LIE ALGEBROID MODULES AND REPRESENTATIONS UP TO HOMOTOPY

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ABSTRACT. We explain how Lie algebroid modules in the sense of Vaintrob provide geometric models for Lie algebroid representations up to homotopy. Specifically, we show that there is a noncanonical way to obtain representations up to homotopy from a given Lie algebroid module, and that any two representations up to homotopy obtained in this way are equivalent in a natural sense. This result extends the relationship between $\mathcal{VB}$-algebroids and 2-term representations up to homotopy, as studied by Gracia-Saz and the author. We also extend the construction of $\mathcal{VB}$-algebroid characteristic classes to the setting of Lie algebroid modules.

1. Introduction

In [Vaĭ97], Vaintrob introduced the notion of a *module over a Lie algebroid*, using the language of supergeometry, and he observed that this notion generalizes that of Lie algebroid representation. Since that time, various other generalizations of Lie algebroid representations have appeared, with the most popular being that of *representation up to homotopy* [AC09]. A representation up to homotopy of a Lie algebroid $A \to M$ is a complex of vector bundles $(\mathcal{E}, \partial)$ over M equipped with an A-connection ∇ and maps $\omega_i : \bigwedge^i \Gamma(A) \to \mathrm{End}_{1-i}(\mathcal{E})$ for $i \geq 2$, satisfying a series of coherence conditions, the first of which says that ω_2 generates chain homotopies controlling the curvature of ∇. The reader should not confuse this with an earlier, different definition of representation up to homotopy in [CF05, ELW99].

In [GSM10], Gracia-Saz and the author studied $\mathcal{VB}$*-algebroids* (i.e. Lie algebroids in the category of vector bundles) and their relationship to representations up to homotopy[1] on 2-term complexes of vector bundles. It was shown there that $\mathcal{VB}$-algebroids provide geometric models for 2-term representations up to homotopy, in the sense that $\mathcal{VB}$-algebroids can be "decomposed" to produce 2-term representations up to homotopy, and that different choices of decomposition lead to "gauge-equivalent" representations up to homotopy. One of the advantages of this point of view is that the category of $\mathcal{VB}$-algebroids over a Lie algebroid A contains canonical objects, namely the tangent and cotangent bundles of A, that respectively play the roles of adjoint and coadjoint representations.

When the definition of $\mathcal{VB}$-algebroid is expressed in supergeometric terms, it becomes clear that $\mathcal{VB}$-algebroids form a special case of Lie algebroid modules in the sense of Vaintrob. The purpose of this paper is to show that the relationship

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[1]In [GSM10], Lie algebroid representations up to homotopy were called *flat superconnections* or *superrepresentations*.

between $\mathcal{VB}$-algebroids and 2-term representations up to homotopy extends to a relationship between Lie algebroid modules and representations up to homotopy. In particular,

(1) Lie algebroid modules can be (noncanonically) decomposed to produce representations up to homotopy,
(2) different choices of decomposition lead to gauge-equivalent representations up to homotopy, and
(3) all semibounded representations up to homotopy arise in this manner.

These results are almost immediate consequences of two structure theorems, proven in §2, for vector bundles over N-manifolds. Since the structure theorems are given in the general setting of N-manifolds, the above statements continue to hold if one replaces $A[1]$ by an arbitrary NQ-manifold. For example, the results of this paper could be applied to the theory of modules over Lie n-algebroids or L_∞-algebras.

Unlike the category of $\mathcal{VB}$-algebroids, the category of Lie algebroid modules has a natural tensor product, giving it the structure of a symmetric monoidal category. It would be interesting to know how much information about a Lie algebroid can be recovered from its category of modules.

The structure of the paper is as follows:

- In §2, we study vector bundles over N-manifolds. We state and prove the structure theorems and introduce the notion of decomposition.
- In §3, we recall the definitions of representation up to homotopy and gauge-equivalence.
- In §4, we recall the definition of Lie algebroid module, and we arrive at the main results relating Lie algebroid modules to representations up to homotopy.
- In §5, we consider the example of the adjoint module of a Lie algebroid A. The cohomology of A with values in the adjoint module is isomorphic to the deformation cohomology of Crainic and Moerdijk [CM08].
- In §6, we describe the constructions of tensor product, direct sum, and dual. We show that there is a cohomology pairing for dual Lie algebroid modules.
- In §7, we recall the construction of characteristic classes in [GSM10], and show that this construction provides well-defined invariants of Lie algebroid modules.

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2. The structure of N-manifold vector bundles

Let $\mathcal{M}$ be a nonnegatively graded manifold, or N-manifold. Recall that there is a natural projection $\pi_{\mathcal{M}}$ onto the underlying degree 0 manifold M, where the pullback map $\pi_{\mathcal{M}}^*$ identifies smooth functions on M with degree 0 functions on $\mathcal{M}$. There is also a natural "zero" embedding $0_{\mathcal{M}} : M \to \mathcal{M}$, whose pullback map annihilates the ideal of positive degree functions on $\mathcal{M}$.

A vector bundle $\mathcal{B}$ over $\mathcal{M}$ is given by its sheaf of sections $\Gamma(\mathcal{B})$, which is, by definition, a locally free graded $C^\infty(\mathcal{M})$-module. We denote the rank of $\Gamma(\mathcal{B})$ in degree i by $\mathrm{rk}_i(\mathcal{B})$. For simplicity, we suppose that $\mathcal{B}$ is degree-bounded, in the sense that there exist integers m, n such that $\mathrm{rk}_i(\mathcal{B}) = 0$ for $i < m$ and for $i > n$.

However, as noted below in Remark 2.6, the results of this section continue to hold if $\mathrm{rk}_i(\mathcal{B})$ is only bounded on one side. In any case, we emphasize that the total space of $\mathcal{B}$ is allowed to be a $\mathbb{Z}$-graded (as opposed to $\mathbb{N}$-graded) manifold.

The pullback bundle $0_{\mathcal{M}}^*\mathcal{B}$ is a graded vector bundle over M. Any graded $C^\infty(M)$-module canonically splits as a direct sum of its homogeneous parts, so we may write $0_{\mathcal{M}}^*\mathcal{B} = \bigoplus E_i[-i]$, where $\{E_i\}$ is a collection of vector bundles over M. We refer to $\mathcal{E} := 0_{\mathcal{M}}^*\mathcal{B} = \bigoplus E_i[-i]$ as the *standard graded vector bundle* associated to $\mathcal{B}$. Obviously, $\mathrm{rk}_i(\mathcal{B}) = \mathrm{rk}(E_i)$.

Since we are assuming that $\mathrm{rk}_i(\mathcal{B}) = 0$ for $i < m$, we have that the map of sections $0_{\mathcal{M}}^* : \Gamma(\mathcal{B}) \to \Gamma(\mathcal{E})$ is an isomorphism in degree m, and we can identify $\Gamma_m(\mathcal{B})$ with $\Gamma(E_m[-m])$. More generally, if we identify the space of sections of $\pi_{\mathcal{M}}^* E_m[-m]$ with $C^\infty(\mathcal{M}) \otimes \Gamma(E_m[-m])$, then the map $\alpha \otimes \varepsilon \mapsto \alpha\varepsilon$ defines a $C^\infty(\mathcal{M})$-linear injection of $\Gamma(\pi_{\mathcal{M}}^* E_m[-m])$ into $\Gamma(\mathcal{B})$ that is isomorphic in degree m.

Let $\mathcal{B}_{m+1}$ be defined as the cokernel of the injection $\pi_{\mathcal{M}}^* E_m[-m] \to \mathcal{B}$. Clearly, $\mathrm{rk}_i(\mathcal{B}_{m+1}) = 0$ if $i < m+1$. By iterating this process, we obtain a tower of vector bundles over $\mathcal{M}$

$$\mathcal{B} := \mathcal{B}_m \longrightarrow \mathcal{B}_{m+1} \longrightarrow \cdots \longrightarrow \mathcal{B}_n \longrightarrow \mathcal{B}_{n+1} = 0\ , \tag{1}$$

where $\mathrm{rk}_i(\mathcal{B}_k) = 0$ for $i < k$, together with canonical isomorphisms $\ker(\mathcal{B}_k \to \mathcal{B}_{k+1}) \cong \pi_{\mathcal{M}}^* E_k[-k]$.

If we choose a splitting of the short exact sequence

$$\pi_{\mathcal{M}}^* E_k[-k] \longrightarrow \mathcal{B}_k \longrightarrow \mathcal{B}_{k+1} \tag{2}$$

for each k, then we obtain an isomorphism $\mathcal{B} \cong \pi_{\mathcal{M}}^*\mathcal{E}$. Thus we have the following structure theorem:

Theorem 2.1. *Let $\mathcal{B}$ be a vector bundle over $\mathcal{M}$, and let $\mathcal{E} \to M$ be the standard graded vector bundle associated to $\mathcal{B}$. Then $\mathcal{B}$ is noncanonically isomorphic to $\pi_{\mathcal{M}}^*(\mathcal{E})$.*

The statement of Theorem 2.1 can be strengthened slightly. We have described a specific procedure for constructing isomorphisms from $\mathcal{B}$ to $\pi_{\mathcal{M}}^*(\mathcal{E})$, and we would like to characterize the isomorphisms that arise from this procedure. To address this issue, we first make the observation that $0_{\mathcal{M}}^*\mathcal{B} = \mathcal{E}$ is canonically isomorphic to $0_{\mathcal{M}}^*\pi_{\mathcal{M}}^*\mathcal{E}$, since $\pi_{\mathcal{M}} \circ 0_{\mathcal{M}} = \mathrm{id}_M$. Any isomorphism $\Theta : \mathcal{B} \to \pi_{\mathcal{M}}^*(\mathcal{E})$ obtained via splittings of (2) is such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\mathrm{id}"] \arrow[d, "\tilde{0}_{\mathcal{M}}"'] & \mathcal{E} \arrow[d, "\tilde{0}_{\mathcal{M}}"] \\ \mathcal{B} \arrow[r, "\Theta"] & \pi_{\mathcal{M}}^*(\mathcal{E}) \end{tikzcd} \tag{3}$$

Here, the vertical maps are the natural maps associated to pullback bundles.

On the other hand, by considering changes of splittings of the sequences (2), we see that the difference between any two isomorphisms obtained by such splittings is given by a collection of maps $\sigma_{k,i} : \Gamma(E_k) \to C_i^\infty(\mathcal{M}) \otimes \Gamma(E_{k-i})$ for $1 \le i \le k-m$. The associated automorphism of $\pi_{\mathcal{M}}^*(\mathcal{E})$ takes $\varepsilon \in \Gamma(E_k)$ to $\varepsilon + \sum_{i=1}^{k-m} \sigma_{k,i}(\varepsilon)$. All automorphisms of $\pi_{\mathcal{M}}^*(\mathcal{E})$ fixing the image of $\tilde{0}_{\mathcal{M}}$ are of this form. In summary, we have the following result, which refines Theorem 2.1:

Theorem 2.2. *Let $\mathcal{B}$ be a vector bundle over $\mathcal{M}$, and let $\mathcal{E} \to M$ be the standard graded vector bundle associated to $\mathcal{B}$. An isomorphism $\Theta : \mathcal{B} \to \pi^*_{\mathcal{M}}(\mathcal{E})$ can be obtained via splittings of* (2) *if and only if the diagram* (3) *commutes.*

For later use, we introduce the following terminology.

Definition 2.3. A *decomposition* of a vector bundle $\mathcal{B} \to \mathcal{M}$ is a choice of isomorphism $\Theta : \mathcal{B} \to \pi^*_{\mathcal{M}}(\mathcal{E})$ such that (3) commutes.

Definition 2.4. A *statomorphism* of a vector bundle $\mathcal{B} \to \mathcal{M}$ is a vector bundle automorphism Ψ such that

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\mathrm{id}"] \arrow[d, "\tilde{0}_{\mathcal{M}}"'] & \mathcal{E} \arrow[d, "\tilde{0}_{\mathcal{M}}"] \\ \mathcal{B} \arrow[r, "\Psi"] & \mathcal{B} \end{tikzcd} \tag{4}$$

commutes.

Remark 2.5. The term "statomorphism" is due to Gracia-Saz and Mackenzie [GSM09], who used it to describe automorphisms of double and triple vector bundles that preserve the underlying structure bundles. We use the term here because there is a natural way to view double vector bundles as graded vector bundles (for example, see [GR09, Meh09, Roy99]), and in this case our definition of statomorphism coincides with theirs.

Remark 2.6. The structure theorems in this section can be extended to the case where $\mathrm{rk}_i(\mathcal{B})$ is only bounded one side. In this "semibounded" case, $\mathcal{B}$ is still realized as a colimit of a tower of vector bundles as in (1), but the tower extends infinitely in one direction. By choosing splittings of the short exact sequences (2), we can obtain an isomorphism $\mathcal{B} \cong \pi^*_{\mathcal{M}}\mathcal{E}$ as a colimit of isomorphisms.

3. Representations up to homotopy of Lie algebroids

Let $A \to M$ be a Lie algebroid. Then $\Omega(A) := \bigwedge \Gamma(A^*)$ is the algebra of A-*forms*, equipped with the differential d_A.

Let $\mathcal{E} = \bigoplus E_i[-i]$ be a graded vector bundle over M. The space of $\mathcal{E}$-*valued A-forms*

$$\Omega(A;\mathcal{E}) := \Omega(A) \otimes_{C^\infty(M)} \Gamma(\mathcal{E})$$

is endowed with a $\mathbb{Z}$-grading where the subspace $\Omega^p(A) \otimes \Gamma(E_i[-i])$ is homogeneous of degree $p + i$.

Definition 3.1. A *representation up to homotopy*, or ∞-*representation*, of A on $\mathcal{E}$ is a degree 1 operator $\mathcal{D}$ on $\Omega(A;\mathcal{E})$ such that $\mathcal{D}^2 = 0$ and such that the Leibniz rule

$$\mathcal{D}(\alpha\omega) = (d_A\alpha)\omega + (-1)^p \alpha(\mathcal{D}\omega) \tag{5}$$

holds for $\alpha \in \Omega^p(A)$ and $\omega \in \Omega(A;\mathcal{E})$.

There is a natural projection map $\mu : \Omega(A;\mathcal{E}) \to \Gamma(\mathcal{E})$ for which the kernel is $\bigoplus_{p>0} \Omega^p(A) \otimes \Gamma(\mathcal{E})$. If $\mathcal{D}$ is an ∞-representation, then the Leibniz rule implies that

$\ker \mu$ is $\mathcal{D}$-invariant. Therefore, there is an induced differential ∂ on $\Gamma(\mathcal{E})$, defined by the property that the following diagram commutes:

$$\begin{tikzcd} \Omega(A;\mathcal{E}) \arrow[r, "\mathcal{D}"] \arrow[d, "\mu"'] & \Omega(A;\mathcal{E}) \arrow[d, "\mu"'] \\ \Gamma(\mathcal{E}) \arrow[r, "\partial"] & \Gamma(\mathcal{E}) \end{tikzcd}$$

If $\mathcal{D}$ and $\mathcal{D}'$ are ∞-representations of A on $\mathcal{E}$ and $\mathcal{E}'$, respectively, then a *morphism* from $\mathcal{D}$ to $\mathcal{D}'$ is an $\Omega(A)$-module morphism $\phi : \Omega(A;\mathcal{E}) \to \Omega(A;\mathcal{E}')$ such that $\phi \circ \mathcal{D} = \mathcal{D}' \circ \phi$. In this case, ϕ induces a chain map from $\Gamma(\mathcal{E})$ to $\Gamma(\mathcal{E}')$.

As usual, an invertible morphism of ∞-representations is called an isomorphism. However, in the case where the graded vector bundle $\mathcal{E}$ is fixed, there is a slightly more refined notion, which we call *gauge equivalence*.

Definition 3.2. A *gauge transformation* of $\Omega(A;\mathcal{E})$ is a degree-preserving $\Omega(A)$-module automorphism u such that the following diagram commutes:

$$\begin{tikzcd} \Omega(A;\mathcal{E}) \arrow[r, "u"] \arrow[d, "\mu"'] & \Omega(A;\mathcal{E}) \arrow[d, "\mu"'] \\ \Gamma(\mathcal{E}) \arrow[r, "\mathrm{id}"] & \Gamma(\mathcal{E}) \end{tikzcd}$$

Under a gauge transformation, an ∞-representation $\mathcal{D}$ transforms as $\mathcal{D}' = u^{-1}\mathcal{D}u$. Two ∞-representations that are related by a gauge transformation are said to be *gauge-equivalent*. Note that gauge-equivalent ∞-representations induce the same differential ∂ on $\Gamma(\mathcal{E})$.

4. Lie algebroid modules

Let $A \to M$ be a Lie algebroid.

Definition 4.1 ([Vaĭ97])**.** A *Lie algebroid module* over A, or *A-module*, is a vector bundle $\mathcal{B} \to A[1]$ equipped with a degree 1 operator Q on $\Gamma(\mathcal{B})$ such that $Q^2 = 0$ and such that the Leibniz rule

$$Q(\alpha\beta) = (d_A\alpha)\beta + (-1)^p\alpha(Q\beta)$$

holds for $\alpha \in C^\infty_p(A[1]) = \Omega^p(A)$ and $\beta \in \Gamma(\mathcal{B})$.

A *morphism* of A-modules from $(\mathcal{B}, Q)$ to $(\mathcal{B}', Q')$ is a linear map $\psi : \mathcal{B} \to \mathcal{B}'$, covering the identity map on $A[1]$, such that $\psi Q = Q'\psi$.

Recall that, for a fixed vector bundle $\mathcal{B} \to A[1]$, we have defined in Definition 2.4 a distinguished class of automorphisms, called *statomorphisms*. We will say that two A-module structures Q and Q' on $\mathcal{B}$ are *statomorphic* if there exists a statomorphism $\psi : \mathcal{B} \to \mathcal{B}$ such that $\psi Q = Q'\psi$.

Remark 4.2. The operator Q in the definition of Lie algebroid module can be equivalently viewed as a linear homological vector field whose base vector field is d_A. In other words, a Lie algebroid module is a special case of a *Q-vector bundle*, i.e. a vector bundle in the category of Q-manifolds.

Remark 4.3. Of particular interest is the special case where the total space of $\mathcal{B}$ is concentrated in degrees 0 and 1 (so that $\mathrm{rk}_i(\mathcal{B})$ vanishes except for $i = -1, 0$). In this case, $\mathcal{B} = D[1]$ for some vector bundle $D \to E$. The fact that $\mathcal{B}$ also has a vector bundle structure over $A[1]$ implies that D is a double vector bundle. In this case, an A-module structure on $\mathcal{B}$ is equivalent to a $\mathcal{VB}$-algebroid structure on D over A.

Let $\mathcal{E} = \bigoplus E_i[-i]$ be a graded vector bundle over M. Assume that $\mathcal{E}$ is bounded in degree (or semibounded, c.f. Remark 2.6). Initially, we consider A-module structures of the form $\pi_A^*\mathcal{E} \to A[1]$, where π_A is the projection map from $A[1]$ to M. In this case, the module of sections $\Gamma(\pi_A^*\mathcal{E})$ is canonically isomorphic to $\Omega(A) \otimes_{C^\infty(M)} \Gamma(\mathcal{E}) = \Omega(A; \mathcal{E})$. Under this identification, Definitions 3.1 and 4.1 become identical, so we immediately have the following:

Lemma 4.4. *∞-representations of A on $\mathcal{E}$ are in one-to-one correspondence with A-modules of the form $\pi_A^*\mathcal{E}$.*

In light of Theorem 2.2, we have a straightforward way to obtain an ∞-representation from an arbitrary A-module $\mathcal{B} \to A[1]$; one simply needs to choose a decomposition $\Theta : \mathcal{B} \to \pi_A^*\mathcal{E}$ (see Definition 2.3), and then the operator Q on $\Gamma(\mathcal{B})$ induces an ∞-representation $\mathcal{D} := \Theta \circ Q \circ \Theta^{-1}$ of A on $\mathcal{E}$.

Furthermore, we observe that a gauge transformation of $\Omega(A; \mathcal{E})$ is precisely the same thing as an automorphism of $\pi_A^*(\mathcal{E})$ that preserves the image of $\tilde{0}_A$. In other words, changes of decomposition correspond to gauge transformations of $\mathcal{D}$. We now have our main result:

Theorem 4.5. *Let $A \to M$ be a Lie algebroid.*

1. *There is a one-to-one correspondence between isomorphism classes of A-modules and isomorphism classes of (semi)bounded ∞-representations of A.*
2. *For any (semi)bounded graded vector bundle $\mathcal{E} = \bigoplus E_i \to M$, there is a one-to-one correspondence between statomorphism classes of A-modules with standard graded vector bundle $\mathcal{E}$ and gauge-equivalence classes of ∞-representations of A on $\mathcal{E}$.*

5. Adjoint module and deformation cohomology

Let $A \to M$ be a Lie algebroid, and let $\mathcal{B} \to A[1]$ be an A-module.

Definition 5.1 ([Vaĭ97]). The *cohomology of A with values in $\mathcal{B}$*, denoted $H^\bullet(A; \mathcal{B})$, is the cohomology of the complex $(\Gamma(\mathcal{B}), Q)$.

The results of §4 imply that $H^\bullet(A; \mathcal{B})$ is isomorphic to the cohomology of A with values in any ∞-representation arising from $\mathcal{B}$.

Example 5.2 (Adjoint module). The *adjoint module* of A is the tangent bundle $T(A[1])$. The sections of $T(A[1])$ are of course vector fields on $A[1]$ (i.e., graded derivations of the algebra $\Omega(A)$), with the operator $Q := [d_A, \cdot]$. The low-degree cohomology with values in the adjoint module was described in [Vaĭ97], but it is worth repeating with additional details.

We first consider degree -1. The degree -1 derivations of $\Omega(A)$ are precisely the contraction operators ι_X for $X \in \Gamma(A)$. The Lie derivative operator $L_X :=$

$[d_A, \iota_X]$ vanishes if and only if X is in the center of the Lie algebra $\Gamma(A)$. Therefore $H^{-1}(A; TA[1])$ can be identified with the center of $\Gamma(A)$.

Next, we consider degree 0. The degree 0 derivations of $\Omega(A)$ are in one-to-one correspondence with linear vector fields on A. A degree 0 derivation ϕ satisfies the equation $[d_A, \phi] = 0$ if and only if ϕ corresponds to a *morphic* vector field [MX98, Meh09], i.e. an infinitesimal automorphism of A. The coboundaries are the Lie derivatives L_X, which may be considered inner infinitesimal automorphisms. The cohomology $H^0(A; TA[1])$ is then the space of outer infinitesimal automorphisms.

Let χ be a degree 1 derivation, and consider the operator $d_A + \chi h$, where h is a formal parameter. Obviously, $(d_A + \chi h)^2$ vanishes to order h^2 if and only if $[d_A, \chi] = 0$. Thus, the degree 1 cocycles correspond to infinitesimal deformations of the Lie algebroid structure on A. The coboundaries consist of those "trivial" infinitesimal deformations that come from pulling back d_A along infinitesimal bundle automorphisms of A. In this sense, $H^1(A; TA[1])$ controls the infinitesimal deformations of A.

The degree 2 cohomology arises when one wants to extend an infinitesimal deformation to higher order. For example, suppose that χ, as above, is a degree 1 cocycle. Then χ^2 is a degree 2 cocycle. If $\chi^2 = -[d_A, \nu]$, then $(d_A + \chi h + \nu h^2)^2$ vanishes to order h^3. More generally, given a formal operator $d_A + \sum_{i=1}^{k} \chi_i h^i$ whose square vanishes to order h^k, one can find a χ_{k+1} such that $(d_A + \sum_{i=1}^{k+1} \chi_i h^i)^2$ vanishes to order h^{k+1} if an obstruction in $H^2(A; TA[1])$ depending on the χ_i vanishes.

It was already observed by Crainic and Moerdijk [CM08] that the differential graded Lie algebra of derivations of $\Omega(A)$ is isomorphic, up to a degree shift, with their *deformation complex* of A, consisting of k-ary antisymmetric brackets on $\Gamma(A)$ satisfying Leibniz rules. The isomorphism can be described in terms of derived brackets, as follows. Let χ be a degree k derivation of $\Omega(A)$. Then we may define a $(k+1)$-ary bracket $[\![\cdot, \dots, \cdot]\!]_\chi$ on $\Gamma(A)$ by

$$\iota_{[\![X_1,\dots,X_{k+1}]\!]_\chi} = [[\cdots[[\chi, \iota_{X_1}], \iota_{X_2}], \cdots], \iota_{X_{k+1}}]$$

for $X_1, \dots, X_{k+1} \in \Gamma(A)$. Antisymmetry of $[\![\cdot, \dots, \cdot]\!]_\chi$ follows from the Jacobi identity and the fact that contraction operators commute. The Leibniz rule follows immediately from the fact that the Lie bracket of derivations satisfies the Leibniz rule.

6. Tensor products, direct sums, and duals

Let $A \to M$ be a Lie algebroid, and let $(\mathcal{B}_1, Q_1)$ and $(\mathcal{B}_2, Q_2)$ be A-modules. Then there is a natural A-module structure on $\mathcal{B}_1 \otimes \mathcal{B}_2$, given by

$$Q(\beta_1 \otimes \beta_2) = (Q_1\beta_1) \otimes \beta_2 + (-1)^{|\beta_1|}\beta_1 \otimes (Q_2\beta_2)$$

for $\beta_i \in \Gamma(\mathcal{B}_i)$. The tensor product is symmetric, in the sense that the Koszul isomorphism from $\mathcal{B}_1 \otimes \mathcal{B}_2$ to $\mathcal{B}_2 \otimes \mathcal{B}_1$, taking $\beta_1 \otimes \beta_2$ to $(-1)^{|\beta_1||\beta_2|}\beta_2 \otimes \beta_1$, is an A-module isomorphism.

Similarly, the direct sum $\mathcal{B}_1 \oplus \mathcal{B}_2$ inherits an A-module structure, given by

$$Q(\beta_1 + \beta_2) = Q_1\beta_1 + Q_2\beta_2.$$

Next, we consider duals. Let $(\mathcal{B}, Q)$ be an A-module, and let $\mathcal{B}^* \to A[1]$ be the vector bundle dual to $\mathcal{B}$. We denote by $\langle \cdot, \cdot \rangle$ the pairing taking $\Gamma(\mathcal{B}^*) \otimes \Gamma(\mathcal{B})$

to $C^\infty(A[1]) = \Omega(A)$. The induced A-module structure Q^* on $\mathcal{B}^*$ is uniquely determined by the equation

$$d_A\langle b, \beta\rangle = \langle Q^* b, \beta\rangle + (-1)^{|b|}\langle b, Q\beta\rangle \tag{6}$$

for $b \in \Gamma(\mathcal{B}^*)$ and $\beta \in \Gamma(\mathcal{B})$. Note that requiring (6) to hold is equivalent to asking that the pairing $\langle\cdot,\cdot\rangle$ be an A-module morphism from $\mathcal{B}^* \otimes \mathcal{B}$ to the trivial rank 1 A-module $(A[1] \times \mathbb{R}, d_A)$.

Dualization takes vector bundles that are bounded in degree from below to those that are bounded from above, and vice versa. The property of being bounded on both sides is preserved by dualization.

Proposition 6.1. *The pairing between $\Gamma(\mathcal{B})$ and $\Gamma(\mathcal{B}^*)$ induces a well-defined cohomology pairing $H^\bullet(A; \mathcal{B}^*) \otimes H^\bullet(A; \mathcal{B}) \to H^\bullet(A)$.*

Proof. From (6), we have that $\langle b, \beta\rangle$ is closed if both b and β are closed, and that $\langle b, \beta\rangle$ is exact if one of b or β is exact and the other is closed. Therefore, the map taking $[b] \otimes [\beta]$ to $[\langle b, \beta\rangle]$ is well-defined at the level of cohomology. □

In the case where M is compact and orientable, one can obtain $\mathbb{R}$-valued pairings parametrized by cohomology with values in the (canonically decomposed) Berezinian A-module $\mathrm{Ber} := \pi_A^*(\wedge^{\mathrm{top}} A \otimes \wedge^{\mathrm{top}} T^*M)$. This is done by composing the pairing of Proposition 6.1 with that of Evens, Lu, and Weinstein [ELW99].

7. Characteristic classes

Chern-Simons type characteristic classes associated to ∞-representations were constructed in [GSM10]. In the 2-term case, it was shown there that the characteristic classes are gauge-invariant, so they can be interpreted as $\mathcal{VB}$-algebroid invariants. We will recall the construction, and we will show that the gauge-invariance property holds in full generality. We note that the arguments here are essentially the same as those in [GSM10]; however, the presentation here is intended to be more geometrically intuitive.

Let $A \to M$ be a Lie algebroid, and let $\mathcal{E} \to M$ be a graded vector bundle that is bounded in degree. We recall the notion of *A-superconnection*.

Definition 7.1 ([GSM10])**.** An *A-superconnection* on $\mathcal{E}$ is a degree 1 operator $\mathcal{D}$ on $\Omega(A; \mathcal{E})$ satisfying the Leibniz rule (5). An A-superconnection is called *flat* if $\mathcal{D}^2 = 0$.

Clearly, a flat A-superconnection is the same thing as an ∞-representation. In general, a version of Chern-Weil theory gives obstructions to the existence of ∞-representations. Specifically, given an A-superconnection $\mathcal{D}$, one can obtain Chern-Weil forms

$$\mathrm{ch}_k(\mathcal{D}) := \mathrm{str}(\mathcal{D}^{2k}) \in \Omega^{2k}(A),$$

where str denotes the supertrace.

Proposition 7.2. *For each k, the form $\mathrm{ch}_k(\mathcal{D})$ is closed, and the cohomology class of $\mathrm{ch}_k(\mathcal{D})$ is independent of $\mathcal{D}$.*

Before proving Proposition 7.2, we give the following Lemma.

Lemma 7.3. *Let $\mathcal{D}$ be an A-superconnection on $\mathcal{E}$, and let θ be an $\mathrm{End}(\mathcal{E})$-valued A-form. Then $[\mathcal{D}, \theta]$ is an $\mathrm{End}(\mathcal{E})$-valued A-form, and*

$$\mathrm{str}([\mathcal{D}, \theta]) = d_A\, \mathrm{str}(\theta).$$

Proof. Locally, $\mathcal{D}$ may be written as $d_A + \eta$, where η is an $\mathrm{End}(\mathcal{E})$-valued A-form. The result follows from the fact that $\mathrm{str}([\eta, \theta]) = 0$. $\square$

Proof of Proposition 7.2. We have that $\mathcal{D}^{2k}$ is an $\mathrm{End}(\mathcal{E})$-valued A-form, so by Lemma 7.3, we have that $d_A \,\mathrm{str}(\mathcal{D}^{2k}) = \mathrm{str}([\mathcal{D}, \mathcal{D}^{2k}]) = 0$. This proves that $\mathrm{ch}_k(\mathcal{D})$ is closed.

The independence of the cohomology class of $\mathrm{ch}_k(\mathcal{D})$ on $\mathcal{D}$ will be an immediate consequence of Proposition 7.4. $\square$

In the case where $\mathcal{D}$ is an ∞-representation, the Chern-Weil forms $\mathrm{ch}_k(\mathcal{D})$ obviously vanish. However, given a pair of ∞-representations, one can construct Chern-Simons type transgression forms, as follows.

Let I be the unit interval, and consider the product Lie algebroid $A \times TI \to M \times I$. Let $\{t, \dot{t}\}$ be the canonical coordinates on $T[1]I$. Any Lie algebroid q-form $\xi \in \Omega^q(A \times TI)$ can be uniquely written as $\xi_0(t) + \dot{t}\xi_1(t)$, where ξ_0 and ξ_1 are t-dependent elements of $\Omega^q(A)$ and $\Omega^{q-1}(A)$, respectively. The Berezin integral

$$\int \xi := \int_{T[1]I} \mathrm{d}t\, \mathrm{d}\dot{t}\, \xi = \int_0^1 \mathrm{d}t\, \xi_1$$

defines a degree -1 map from $\Omega(A \times TI)$ to $\Omega(A)$. The differential on $\Omega(A \times TI)$ is

$$d_{A\times TI} = d_A + \dot{t}\frac{\partial}{\partial t},$$

and a straightforward computation shows that the equation

$$\int d_{A\times TI}\xi + d_A \int \xi = \xi_0(1) - \xi_0(0) \tag{7}$$

holds for all $\xi \in \Omega(A \times TI)$.

Let p be the projection map from $M \times I$ to M. Given a pair of A-superconnections $\mathcal{D}_0$ and $\mathcal{D}_1$ on $\mathcal{E}$, we can form an $(A \times TI)$-superconnection $\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1}$ on $p^*\mathcal{E}$, given by

$$\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1}(a) = t\mathcal{D}_1(a) + (1-t)\mathcal{D}_0(a),$$

where $a \in \Gamma(\mathcal{E})$ is viewed as a t-independent section of $p^*\mathcal{E}$. The *transgression forms* $\mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_1) \in \Omega^{2k-1}(A)$ are defined as

$$\mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_1) := \int \mathrm{ch}_k(\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1}) = \int \mathrm{str}(\mathcal{T}^{2k}_{\mathcal{D}_0,\mathcal{D}_1}).$$

Proposition 7.4. $d_A\mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_1) = \mathrm{ch}_k(\mathcal{D}_1) - \mathrm{ch}_k(\mathcal{D}_0)$. *In particular, if* $\mathcal{D}_0$ *and* $\mathcal{D}_1$ *are* ∞*-representations, then* $\mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_1)$ *is closed.*

Proof. Let $\xi := \mathrm{ch}_k(\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1}) = \mathrm{str}(\mathcal{T}^{2k}_{\mathcal{D}_0,\mathcal{D}_1}) \in \Omega^{2k}(A \times TI)$. By Proposition 7.2, we have that $d_{A\times TI}\xi = 0$. Equation (7) then implies that

$$d_A\mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_1) = \xi_0(1) - \xi_0(0). \tag{8}$$

To compute the right side of (8), we first calculate

$$\mathcal{T}^2_{\mathcal{D}_0,\mathcal{D}_1} = \dot{t}(\mathcal{D}_1 - \mathcal{D}_0) + t^2\mathcal{D}_1^2 + (1-t)^2\mathcal{D}_0^2 + t(1-t)[\mathcal{D}_0, \mathcal{D}_1], \tag{9}$$

so that

$$\mathcal{T}^2_{\mathcal{D}_0,\mathcal{D}_1}(1) = \dot{t}(\mathcal{D}_1 - \mathcal{D}_0) + \mathcal{D}_1^2,$$
$$\mathcal{T}^2_{\mathcal{D}_0,\mathcal{D}_1}(0) = \dot{t}(\mathcal{D}_1 - \mathcal{D}_0) + \mathcal{D}_0^2.$$

It follows that

$$\mathcal{T}^{2k}_{\mathcal{D}_0,\mathcal{D}_1}(1) = \mathcal{D}_1^{2k} + \mathcal{O}(\dot{t}),$$
$$\mathcal{T}^{2k}_{\mathcal{D}_0,\mathcal{D}_1}(0) = \mathcal{D}_0^{2k} + \mathcal{O}(\dot{t}).$$

We conclude that the right side of (8) is $\mathrm{str}(\mathcal{D}_1^{2k}) - \mathrm{str}(\mathcal{D}_0^{2k}) = \mathrm{ch}_k(\mathcal{D}_1) - \mathrm{ch}_k(\mathcal{D}_0)$. □

Remark 7.5. If $\mathcal{D}_0$ and $\mathcal{D}_1$ are ∞-representations, then (9) reduces to

$$\mathcal{T}^2_{\mathcal{D}_0,\mathcal{D}_1} = \dot{t}(\mathcal{D}_1 - \mathcal{D}_0) + t(1-t)[\mathcal{D}_0, \mathcal{D}_1].$$

Using the fact that $\mathcal{D}_0$ and $\mathcal{D}_1$ both commute with $[\mathcal{D}_0, \mathcal{D}_1]$, we see that

$$\mathcal{T}^{2k}_{\mathcal{D}_0,\mathcal{D}_1} = k\dot{t}t^{k-1}(t-1)^{k-1}(\mathcal{D}_1 - \mathcal{D}_0)[\mathcal{D}_0, \mathcal{D}_1]^{k-1} + t^k(1-t)^k[\mathcal{D}_0, \mathcal{D}_1]^k.$$

The Berezin integral of $\mathcal{T}^{2k}_{\mathcal{D}_0,\mathcal{D}_1}$ can then be explicitly computed, giving us the simple formula

$$\mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_1) = P_k \,\mathrm{str}\left((\mathcal{D}_1 - \mathcal{D}_0)[\mathcal{D}_0, \mathcal{D}_1]^{k-1}\right), \tag{10}$$

where the constant P_k is

$$P_k = k\int_0^1 t^{k-1}(1-t)^{k-1}\mathrm{d}t = \frac{k!(k-1)!}{(2k-1)!}.$$

The following two propositions describe important properties satisfied by the forms $\mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_1)$. The first is a sort of "triangle identity", and the second asserts that the cohomology classes are stable under $\Omega(A)$-module automorphisms.

Proposition 7.6. *Let $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_2$ be ∞-representations of A on $\mathcal{E}$. Then*

$$\mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_1) + \mathrm{cs}_k(\mathcal{D}_1, \mathcal{D}_2) - \mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_2)$$

is exact.

Proof. Consider the transgression form $\xi := \mathrm{cs}_k(\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1}, \mathcal{T}_{\mathcal{D}_0,\mathcal{D}_2}) \in \Omega^{2k-1}(A \times TI)$. By Proposition 7.4, we have that

$$d_{A\times TI}\xi = \mathrm{ch}_k(\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_2}) - \mathrm{ch}_k(\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1}).$$

Applying the Berezin integral to both sides and using (7), we get

$$-d_A \int \xi + \xi_0(1) - \xi_0(0) = \mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_2) - \mathrm{cs}_k(\mathcal{D}_0, \mathcal{D}_1).$$

To complete the proof, we need to compute the terms $\xi_0(1)$ and $\xi_0(0)$.

Letting s be the coordinate on the second copy of I, we write

$$\mathcal{T}_{\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1},\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_2}}(a) = st\mathcal{D}_2(a) + (1-s)t\mathcal{D}_1(a) + (1-t)\mathcal{D}_0(a)$$

for a an s- and t-independent section of the pullback of $\mathcal{E}$ to $M \times I \times I$. Then

$$\begin{aligned}\mathcal{T}^2_{\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1},\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_2}} =& \dot{s}t(\mathcal{D}_2 - \mathcal{D}_1) + \dot{t}(s\mathcal{D}_2 + (1-s)\mathcal{D}_1 - \mathcal{D}_0) + s(1-s)t^2[\mathcal{D}_2, \mathcal{D}_1] \\ &+ st(1-t)[\mathcal{D}_2, \mathcal{D}_0] + (1-s)t(1-t)[\mathcal{D}_1, \mathcal{D}_0].\end{aligned}$$

We observe that integration with respect to s and $\dot{s}$ commutes with evaluation of t and $\dot{t}$, so we may evaluate first. We see that

$$\mathcal{T}^2_{\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1},\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_2}}(t=1) = \dot{s}(\mathcal{D}_2 - \mathcal{D}_1) + \dot{t}(s\mathcal{D}_2 + (1-s)\mathcal{D}_1 - \mathcal{D}_0) + s(1-s)[\mathcal{D}_2, \mathcal{D}_1],$$
$$\mathcal{T}^2_{\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1},\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_2}}(t=0) = \dot{t}(s\mathcal{D}_2 + (1-s)\mathcal{D}_1 - \mathcal{D}_0),$$

so that

$$\mathcal{T}^{2k}_{\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1},\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_2}}(t=1) = (\dot{s}(\mathcal{D}_2 - \mathcal{D}_1) + s(1-s)[\mathcal{D}_2, \mathcal{D}_1])^k + \mathcal{O}(\dot{t}),$$
$$\mathcal{T}^{2k}_{\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_1},\mathcal{T}_{\mathcal{D}_0,\mathcal{D}_2}}(t=0) = \mathcal{O}(\dot{t}).$$

Therefore,

$$\xi_0(1) = \int \operatorname{str}(\dot{s}(\mathcal{D}_2 - \mathcal{D}_1) + s(1-s)[\mathcal{D}_2, \mathcal{D}_1])^k = \int \operatorname{str}(\mathcal{T}^{2k}_{\mathcal{D}_1,\mathcal{D}_2}) = \operatorname{cs}_k(\mathcal{D}_1, \mathcal{D}_2)$$

and $\xi_0(0) = 0$. □

Proposition 7.7. *Let u_r be a smooth path of degree-preserving $\Omega(A)$-module automorphisms of $\Omega(A;\mathcal{E})$ such that $u_0 = \mathrm{id}$. Let $\mathcal{D}$ be an ∞-representation of A on $\mathcal{E}$, and let $\mathcal{D}_r := u_r^{-1}\mathcal{D}_0 u_r$. Then $\operatorname{cs}_k(\mathcal{D}_0, \mathcal{D}_1)$ is exact.*

Proof. Write $\mathcal{D}_r = \mathcal{D}_0 + \theta_r$, where θ_r is a path of $\operatorname{End}(\mathcal{E})$-valued A-forms. Since $\mathcal{D}_r^2 = 0$, we have that

$$[\mathcal{D}_0, \mathcal{D}_r] = [\mathcal{D}_0, \theta_r] = -\theta_r^2. \tag{11}$$

From (9) we have

$$\mathcal{T}^2_{\mathcal{D}_0,\mathcal{D}_r} = \dot{t}\theta_r + t(1-t)[\mathcal{D}_0, \mathcal{D}_r] = \dot{t}\theta_r - t(1-t)\theta_r^2,$$

so

$$\mathcal{T}^{2k}_{\mathcal{D}_0,\mathcal{D}_r} = k(t^2-t)^{k-1}\dot{t}\theta_r^{2k-1} + (t^2-t)^k\theta_r^{2k}.$$

It follows that $\operatorname{cs}_k(\mathcal{D}_0, \mathcal{D}_r)$ is proportional to $\operatorname{str}(\theta_r^{2k-1})$, so that $\frac{\mathrm{d}}{\mathrm{d}r}\operatorname{cs}_k(\mathcal{D}_0, \mathcal{D}_r)$ is proportional to

$$(2k-1)\operatorname{str}\left(\frac{\mathrm{d}\theta_r}{\mathrm{d}r}\theta_r^{2k-2}\right). \tag{12}$$

We wish to show that (12) is exact. First, we compute that

$$\begin{aligned}
\frac{\mathrm{d}\theta_r}{\mathrm{d}r} = \frac{\mathrm{d}\mathcal{D}_r}{\mathrm{d}r} &= \frac{\mathrm{d}u_r^{-1}}{\mathrm{d}r}\mathcal{D}_0 u_r + u_r^{-1}\mathcal{D}_0\frac{\mathrm{d}u_r}{\mathrm{d}r} \\
&= \frac{\mathrm{d}u_r^{-1}}{\mathrm{d}r}u_r\mathcal{D}_r + \mathcal{D}_r u_r^{-1}\frac{\mathrm{d}u_r}{\mathrm{d}r} \\
&= \left[\mathcal{D}_r, u_r^{-1}\frac{\mathrm{d}u_r}{\mathrm{d}r}\right].
\end{aligned}$$

Using the property $[\mathcal{D}_r, \theta_r^2] = 0$, which follows from (11), we deduce that

$$\frac{\mathrm{d}\theta_r}{\mathrm{d}r}\theta_r^{2k-2} = \left[\mathcal{D}_r, u_r^{-1}\frac{\mathrm{d}u_r}{\mathrm{d}r}\theta^{2k-2}\right].$$

Therefore, using Lemma 7.3, we have that (12) equals

$$(2k-1)\operatorname{str}\left(\left[\mathcal{D}_r, u_r^{-1}\frac{\mathrm{d}u_r}{\mathrm{d}r}\theta^{2k-2}\right]\right) = (2k-1)d_A\operatorname{str}\left(u_r^{-1}\frac{\mathrm{d}u_r}{\mathrm{d}r}\theta^{2k-2}\right),$$

which is exact, as desired. □

The transgression form construction allows us to define characteristic classes associated to a single ∞-representation $\mathcal{D}$, as follows. Choose a metric on E_i for each i. We may use the metric to obtain an adjoint operator $\mathcal{D}^\dagger$ on $\Omega(A;\mathcal{E})$, given by the equation

$$d_A\langle\omega_1, \omega_2\rangle = \langle\mathcal{D}\omega_1, \omega_2\rangle + (-1)^{|\omega_1|}\langle\omega_1, \mathcal{D}^\dagger\omega_2\rangle.$$

The operator $\mathcal{D}^\dagger$ satisfies the Leibniz rule and squares to zero, but it is generally not homogeneous of degree 1; we say that it is a *nonhomogeneous ∞-representation* or *nonhomogeneous flat A-superconnection*. We observe that the definitions and proofs from earlier in this section carry over verbatim to the nonhomogeneous case, with the only difference being that the Chern-Weil and Chern-Simons forms may be nonhomogeneous.

Given an ∞-representation $\mathcal{D}$, define the Chern-Simons forms associated to $\mathcal{D}$ as

$$\mathrm{cs}_k(\mathcal{D}) := \mathrm{cs}_k(\mathcal{D}, \mathcal{D}^\dagger).$$

The following theorem implies that the cohomology classes of $\mathrm{cs}_k(\mathcal{D})$ are well-defined invariants of Lie algebroid modules.

Theorem 7.8. *The cohomology classes $[\mathrm{cs}_k(\mathcal{D})]$ are independent of the choice of metric and invariant with respect to gauge transformations.*

Proof. The space of metrics is convex, hence path-connected. Given a path of metrics $\langle \cdot, \cdot \rangle_r$, let $u_r \in \mathrm{End}(\mathcal{E})$ be given by

$$\langle a, a' \rangle_r = \langle u_r(a), a' \rangle_0.$$

Then the corresponding adjoint operators satisfy the equation

$$\mathcal{D}^{\dagger_r} = u_r^{-1} \mathcal{D}^{\dagger_0} u_r.$$

Metric-independence then follows directly from Propositions 7.6 and 7.7.

Similarly, gauge-invariance follows from Propositions 7.6 and 7.7. For this, we use the fact that the space of gauge transformations is an affine space modeled on $\bigoplus_{i,k} \Omega^i(A) \otimes \mathrm{Hom}(E_k, E_{k-i})$, hence path-connected. □

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