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The $M/M/\infty$ Service System with Ranked Servers in Heavy Traffic

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Abstract: We consider an $M/M/\infty$ service system in which an arriving customer is served by the first idle server in an infinite sequence $S_1, S_2, \ldots$ of servers. We determine the first two terms in the asymptotic expansions of the moments of L as $\lambda \to \infty$, where L is the index of the server S_L serving a newly arriving customer in equilibrium, and λ is the ratio of the arrival rate to the service rate. The leading terms of the moments show that L/λ tends to a uniform distribution on $[0, 1]$.

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1. Introduction

We consider a stream of customers, with independent exponentially distributed interarrival times, arriving at rate λ to an infinite sequence $S_1, S_2, \ldots$ of servers. Each arriving customer engages the server S_l having the lowest index among currently idle servers, and renders that server busy for an independent exponentially distributed service time with mean 1. This stochastic service system, which is conventionally denoted $M/M/\infty$, has been extensively studied in the limit $\lambda \to \infty$; see Newell [N]. We shall be interested in a question mentioned only tangentially by Newell: what is the distribution of the random variable L defined as the index of the server S_L serving a newly arriving customer when the system is in equilibrium? Newell [N, p. 9] states that L "is approximately uniformly distributed over the interval" $[1, \lambda]$, basing this assertion on the approximation

$$\Pr[L > l] \approx \begin{cases} 1 - \dfrac{l}{\lambda}, & \text{if } l < \lambda, \\ 0, & \text{if } l > \lambda. \end{cases} \tag{1.1}$$

But no error bounds are given for this or other approximations stated by Newell, and not even the fact that the first moment has the asymptotic behavior

$$\mathrm{Ex}[L] \sim \frac{\lambda}{2} \tag{1.2}$$

that it would have under the uniform distribution is established rigorously. Our goal in this paper is to give a rigorous version of (1.1) that will suffice to establish not only (1.2), but also the next term,

$$\mathrm{Ex}[L] = \frac{\lambda}{2} + \frac{1}{2}\log\lambda + O(1), \tag{1.3}$$

and more generally

$$\mathrm{Ex}[L^m] = \frac{\lambda^m}{m+1} + \frac{m\,\lambda^{m-1}\log\lambda}{2} + O\left(\lambda^{m-1}\right) \tag{1.4}$$

for $m \geq 1$. In particular, we have

$$\begin{aligned} \mathrm{Var}[L] &= \mathrm{Ex}[L^2] - \mathrm{Ex}[L]^2 \\ &= \frac{\lambda^2}{12} + \frac{\lambda\log\lambda}{2} + O(\lambda). \end{aligned}$$

Since the interval $[0, 1]$ is bounded, formula (1.4) shows that the m-th moment of L/λ tends to $1/(m+1)$ as $\lambda \to \infty$ for all $m \geq 1$, and thus suffices to show that the distribution of L/λ tends to the uniform distribution on the interval $[0, 1]$. We note that a problem that is in a sense dual to ours (finding the largest index of a busy server, rather than the smallest index of an idle server) has been treated by Coffman, Kadota and Shepp [C].

The key to our results is the probability $\Pr[L > l]$, which is simply the probability that the first l servers $S_1, \ldots, S_l$ are all busy. It is well known that this probability is given by the Erlang loss formula

$$\begin{aligned} \Pr[L > l] &= \frac{\lambda^l/l!}{\sum_{0 \leq k \leq l} \lambda^k/k!} \\ &= \frac{1}{D_l}, \end{aligned}$$

where

$$D_l = \sum_{0 \leq k \leq l} \frac{l!}{(l-k)!\,\lambda^k} \tag{1.5}$$

(see for example Newell [N, p. 3]). The sum D_l can be expressed as an integral,

$$D_l = \int_0^\infty \left(1 + \frac{x}{\lambda}\right)^l e^{-x}\, dx$$

(see for example Newell [N, p. 7]), and most of Newell's analysis is based on such a representation. But we shall work directly with the expression of D_l as the sum in (1.5).

We shall divide the range of summation in (1.5) into two parts. The first, which we shall call the "body" of the distribution, will be $0 \le k \le l_0 = \lambda - s$, where $s = \sqrt{\lambda}$. The second, which we shall call the "tail", will be $l > l_0$. In Section 2, we shall derive an estimate for $\Pr[L > l]$ in the body, and in Section 3, we shall derive an estimate for the tail. In Section 4, we shall combine these estimates to establish (1.4).

2. The Body

In this section we shall establish the estimate

$$\Pr[L > l] = (1 - l/\lambda) + \frac{1}{\lambda(1 - l/\lambda)} + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{\lambda^2(1 - l/\lambda)^3}\right) \tag{2.1}$$

for $l \le l_0 = \lambda - s$, where $s = \sqrt{\lambda}$. We begin by using the principle of inclusion-exclusion to derive bounds on the denominator D_l.

We begin with a lower bound. Since

$$\begin{aligned} l(l-1)\cdots(l-k+1) &\ge l^k - \left(\sum_{0 \le j \le k-1} j\right) l^{k-1} \\ &= l^k - \binom{k}{2} l^{k-1}, \end{aligned}$$

we have

$$\begin{aligned} D_l &= \sum_{0 \le k \le l} \frac{l(l-1)\cdots(l-k+1)}{\lambda^k} \\ &\ge \sum_{0 \le k \le l} \left(\frac{l}{\lambda}\right)^k - \frac{1}{\lambda} \sum_{0 \le k \le l} \binom{k}{2} \left(\frac{l}{\lambda}\right)^{k-1}. \end{aligned}$$

For the first sum we have

$$\sum_{0 \le k \le l} \left(\frac{l}{\lambda}\right)^k = \frac{1 + O\big((l/\lambda)^l\big)}{1 - l/\lambda}.$$

We note that the logarithm of $(l/\lambda)^l$ has a non-negative second derivative for $l \ge 1$. Thus $(l/\lambda)^l$ assumes its maximum in the interval $0 \le l \le l_0$ for $l = 0$, $l = 1$ or $l = l_0$. Its values there are 0, $1/\lambda$ and $(1 - s/\lambda)^{\lambda - s} = (1 - 1/\sqrt{\lambda})^{\lambda - \sqrt{\lambda}} \le e^{-\sqrt{\lambda}+1}$, respectively. As $\lambda \to \infty$, the largest of these values is $1/\lambda$, so we have $O\big((l/\lambda)^l\big) = O(1/\lambda)$ for $0 \le l \le l_0$. Thus the first sum is

$$\sum_{0 \le k \le l} \left(\frac{l}{\lambda}\right)^k = \frac{1 + O(1/\lambda)}{1 - l/\lambda}.$$

For the second sum we have

$$\sum_{0 \le k \le l} \binom{k}{2} \left(\frac{l}{\lambda}\right)^{k-1} = \frac{1 + O\big(l^2 (l/\lambda)^l\big)}{(1 - l/\lambda)^3}.$$

The logarithm of $l^2(l/\lambda)^l$ has a non-negative second derivative for $l \geq 3$, so an argument similar to that used for the first sum shows that $O\big(l^2(l/\lambda)^l\big) = O(1/\lambda)$ for $0 \leq l \leq l_0$. Thus we have

$$\sum_{0\leq k\leq l} \binom{k}{2} \left(\frac{l}{\lambda}\right)^{k-1} = \frac{1+O(1/\lambda)}{(1-l/\lambda)^3}$$

and the lower bound

$$D_l \geq \frac{1+O(1/\lambda)}{1-l/\lambda} - \frac{1+O(1/\lambda)}{\lambda(1-l/\lambda)^3}. \tag{2.2}$$

For an upper bound, we have

$$\begin{aligned} l(l-1)\cdots(l-k+1) &\leq l^k - \left(\sum_{0\leq j\leq k-1} j\right) l^{k-1} + \left(\sum_{0\leq i<j\leq k-1} ij\right) l^{k-2} \\ &\leq l^k - \binom{k}{2} l^{k-1} + \frac{1}{2}\binom{k}{2}^2 l^{k-2} \end{aligned}$$

(because $\sum_{0\leq i<j\leq k-1} ij = \left(\left(\sum_{0\leq j\leq k-1} j\right)^2 - \sum_{0\leq j\leq k-1} j^2\right) \Big/ 2 \leq \left(\sum_{0\leq j\leq k-1} j\right)^2 /2 = \binom{k}{2}^2/2$). Thus we have

$$D_l \leq \sum_{0\leq k\leq l} \left(\frac{l}{\lambda}\right)^k - \frac{1}{\lambda} \sum_{0\leq k\leq l} \binom{k}{2} \left(\frac{l}{\lambda}\right)^{k-1} + \frac{1}{2\lambda^2} \sum_{0\leq k\leq l} \binom{k}{2}^2 \left(\frac{l}{\lambda}\right)^{k-2}.$$

For the third sum we have

$$\begin{aligned} \sum_{0\leq k\leq l} \binom{k}{2}^2 \left(\frac{l}{\lambda}\right)^{k-2} &\leq \sum_{k\geq 0} \binom{k}{2}^2 \left(\frac{l}{\lambda}\right)^{k-2} \\ &= O\left(\frac{1}{(1-l/\lambda)^5}\right). \end{aligned}$$

and thus the upper bound

$$D_l \leq \frac{1+O(1/\lambda)}{1-l/\lambda} - \frac{1+O(1/\lambda)}{\lambda(1-l/\lambda)^3} + O\left(\frac{1}{\lambda^2(1-l/\lambda)^5}\right).$$

Combining this upper bound with the lower bound (2.2) yields

$$D_l = \frac{1+O(1/\lambda)}{1-l/\lambda} - \frac{1+O(1/\lambda)}{\lambda(1-l/\lambda)^3} + O\left(\frac{1}{\lambda^2(1-l/\lambda)^5}\right).$$

To obtain $\Pr[L > l]$, we take the reciprocal of D_l:

$$\begin{aligned} \Pr[L > l] &= \left(\frac{1+O(1/\lambda)}{1-l/\lambda} - \frac{1+O(1/\lambda)}{\lambda(1-l/\lambda)^3} + O\left(\frac{1}{\lambda^2(1-l/\lambda)^5}\right)\right)^{-1} \\ &= \big(1+O(1/\lambda)\big)\,(1-l/\lambda)\,\left(1 - \frac{1}{\lambda(1-l/\lambda)^2} + O\left(\frac{1}{\lambda^2(1-l/\lambda)^4}\right)\right)^{-1} \\ &= \big(1+O(1/\lambda)\big)\,(1-l/\lambda)\,\left(1 + \frac{1}{\lambda(1-l/\lambda)^2} + O\left(\frac{1}{\lambda^2(1-l/\lambda)^4}\right)\right) \\ &= \big(1+O(1/\lambda)\big)\,\left((1-l/\lambda) + \frac{1}{\lambda(1-l/\lambda)} + O\left(\frac{1}{\lambda^2(1-l/\lambda)^3}\right)\right). \end{aligned}$$

Observing that $O(1/\lambda)\,(1-l/\lambda) = O(1/\lambda)$ and $O(1/\lambda)/\lambda(1-l/\lambda) = O\big(1/\lambda^2(1-l/\lambda)^3\big)$, we obtain (2.1).

3. The Tail

In this section we shall establish the estimate

$$\Pr[L > l] = O(e^{-\lambda}\,\lambda^l/l!) \tag{3.1}$$

for $l \geq \lambda - s$, where $s = \sqrt{\lambda}$. To obtain an upper bound on $\Pr[L > l]$, we obtain a lower bound on D_l. We have

$$\begin{aligned} D_l &= \sum_{0\leq k\leq l} \frac{l!}{(l-k)!\,\lambda^k} \\ &\geq \frac{l!}{\lfloor\lambda - s\rfloor!\,\lambda^{l-\lfloor\lambda-s\rfloor}} + \cdots + \frac{l!}{\lfloor\lambda - 2s\rfloor!\,\lambda^{l-\lfloor\lambda-2s\rfloor}}, \end{aligned} \tag{3.2}$$

because $l - \lfloor\lambda - s\rfloor \geq l - (\lambda - s) \geq 0$ by assumption and $\lfloor\lambda - 2s\rfloor \geq 0$ for all sufficiently large λ. There are $\lfloor\lambda - 2s\rfloor - \lfloor\lambda - 2s\rfloor + 1 \geq s$ terms in the sum (3.2). Furthermore, the smallest of these terms is the last, because its denominator contains factors of λ where the preceding terms contain factors smaller than λ. Thus we have

$$D_l \geq \frac{s\,l!}{\lfloor\lambda - 2s\rfloor!\,\lambda^{l-\lfloor\lambda-2s\rfloor}}.$$

For the factorial in the denominator of this bound, we shall use the estimate $n! \leq e\sqrt{n}\,e^{-n}\,n^n$, which holds for all $n \geq 1$ (because the trapezoidal rule underestimates the integral $\int_1^n \log x\,dx$ of the concave function $\log x$). This estimate yields

$$D_l \geq \frac{s\,l!\,e^{\lfloor\lambda-2s\rfloor}}{e\sqrt{\lfloor\lambda - 2s\rfloor}\,\lfloor\lambda - 2s\rfloor^{\lfloor\lambda-2s\rfloor}\,\lambda^{l-\lfloor\lambda-2s\rfloor}}. \tag{3.3}$$

We have

$$\begin{aligned} e^{\lfloor\lambda-2s\rfloor} &\geq e^{\lambda-2s-1}, \\ \lfloor\lambda - 2s\rfloor^{\lfloor\lambda-2s\rfloor} &\leq (\lambda - 2s)^{\lfloor\lambda-2s\rfloor} \\ &= \lambda^{\lfloor\lambda-2s\rfloor}\,(1 - 2s/\lambda)^{\lfloor\lambda-2s\rfloor} \\ &\leq \lambda^{\lfloor\lambda-2s\rfloor}\,(1 - 2s/\lambda)^{\lambda-2s-1} \\ &\leq \lambda^{\lfloor\lambda-2s\rfloor}\,e^{(-2s/\lambda)(\lambda-2s-1)} \\ &\leq \lambda^{\lfloor\lambda-2s\rfloor}\,e^{-2s+4s^2/\lambda+1} \\ &\leq \lambda^{\lfloor\lambda-2s\rfloor}\,e^{-2s+5} \end{aligned}$$

and

$$\sqrt{\lfloor\lambda - 2s\rfloor} \leq s.$$

Substituting these bounds into (3.3) yields

$$D_l \geq \frac{l!\,e^{\lambda}}{e^7\,\lambda^l}.$$

Taking the reciprocal of this bound yields (3.1).

4. The Moments

In this section we shall use (2.1) and (3.1) to prove (1.4). We write

$$\begin{aligned}\Delta_m(l) &= l^m - (l-1)^m \\ &= m\, l^{m-1} + O(l^{m-2})\end{aligned}$$

for the backward differences of the m-th powers of l. Then partial summation yields

$$\begin{aligned}\mathrm{Ex}[L^m] &= \sum_{l\geq 0} l^m \;\Pr[L = l] \\ &= \sum_{l\geq 0} \Delta_m(l) \;\Pr[L > l] \\ &= \sum_{l\geq 0} m\, l^{m-1} \;\Pr[L > l] + O\left(\sum_{l\geq 0} l^{m-2} \;\Pr[L > l]\right) \qquad (4.1)\end{aligned}$$

This formula shows that we should evaluate sums of the form

$$T_n = \sum_{l\geq 0} l^n \;\Pr[L > l]. \qquad (4.2)$$

We shall show that

$$T_n = \frac{\lambda^{n+1}}{(n+1)(n+2)} + \frac{\lambda^n \;\log\lambda}{2} + O(\lambda^n). \qquad (4.3)$$

Substitution of this formula into (4.1) will then yield (1.4).

We shall break the range of summation in (4.2) at $l_0 = \lambda - s$, where $s = \sqrt{\lambda}$, using (2.1) for $0 \leq l \leq l_0$ and (3.1) for $l > l_0$. Summing the first term in (2.1), we have

$$\begin{aligned}\sum_{0\leq l\leq l_0} l^n(1 - l/\lambda) &= \frac{1}{\lambda} \sum_{0\leq l\leq l_0} (\lambda\, l^n - l^{n+1}) \\ &= \frac{1}{\lambda}\left(\left(\frac{\lambda\, l_0^{n+1}}{n+1} + O(l_0^n)\right) - \left(\frac{\lambda^{n+2}}{n+2} + O(l_0^{n+1})\right)\right) \\ &= \frac{1}{\lambda}\left(\left(\frac{\lambda\,(\lambda^{n+1} - (n+1)\lambda^n s)}{n+1} + O(\lambda^n)\right) - \left(\frac{\lambda^{n+2} - (n+2)\lambda^{n+1} s}{n+2} + O(\lambda^{n+1})\right)\right) \\ &= \frac{\lambda^{n+1}}{(n+1)(n+2)} + O(\lambda^n).\end{aligned}$$

Summing the second term in (2.1), we have

$$\begin{aligned}\sum_{0\leq l\leq l_0} \frac{l^n}{\lambda - l} &= \sum_{s\leq k\leq\lambda} \frac{(\lambda - k)^n}{k} \\ &= \sum_{s\leq k\leq\lambda} \left(\frac{\lambda^n}{k} + O(\lambda^{n-1})\right) \\ &= \lambda^n \log\frac{\lambda}{s} + O(\lambda^n) \\ &= \frac{\lambda^n \log\lambda}{2} + O(\lambda^n),\end{aligned}$$

where we have used $\sum_{1\le k\le n} 1/k = \log n + O(1)$. Summing the third term in (2.1) of course yields $O(\lambda^n)$. Summing the last term in (2.1), we have

$$\begin{aligned}
\lambda \sum_{0\le l\le l_0} \frac{l^n}{(\lambda-l)^3} &= \lambda \sum_{s\le k\le \lambda} \frac{(\lambda-k)^n}{k^3} \\
&\le \lambda^{n+1} \sum_{s\le k\le \lambda} \frac{1}{k^3} \\
&\le \lambda^{n+1} \sum_{k\ge s} \frac{1}{k^3} \\
&= \lambda^{n+1} \left(\frac{2}{s^2} + O\left(\frac{1}{s^3}\right)\right) \\
&= O(\lambda^n),
\end{aligned}$$

where we have used $\sum_{k\ge n} 1/k^3 = 2/n^2 + O(1/n^3)$. Combining these estimates, we obtain

$$\sum_{0\le l\le l_0} l^n \Pr[L > l] = \frac{\lambda^{n+1}}{(n+1)(n+2)} + \frac{\lambda^n \log\lambda}{2} + O(\lambda^n). \tag{4.4}$$

Finally, summing (3.1) we have

$$\begin{aligned}
\sum_{l>l_0} \frac{l^n e^{-\lambda} \lambda^l}{l!} &\le \sum_{l\ge 0} \frac{l^n e^{-\lambda} \lambda^l}{l!} \\
&= O(\lambda^n),
\end{aligned}$$

because the summation on the right-hand side is the n-th moment of a Poisson random variable with mean λ, which is a polynomial of degree n in λ. Thus

$$\sum_{l>l_0} l^n \Pr[L > l] = O(\lambda^n).$$

Combining this estimate with (4.4) yields (4.3) and completes the proof of (1.4).

5. Conclusion

We have obtained the first two terms in the asymptotic expansions of the moments of L as $\lambda \to \infty$. An obvious open question is whether one can obtain a complete asymptotic expansion, or even just the constant term in (1.3) and the corresponding terms in (1.4). While our estimates for the contributions to the $O(1)$ term in (1.3) could be improved (for example, by a better choice of the parameter s), it is clear that new techniques will be needed to obtain an error term tending to zero.

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7. References

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