightarrow{\nabla} \cdot \overrightarrow{J} = 0 \tag{41}$$

due to the equation of continuity and

$$\overrightarrow{S} = \partial_0 \overrightarrow{J} + \overrightarrow{\nabla} J_0 + i \left(\overrightarrow{\nabla} \times \overrightarrow{J} \right). \tag{42}$$

As such, we may express the generalized gravi-electric $\overrightarrow{\mathcal{E}}$ and gravi-magnetic $\overrightarrow{\mathcal{H}}$ fields in terms of two four-potentials namely the gravi-electric $\{\boldsymbol{B_\mu}\}$and gravi-magnetic $\{\boldsymbol{A_\mu}\}$ as

$$\begin{aligned} \overrightarrow{\mathcal{E}} &= \frac{\partial \overrightarrow{\boldsymbol{B}}}{\partial t} + \overrightarrow{\nabla} \text{Ø-} \overrightarrow{\nabla} \times \overrightarrow{\boldsymbol{A}}; \\ \overrightarrow{\mathcal{H}} &= -\frac{\partial \overrightarrow{\boldsymbol{A}}}{\partial t} - \overrightarrow{\nabla} \Phi + \overrightarrow{\nabla} \times \overrightarrow{\boldsymbol{B}}. \end{aligned} \tag{43}$$

These generalized gravi-electric $\overrightarrow{\mathcal{E}}$ and gravi-magnetic $\overrightarrow{\mathcal{H}}$ fields have the similar expressions discussed earlier for the generalized electromagnetic fields of dyons [7, 10, 11]in terms of two four-potentials. As such, the theory of generalized gravito- electromagnetism discussed here works in the same footing of dyons where the real part is gravitational sector and imaginary part as electromagnetism leading to striking symmetry between linear gravitation and electromagnetism.Here dyon has been considered as the particles carrying simultaneously the existence of gravitational mass(charge) and the electronic charge. Equation (43) thus satisfies the following differential equations like the generalized Dirac-Maxwell's (GDM) of dyons [7, 10, 11] i.e.

$$\begin{aligned} \overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{E}} &= -\rho_G; \\ \overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{H}} &= \rho_{\mathcal{E}} \\ \overrightarrow{\nabla} \times \overrightarrow{\mathcal{E}} &= \frac{\partial \overrightarrow{\mathcal{H}}}{\partial t} + \overrightarrow{j_{\mathcal{E}}}; \\ \nabla \times \overrightarrow{\mathcal{H}} &= -\overrightarrow{j_G} - \frac{\partial \overrightarrow{\mathcal{E}}}{\partial t}; \end{aligned} \tag{44}$$

where ρ_G ; $\rho_{\mathcal{E}}$ are charge source densities and $\overrightarrow{j_G}$; $\overrightarrow{j_{\mathcal{E}}}$ are current source densities respectively associated with gravitational and electronic charged particles. These GDM type equations (44) are invariant under the duality transformations

$$\begin{aligned} \overrightarrow{\mathcal{E}} &\longmapsto \overrightarrow{\mathcal{E}} \cos\theta + \overrightarrow{\mathcal{H}} \sin\theta; \\ \overrightarrow{\mathcal{H}} &\longmapsto -\overrightarrow{\mathcal{E}} \sin\theta + \overrightarrow{\mathcal{H}} \cos\theta; \\ \rho_G &\longmapsto \rho_G \cos\theta + \rho_{\mathcal{E}} \sin\theta; \\ \rho_{\mathcal{E}} &\longmapsto -\rho_G \sin\theta + \rho_{\mathcal{E}} \cos\theta; \\ \overrightarrow{j_G} &\longmapsto \overrightarrow{j_G} \cos\theta + \overrightarrow{j_{\mathcal{E}}} \sin\theta \\ \overrightarrow{j_{\mathcal{E}}} &\longmapsto -\overrightarrow{j_{\mathcal{E}}} \sin\theta + \overrightarrow{j_G} \cos\theta. \end{aligned} \tag{45}$$

On the other hand, the quaternion field equations (30), (37) and (39) are considered as self-dual. These are also invariant under quaternion transformations. As such, our theory is compact, simpler, consistent and manifestly covariant one. Our theory also reduces to the theory of linear gravity (electromagnetism) in the absence of electromagnetism (gravitation) or vice versa. It leads to the dynamics of gravitational (electric) mass (charge) in the absence of electric (gravitational) charge (mass) or vice versa. Here dyon has been considered as the combination of electric charge and gravitational mass.

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