EXTRINSIC HYPERSPHERES IN MANIFOLDS WITH SPECIAL HOLONOMY

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ABSTRACT. We describe extrinsic hyperspheres and totally geodesic hypersurfaces in manifolds with special holonomy. In particular we prove the nonexistence of extrinsic hyperspheres in quaternion-Kähler manifolds. We develop a new approach to extrinsic hyperspheres based on the classification of special Killing forms.

1. INTRODUCTION

A submanifold of a Riemannian manifold is called an extrinsic sphere if it is totally umbilical and has non-zero parallel mean curvature vector field. This concept was introduced by Nomizu and Yano in [15] as a natural analogue to ordinary spheres in Euclidean spaces. Extrinsic spheres have been studied intensively during the last fifty years. In several cases it was shown that extrinsic spheres have to be Euclidean spheres and partial classifications were obtained.

In this article we will only consider the case of extrinsic hyperspheres, i.e. extrinsic spheres of codimension one. In general, the existence of extrinsic hyperspheres seems to impose strong restrictions on the geometry of the ambient manifold, e.g. Chen and Nagano [6] showed that a locally irreducible symmetric space admitting an extrinsic hyperspheres has to be of constant curvature. However there are also interesting examples of extrinsic spheres which are not isometric to ordinary spheres. In particular Sasakian manifolds appear as extrinsic hyperspheres of Kähler manifolds (cf. [20]). Hence it is natural to ask for the existence of extrinsic hyperspheres in manifolds with *special* holonomy, i.e. manifolds whose restricted holonomy group is strictly contained in the corresponding special orthogonal group. By the Berger-Simons holonomy theorem, we have to consider the following cases: The manifold can be locally a Riemannian product, a locally symmetric space or its restricted holonomy group is one of $U(m)$, $SU(m)$, $Sp(m)$, $Sp(m) \cdot Sp(1)$, G_2 or $Spin(7)$. Our first result concerns quaternion-Kähler manifolds, i.e. Riemannian manifolds with restricted holonomy contained in $Sp(m) \cdot Sp(1)$, with $m \geq 2$. We prove the following

Theorem 1.1. *A quaternion-Kähler manifold of non-vanishing scalar curvature does not admit an extrinsic hypersphere.*

Similarly we show that there are no extrinsic hyperspheres in complete Riemannian products without one-dimensional factors and in complete manifolds with holonomy G_2 or Spin_7. However we give non-complete examples, as metric cones over manifolds with special geometric structures, such as Sasakian, nearly Kähler or nearly parallel G_2-structures. In fact every manifold is an extrinsic hypersphere in its (non-complete) metric cone.

Our main observation in the proof is that a parallel form on the ambient manifold naturally defines a so-called special Killing form on any extrinsic hypersphere. Then we use the classification of special Killing forms (cf. [17]) and in particular the fact that these forms define parallel forms on the metric cone. This gives a unified approach to the investigation of extrinsic hyperspheres, which also reproves some of the known results in the local product and Kähler case.

At some points we also use a remarkable theorem of Koiso (cf. [11] or Theorem 2.3). In particular this theorem states that a complete Einstein manifold of non-constant sectional curvature does not admit any extrinsic hypersphere which is itself Einstein and has positive scalar curvature. As a striking consequence we note that, contrary to the general expectation, it is not possible to construct new examples of 6-dimensional nearly Kähler manifolds as totally umbilical hypersurfaces of complete nearly parallel G_2-manifolds.

Finally we consider totally geodesic hypersurfaces in manifolds with special holonomy. We first show that the problem of finding totally geodesic hypersurfaces in a locally reducible manifold reduces to the same problem for one of the locally defined factors, see Theorem 4.2. Our main result in the irreducible case is then the following

Theorem 1.2. *There do not exist any totally geodesic hypersurfaces in*

(1) *locally irreducible Kähler-Einstein manifolds (including Calabi-Yau and hyperkähler manifolds);*
(2) *Quaternion-Kähler manifolds*
(3) *manifolds with holonomy* G_2 *or* $\mathrm{Spin}(7)$.
(4) *locally irreducible symmetric spaces of non-constant sectional curvature.*

In particular, Theorems 1.1 and 1.2 imply that a complete quaternion-Kähler manifold does not admit any (possibly non-complete) totally umbilical hypersurface.

2. Preliminaries

Let $(\bar{M}, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold and let $i : M \subset \bar{M}$ be a submanifold with induced Riemannian metric g. The second fundamental form is defined as $\mathrm{II}(X, Y) = \bar{\nabla}_X Y - \nabla_X Y$ where X and Y are vector fields tangent to M and ∇ resp. $\bar{\nabla}$ denote the Levi-Civita connections of g resp. $\bar{g}$. Let N be a normal vector field on M then the shape operator $A_N X := (\bar{\nabla}_X N)^T$ is related to the second fundamental form via

$$\bar{g}(\mathrm{II}(X, Y), N) = \bar{g}(A_N X, Y) \ ,$$

for any vector fields X, Y on M. A submanifold $M \subset \bar{M}$ is said to be *totally umbilical* if $\mathrm{II}(X, Y) = g(X, Y)H$, with $H = \frac{1}{n}\mathrm{tr}\,\mathrm{II}$ denoting the *mean curvature* vector field of M in

$\bar{M}$. Choosing a parallel unit length normal vector field N, this condition can be written as $\mathrm{II}(X,Y) = \lambda\, g(X,Y)\, N$ for some function λ on M. The manifold M is called *totally geodesic* in $\bar{M}$ if the equation $\mathrm{II} = 0$ holds, corresponding to the special case $\lambda = 0$.

In this article we are especially interested in *extrinsic hyperspheres*, i.e. complete hypersurfaces such that $\mathrm{II}(X,Y) = \lambda g(X,Y)N$ for some real constant $\lambda \neq 0$.

Let $M \subset \bar{M}$ be a totally umbilical hypersurface, with unit length normal vector field N, then the covariant derivative $\bar{\nabla}$ may be written as

$$(1) \qquad \bar{\nabla}_X Y = \nabla_X Y + \lambda g(X,Y)N, \qquad \bar{\nabla}_X N = -\lambda X \ ,$$

where X, Y denote vector fields tangent to M. For totally umbilical hypersurfaces the curvature equations of Gauß and Codazzi take the following form:

$$
\begin{aligned}
\bar{R}(X,Y,Z,W) &= R(X,Y,Z,W) + \lambda^2 g(X \wedge Y, Z \wedge W) \\
\bar{R}(X,Y,Z,N) &= X(\lambda)\, g(Y,Z) - Y(\lambda)\, g(X,Z) = (d\lambda \wedge Z)(X,Y)
\end{aligned}
$$

where X, Y, Z, W are vector fields on M, and $\bar{R}$ resp. R denote the Riemannian curvature tensors of $\bar{g}$ resp. g, and Z is identified with its dual 1-form using the metric g. Let the curvature operator R on 2-vectors be defined by

$$g(R(X \wedge Y), Z \wedge W) = -R(X,Y,Z,W) \ ,$$

so that the curvature operator of the standard sphere is the identity. Then the Gauß equation may also be written as $R = \bar{R} + \lambda^2 \,\mathrm{id}$. Hence the sectional curvatures $\bar{K}$ resp. K of $\bar{g}$ resp. g are related by $K = \bar{K} + \lambda^2$.

The following well-known lemma will be helpful below (cf. [11]).

Lemma 2.1. *Let $(M^n, g), n \geq 2$, be a totally umbilical hypersurface of an Einstein manifold $(\bar{M}, \bar{g})$. Then $\mathrm{II} = \lambda g N$, for some constant λ and a parallel unit length normal vector field N, i.e. complete totally umbilical hypersurfaces in Einstein manifolds are extrinsic spheres. Moreover,*

$$\lambda^2 = \frac{\mathrm{scal}_g}{n(n-1)} - \frac{\mathrm{scal}_{\bar{g}}}{n(n+1)}$$

and scal_g is constant. In particular the inequality $(n+1)\,\mathrm{scal}_g \geq (n-1)\,\mathrm{scal}_{\bar{g}}$ holds, with equality in the case of a totally geodesic hypersurface, i.e. for $\lambda = 0$.

Proof. Let $M \subset \bar{M}$ be a totally umbilical hypersurface in $\bar{M}$. By definition we have $\mathrm{II} = \lambda g N$ for some function λ on M. Let $\{e_i\}, i = 1, \ldots, n+1$, with $e_{n+1} := N$, be a local orthonormal frame for $T\bar{M}$ restricted to M. Then the Ricci curvature $\overline{\mathrm{Ric}}$ of $\bar{g}$ applied to a vector field X tangent to M can be computed using the Gauß equation:

$$
\begin{aligned}
\overline{\mathrm{Ric}}(X, X) \ &= \textstyle\sum_{i=1}^{n} \bar{R}(X, e_i, e_i, X) \ + \ \bar{R}(X, N, N, X) \\
&= \textstyle\sum_{i=1}^{n} (R(X, e_i, e_i, X) \ + \ \lambda^2 [g(X, e_i)^2 - g(X, X) g(e_i, e_i)]) \ + \ \bar{R}(X, N, N, X) \\
&= \mathrm{Ric}(X, X) \ - \ \lambda^2 (n-1) |X|^2 \ + \ \bar{R}(X, N, N, X)
\end{aligned}
$$

In this equation we take the trace over an orthonormal base in TM and use the assumption that $(\bar{M}, \bar{g})$ is Einstein to obtain

$$
n \frac{\mathrm{scal}_{\bar{g}}}{n+1} \ = \ \mathrm{scal}_g \ - \ n(n-1)\lambda^2 \ + \ \frac{\mathrm{scal}_{\bar{g}}}{n+1}
$$

This proves the equation for λ^2, the inequality and the characterization of the case of equality. It remains to show that λ, and thus also scal_g, is constant. This immediately follows from the Codazzi equation. Indeed if we take the trace over a local orthonormal frame on M we obtain for any vector field X on M

$$
\overline{\mathrm{Ric}}\,(X, N) \ = \ (n-1)\, d\lambda(X) \cdot
$$

Hence, $d\lambda = 0$ and we conclude that λ as well as scal_g have to be constant on M.

$\square$

Remark 2.2. Using a result of [13] on the existence of submanifolds with parallel second fundamental form (e.g. totally umbilical submanifolds with $\lambda = const$), one can show that in a complete manifold with real analytic metric every (possibly non-complete) submanifold with parallel second fundamental is contained in a complete one. Further, we recall that every Einstein metric is real analytic with respect to normal coordinates according to a theorem of DeTurck and Kazdan, see [11]. It follows that in a complete Einstein manifold every totally umbilical submanifold is an open part of an extrinsic sphere.

In general we will not assume that the ambient manifold $\bar{M}$ has to be complete. However if we assume completeness, as well as the Einstein condition for g and $\bar{g}$, the following theorem of Koiso (cf. [11]) gives a rather strong restriction for extrinsic hyperspheres.

Theorem 2.3 (Koiso). *Let (M, g) be a totally umbilical Einstein hypersurface in a complete Einstein manifold $(\bar{M}, \bar{g})$. Then the only possible cases are:*

(a) g has positive Ricci curvature. Then g and $\bar{g}$ have constant sectional curvature

(b) $\bar{g}$ has negative Ricci curvature. If $\bar{M}$ is compact or $(\bar{M}, \bar{g})$ homogeneous, then g and $\bar{g}$ have constant sectional curvature

(c) g and $\bar{g}$ have zero Ricci curvature. If $(\bar{M}, \bar{g})$ is simply connected, then $(\bar{M}, \bar{g})$ decomposes as $(\tilde{M}, \tilde{g}) \times \mathbb{R}$, where $(\tilde{M}, \tilde{g})$ is a totally geodesic hypersurface in $(\bar{M}, \bar{g})$ which contains M.

3. Extrinsic hyperspheres

In this section we will study totally umbilical submanifolds in ambient spaces with special holonomy.

3.1. **Special Killing forms.** Let $i : M \subset \bar{M}$ be an n-dimensional extrinsic hypersphere in a manifold $(\bar{M}, \bar{g})$ with special holonomy. Except for the case of symmetric spaces, the restriction of holonomy is directly linked to the existence of certain parallel differential forms $\sigma \in \Omega^k(\bar{M})$. The main tool in our investigation of extrinsic hyperspheres is the observation that the pull-back forms $i^*(N \lrcorner \sigma)$ and $i^*\sigma$ are special Killing resp. $*$-Killing forms on M (cf. [17]). Here and henceforth N denotes a unit normal vector field along M.

Lemma 3.1. *Let $i : M \subset \bar{M}$ be an n-dimensional extrinsic hypersphere and σ be a non-trivial parallel k-form on $\bar{M}$ and let $\gamma := i^*(N \lrcorner \sigma)$ and $\beta := i^*\sigma$ be the pull-back forms on M. Then for every vector field X on M the following equations hold:*

$$
\begin{aligned}
(i) && \nabla_X \gamma &= \tfrac{1}{k} X \lrcorner d\gamma \\
(ii) && \nabla_X d\gamma &= -k\lambda^2 X \wedge \gamma \\
(iii) && \nabla_X \beta &= -\tfrac{1}{n-k+1} X \wedge d^*\beta \\
(iv) && \nabla_X d^*\beta &= (n-k+1)\lambda^2 X \lrcorner \beta,
\end{aligned}
$$

where the non-zero constant λ is given by (1). In particular, it follows that γ is coclosed and β is closed. Moreover, the forms γ and β are related by

$$
d\gamma = -k\,\lambda\,\beta, \qquad d^*\beta = -(n-k+1)\lambda\,\gamma\,.
$$

Furthermore, γ is a non-parallel $k-1$-form on M.

Proof. Let $i : M \to \bar{M}$ the inclusion map of the extrinsic hypersphere M, then the differential i_* identifies T_pM with a subspace of $T_p\bar{M}$. We have $i_*(\nabla_X Y) = \bar{\nabla}_X Y - \lambda g(X,Y)N$, where X, Y are vector fields tangent to M. Let $X, X_1, \dots, X_k$ be vector fields on M then

$$
\begin{aligned}
(\nabla_X i^*\sigma)(X_1, \dots, X_k) &= X(\sigma(X_1, \dots, X_k)) - \sum_j \sigma(\dots, i_*(\nabla_X X_j), \dots) \\
&= (\bar{\nabla}_X \sigma)(X_1, \dots, X_k) + \lambda \sum_j g(X, X_j)\, \sigma(\dots, N, \dots) \\
&= \lambda\,(X \wedge i^*[N \lrcorner \sigma])(X_1, \dots, X_k)
\end{aligned}
$$

It immediately follows that $i^*\sigma$ is closed. Moreover, contracting with $X = e_k$ and summing over a local orthonormal base $\{e_k\}$ of TM yields $d^*(i^*\sigma) = -(n-k+1)\,\lambda\,i^*[N \lrcorner \sigma]$. Substituting this into the equation for $\nabla_X i^*\sigma$ proves (iii). Similarly we find

$$
\nabla_X i^*(N \lrcorner \sigma) = \bar{\nabla}_X(N \lrcorner \sigma) = (\bar{\nabla}_X N) \lrcorner i^*\sigma = -\lambda X \lrcorner i^*\sigma\,.
$$

This implies that $i^*(N \lrcorner \sigma)$ is coclosed and that $di^*(N \lrcorner \sigma) = -k\,\lambda\,i^*\sigma$, completing also the proof of equation (i). Finally we use the calculations above to conclude the proof of equations (ii) and (iv):

$$
\begin{aligned}
\nabla_X d\gamma &= -k\,\lambda\,\nabla_X \beta = -k\lambda^2 X \wedge \gamma, \\
\nabla_X d^*\beta &= -(n-k+1)\lambda \nabla_X \gamma = (n-k+1)\lambda^2 X \lrcorner \beta\,.
\end{aligned}
$$

Suppose, by contradiction, that γ is a parallel $k-1$-form on M. Then $d\gamma = 0$, hence $\beta = -\tfrac{1}{k\lambda}d\gamma = 0$ and thus $\gamma = -\tfrac{1}{(n-k+1)\lambda}d^*\beta = 0$. Therefore, $\sigma|_M = 0$ which is not possible since σ is a non-trivial parallel k-form on $\bar{M}$. We conclude that γ is not parallel. $\square$

Equations (i) and (ii) define a non-parallel special $(k-1)$-Killing form γ. Complete manifolds admitting such forms were classified in [17]. It turns out that special Killing forms can only exist on Euclidian spheres, Sasakian- and 3-Sasakian manifolds, nearly Kähler manifolds in dimension 6 or nearly parallel G_2-manifolds in dimension 7.

The classification is based on the fact that every special Killing $(k-1)$-form ψ defines a parallel k-form $\tilde{\psi}$ on the metric cone $\tilde{M}$, i.e. the manifold $\tilde{M} = M \times \mathbb{R}_+$ with the cone metric $\tilde{g} = t^2 g + dt^2$. Recall that the metric cone is a non-complete manifold, which contains the complete manifold M as an extrinsic hypersphere. The parallel form on $\tilde{M}$ is defined as $\tilde{\psi} = \frac{1}{k} d(t^k \psi) = t^{k-1} dt \wedge \psi + \frac{1}{k} t^k d\psi$

It is important to note that this construction assumes a certain normalization in equations (ii) and (iv), which in our case is equivalent to $\lambda^2 = 1$. Clearly, after a constant rescaling of the metric $\bar{g}$ and replacing N with $-N$ if necessary, one can even assume that $\lambda = -1$ in (1).

Returning to our situation, let $i : M \subset \bar{M}$ be an n-dimensional extrinsic hypersphere and σ be a non-trivial parallel k-form on $\bar{M}$. Restricted to the submanifold $M \subset \bar{M}$, we may write σ with the notation from above as

$$\sigma \;=\; N \wedge i^*(N \lrcorner \sigma) \,+\, i^*\sigma \;=\; N \wedge \gamma + \beta \,.$$

where γ is a non-parallel special $(k-1)$- Killing form such that $d\gamma = k\beta$. Thus we obtain that

$$\tilde{\sigma} \;=\; t^{k-1} dt \wedge \gamma \,+\, t^k \beta \,,$$

is a non-trivial parallel k-form on the cone $\tilde{M}$. Obviously, the k-forms σ and $\tilde{\sigma}$ have the same algebraic type. This implies that their stabilizer under the $\mathrm{SO}(n+1)$-action on Λ^k has to be the same.

It is well known that if (M, g) is complete, the metric cone $\tilde{M}$ has reducible holonomy only if it is flat, in which case M is isometric to the standard sphere [7]. Moreover, if the cone metric $\tilde{g}$ is Einstein, then it has to be Ricci flat. In particular, the metric cone can be symmetric only if it is flat. Indeed an irreducible symmetric space is Einstein, thus the cone is then Ricci flat and also flat. Similarly, the metric cone can not be a quaternion-Kähler manifold, since these manifolds are automatically Einstein. Thus the scalar curvature of the cone vanishes and the holonomy is reduced to $\mathrm{Sp}(m)$, i.e. the cone is in fact hyperkähler. According to the the Berger list, there remain five cases of irreducible cones $\tilde{M}$ admitting parallel forms: Kähler, Calabi-Yau, hyperkähler manifolds, and manifolds with holonomy G_2 resp. $\mathrm{Spin}(7)$, in dimensions 7 resp. 8. It follows that M has a Sasakian, Einstein-Sasakian, 3-Sasakian, nearly Kähler or nearly parallel G_2 structure, respectively.

3.2. **Quaternion-Kähler manifolds.** Let $(\bar{M}^{4m}, \bar{g})$ be a quaternion-Kähler manifold, i.e. a Riemannian manifolds with (restricted) holonomy contained in $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$. Since for $m = 1$ the holonomy condition is empty, one usually assumes $m \geq 2$. On quaternion-Kähler manifolds one has a parallel 4-form σ, the so-called Kraines form. Its stabilizer is the group $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4m)$.

Let $M \subset \bar{M}$ be an extrinsic hypersphere. Thus it admits a special Killing form and carries one of the special geometric structures mentioned above. Let us first assume that M is Sasakian, but not Einstein. Then the cone $\tilde{M}$ is an irreducible Kähler manifold with holonomy equal to $\mathrm{U}(2m)$. The parallel forms are powers of the Kähler form, whose stabilizers contain $\mathrm{U}(2m)$. But for $m \geq 2$ the unitary group $\mathrm{U}(2m)$ is not contained in $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$. Thus this case is not possible.

In the remaining cases, M is the standard sphere, Einstein-Sasakian, 3-Sasakian, nearly Kähler, or nearly parallel G_2, and is Einstein with positive scalar curvature $\mathrm{scal}_g = n(n-1)$. If $\bar{M}$ would be complete then we could apply the result of Koiso, i.e. Theorem 2.3, to rule out these cases. However, even if $\bar{M}$ is not complete, we may exclude the remaining possibilities. Indeed, the cone over M has to be Ricci flat and Lemma 2.1 shows that $\mathrm{scal}_{\bar{g}} = 0$. Thus the holonomy of $(\bar{M}, \bar{g})$ reduces further to $\mathrm{Sp}(m)$, which is a different case.

This proves Theorem 1.1.

$\square$

3.3. Kähler manifolds.
This case also includes Calabi-Yau and hyperkähler manifolds. It is well known that a Kähler form $\sigma \in \Omega^2(\bar{M})$ induces a Sasakian structure on any extrinsic hypersphere $M \subset \bar{M}$. The Killing vector field of the Sasakian structure is given by $\xi = JN = N \lrcorner \sigma$ (cf. [20]). Non-complete examples are obtained as metric cones over Sasakian, Einstein-Sasakian resp. 3-Sasakian manifolds.

However we do not know of any example of a complete Kähler manifold admitting an extrinsic hypersphere.

3.4. Manifolds with holonomy G_2 or $\mathrm{Spin}(7)$.
Let $(\bar{M}, \bar{g})$ be a manifold with holonomy contained in G_2 or $\mathrm{Spin}(7)$. Then $\bar{M}$ carries a parallel 3- resp. 4-form σ and the 2- resp. 3-form $N \lrcorner \sigma$ defines a nearly Kähler resp. nearly parallel G_2-structure on any extrinsic hypersphere $M \subset \bar{M}$. These manifolds are Einstein with positive scalar curvature and we may use the result of Koiso from Theorem 2.3 to exclude them as hypersurfaces of complete manifold $\bar{M}$.

Again there are non-complete examples $\bar{M}$, as metric cones over nearly Kähler resp. nearly parallel G_2-manifolds. Conversely it follows from [11, Eq. (2.3.b)] that any $\bar{M}$ with holonomy G_2 or $\mathrm{Spin}(7)$ admitting an extrinsic hypersphere M is locally isometric to the cone over M.

More generally, it is an old and well known observation of Gray [8] that already the existence of a nearly parallel G_2-structure on $\bar{M}^7$ implies the existence of a nearly Kähler structure on any totally umbilical hypersurface $M \subset \bar{M}$. However, as we have just seen, it is a striking consequence of Koiso's Theorem 2.3 that if $\bar{M}$ is complete, then no new examples of nearly Kähler manifolds can be produced in this way.

3.5. Local product manifolds.
Let $\bar{g}$ be a Riemannian product metric on $\bar{M} = \bar{M}_1 \times \bar{M}_2$. We assume that not both factors have dimension one. The volume forms $\mathrm{vol}_{M_1}, \mathrm{vol}_{M_2}$ are parallel forms on $\bar{M}$. Let $\bar{M}_1$ be the factor with a non-vanishing projection of the normal vector N. Then $N \lrcorner \mathrm{vol}_{\bar{M}_1}$ is different from zero and defines, as described above, a parallel form

of degree dim $\bar{M}_1$ on the cone over M. However this form has a non-trivial kernel (the vectors from $T\bar{M}_2 \cap TM$), which is at least one-dimensional and defines a parallel distribution on the cone. Hence the cone is reducible and, if M is complete, we apply the theorem of Gallot [7] to conclude that the cone is flat and M is isometric to the sphere. This result was first obtained by M. Okumura [16] (using the Obata Theorem).

Finally we remark that if we do not require the completeness condition for M and $\bar{M}$, then the situation is much more flexible and one can construct lots of examples by taking $(\bar{M}, \bar{g})$ to be a product of two Riemannian cones $(M_1 \times \mathbb{R}, t^2 g_1 + dt^2)$ and $(M_2 \times \mathbb{R}, s^2 g_2 + ds^2)$. Indeed, such a product is always a Riemannian cone over the manifold $M = M_1 \times M_2 \times \mathbb{R}$ endowed with the incomplete Riemannian metric $g := \sin^2 \theta \, g_1 + \cos^2 \theta \, g_2 + d\theta^2$, as shown by the formula (cf. [12])

$$(t^2 g_1 + dt^2) + (s^2 g_2 + ds^2) = r^2 (\sin^2 \theta \, g_1 + \cos^2 \theta \, g_2 + d\theta^2) + dr^2, \qquad (s, t) = (r \cos \theta, r \sin \theta).$$

The manifold (M, g) is thus embedded as a totally umbilical hypersurface in $(\bar{M}, \bar{g})$.

3.6. Locally symmetric spaces.

Extrinsic spheres in locally symmetric spaces are well understood. It follows from results of Chen [3] that the real space forms are the only irreducible locally symmetric spaces admitting extrinsic hyperspheres. Since every locally irreducible symmetric space is a complete Einstein manifold, this result is also implied by Theorem 2.3 of Koiso. Moreover, any extrinsic hypersphere in a symmetric space is a symmetric submanifold in the sense of [2, Ch. 9.3] (cf. [2, Proposition 9.3.1]). Therefore, if $\bar{M}$ is a product $\bar{M}_1 \times \cdots \times \bar{M}_k$ where $\bar{M}_i$ are simply connected irreducible symmetric spaces, it follows from a result of Naitoh [14] that any extrinsic hypersphere is of the form $M_1 \times \bar{M}_2 \times \cdots \times \bar{M}_k$ where M_1 is an extrinsic hypersphere in a space $\bar{M}_1$ of constant curvature.

Similar results are true for certain classes of homogeneous spaces. In [18] Tojo proves that compact normally homogeneous spaces admitting extrinsic hyperspheres have constant sectional curvature. The same conclusion is proved by Tsukada in [19] for isotropy irreducible homogeneous spaces admitting totally umbilical hypersurfaces.

4. Totally geodesic hypersurfaces

There are many examples of totally geodesic hypersurfaces in (possibly non-complete) Einstein manifolds. In fact Koiso proves in [11] the following

Theorem 4.1. *Let (M, g) be a real analytic Riemannian manifold with constant scalar curvature. Then there exists a (possibly non-complete) Einstein manifold $(\bar{M}, \bar{g})$ such that (M, g) is isometrically embedded into $(\bar{M}, \bar{g})$ as a totally geodesic hypersurface. Moreover, such $(\bar{M}, \bar{g})$ is essentially uniquely determined. More precisely, if $(\tilde{M}, \tilde{g})$ is a second Einstein manifold which contains M as a totally geodesic hypersurface, then there exist open neighborhoods $\bar{U}$ and $\tilde{U}$ of M in $\bar{M}$ and $\tilde{M}$, respectively, and an isometry $I : \tilde{U} \to \bar{U}$ with $I|_M = \mathrm{id}$.*

In this section we will show that the Einstein manifold $\bar{M}^{n+1}$ given by Koiso's Theorem 4.1 can never have special holonomy if (M, g) is locally irreducible. In fact, then $\bar{M}$ is locally

irreducible, too, because of Theorem 4.2. Thus we can apply Theorem 4.3 in order to obtain that the restricted holonomy group of $\bar{M}$ is given by $SO(n+1)$.

4.1. **Local Products.** We will first show that if $(\bar{M}, \bar{g})$ is locally reducible and complete, then the problem of finding totally geodesic hypersurfaces in $\bar{M}$ reduces to the same problem on one of the factors. More precisely, we will prove the following:

Theorem 4.2. *Let $(\bar{M}, \bar{g})$ be a complete, simply connected manifold with reducible holonomy, and assume that (M, g) is a complete totally geodesic hypersurface of $\bar{M}$. Then $\bar{M}$ can be written as a Riemannian product $(\bar{M}, \bar{g}) = (\bar{M}_1, \bar{g}_1) \times (\bar{M}_2, \bar{g}_2)$ such that M is equal to $M'_1 \times \bar{M}_2$, where M'_1 is a complete totally geodesic hypersurface of $\bar{M}_1$.*

Proof. Since $\bar{M}$ is complete, simply connected and has reducible holonomy, the de Rham decomposition theorem shows that it is isometric to a Riemannian product $\bar{M} = \bar{M}_1 \times \bar{M}_2$, with $\bar{g} = \bar{g}_1 + \bar{g}_2$. The exponential function clearly satisfies

$$(2) \qquad \exp^{\bar{M}}_{(x_1,x_2)}(X_1, X_2) = (\exp^{\bar{M}_1}_{x_1}(X_1), \exp^{\bar{M}_2}_{x_2}(X_2))$$

for all $(x_1, x_2) \in \bar{M}$ and $(X_1, X_2) \in T_{(x_1,x_2)}\bar{M}$.

Let $M \subset \bar{M}$ be a totally geodesic hypersurface with unit length normal vector field N. With respect to the decomposition $T\bar{M} = T\bar{M}_1 \oplus T\bar{M}_2$, the vector field N can be written as $N = X_1 + X_2$ at every point of M. If at some point $x = (x_1, x_2) \in M$ one component, e.g. X_1, vanishes, then $T_x M = T_{x_1}\bar{M}_1 \times X_2^\perp$, thus by (2) $M = \bar{M}_1 \times \exp_{x_2}(X_2^\perp)$, where the second factor is clearly a totally geodesic hypersurface in $\bar{M}_2$. In the following we will assume that both components of N are different from zero.

For every $x = (x_1, x_2) \in M$ we write $N_x = aN_1 + bN_2$ where a and b are functions on M and N_1, N_2 are unit vectors in $T_{x_1}\bar{M}_1$ and $T_{x_2}\bar{M}_2$ depending a priori on x_2 and x_1 respectively. We will show later on that they actually do not depend on these variables.

Let ω_i denote the restriction to M of the volume forms of the two factors of $\bar{M}$. Consider the vector field H on M defined by $g(H, \cdot) = *(*\omega_1 \wedge *\omega_2)$ (the Hodge dual $*$ is that of M). Up to a sign, depending on the orientation of M, one has $H = bN_1 - aN_2$. Since ω_i are parallel, H is a parallel vector field on M, so if φ_t denotes its flow, then $f_t(x)$ is a geodesic for all $x \in M$.

Let us fix some $x = (x_1, x_2) \in M$ and consider the totally geodesic surfaces $M_1 = (\bar{M}_1 \times \{x_2\}) \cap M$ and $M_2 = (\{x_1\} \times \bar{M}_2) \cap M$ of $\bar{M}_1$ and $\bar{M}_2$ respectively. The projection of $T_x M$ to $T_{x_1}\bar{M}_1$ is onto, therefore the projection $\pi_1 : M \to \bar{M}_1$ is onto. Indeed, for every $y_1 \in \bar{M}_1$ there exists $Y_1 \in T_{x_1}\bar{M}_1$ such that $y_1 = \exp^{\bar{M}_1}_{x_1}(Y_1)$, so by (2), $y_1 = \pi_1(\exp^{\bar{M}}_{(x_1,x_2)}(Y_1, Y_2))$, where Y_2 is chosen so that $(Y_1, Y_2) \in T_x M$.

We will now show that aN_1 only depends on x_1. Indeed, the set of $y_2 \in \bar{M}_2$ such that $(x_1, y_2) \in M$ is just M_2, and for every vector $Y_2 \in T_{x_2}M_2$ we have $\bar{g}(Y_2, N) = 0$, so $0 = \nabla^{\bar{g}}_{Y_2} N = \nabla^{\bar{g}}_{Y_2}(aN_1) + \nabla^{\bar{g}}_{Y_2}(bN_2)$. Since the two terms in the right hand factor are tangent to $\bar{M}_1$ and $\bar{M}_2$ respectively, they both vanish. In particular, $\nabla^{\bar{g}}_{Y_2}(aN_1) = 0$, and since N_1 has unit length, a and N_1 are both constant along M_2. This fact, together with the previous

observation that the projections of M on $\bar{M}_1$ and $\bar{M}_2$ are onto, show that there exist globally defined functions a on $\bar{M}_1$, b on $\bar{M}_2$ and vector fields $\bar{N}_1$ on $\bar{M}_1$, $\bar{N}_2$ on $\bar{M}_2$, such that $N_x = a(x_1)\bar{N}_1(x_1) + b(x_2)\bar{N}_2(x_2)$ for all $x = (x_1, x_2) \in M$.

We claim that $\bar{N}_1$ is parallel on $\bar{M}_1$. First, if $X \in T_{x_1}M_1$, then X is orthogonal to N so $\nabla_X(a\bar{N}_1) = 0$ like before. Since $\bar{N}_1$ has unit length, this shows that $X(a) = 0$, so a is constant along M_1 and $\nabla_X \bar{N}_1 = 0$. It remains to check the parallelism in the direction of $\bar{N}_1$ itself. Equation (2) shows that the geodesic $\exp_{x_1}(tb\bar{N}_1)$ is the projection in $\bar{M}_1$ of $\exp_x(tH)$, whose tangent vector at every t is H. Thus the tangent vector of $\exp_{x_1}(tbN_1)$ is $(bN_1)(\exp_{x_1}(tbN_1))$, showing that bN_1 is parallel in the direction of N_1. On the other hand, we have already seen that b only depends on the second variable, so N_1 is parallel at x_1, and thus everywhere on $\bar{M}_1$. Similarly, N_2 is parallel on $\bar{M}_2$. By the de Rham theorem again, one can write $\bar{M}_1 = M_1 \times \mathbb{R}$, $\bar{g}_1 = g_1 + dt^2$, $N_1 = \partial/\partial t$ and $\bar{M}_2 = M_2 \times \mathbb{R}$, $\bar{g}_2 = g_2 + ds^2$, $N_2 = \partial/\partial s$. From the above, the functions a and b only depend on t and s respectively, but since $a^2 + b^2 = 1$, they are both constant. This shows that identifying $\bar{M}$ with $\mathbb{R} \times (M_1 \times M_2 \times \mathbb{R})$ by the isometry $((t, x_1), (s, x_2)) \mapsto (at + bs, (x_1, x_2, bt - as))$, N is identified to the unit tangent vector to the $\mathbb{R}$-factor, and thus M is isometric to the second factor $M_1 \times M_2 \times \mathbb{R}$. This finishes the proof of the theorem.   □

4.2. Irreducible manifolds with totally geodesic hypersurfaces.
Now we turn our attention to the case where $(\bar{M}, \bar{g})$ is locally irreducible. Recall that a hypersurface M of a Riemannian manifold $(\bar{M}, \bar{g})$ is called *locally reflective* if the geodesic reflection r in M defines an isometry of a suitable open neighborhood U of M in $\bar{M}$. Then r is locally given by $r(\exp(tN_p)) = \exp(-tN_p)$ (where N_p denotes the normal vector at $p \in M$). Moreover, we recall that a locally reflective submanifold is automatically totally geodesic (cf. [2]).

If M is a totally geodesic hypersurface of an Einstein manifold $(\bar{M}, \bar{g})$, then M has constant scalar curvature according to Lemma 2.1. In this situation, N. Koiso has shown that M is a locally reflective submanifold, cf. Remark 7 of [11].

Theorem 4.3. *Let $(\bar{M}, \bar{g})$ be a locally irreducible Riemannian manifold. If there exists an n-dimensional locally reflective hypersurface $M \subset \bar{M}$, then the restricted holonomy group of $\bar{M}$ is equal to $\mathrm{SO}(n + 1)$. In particular, there are no totally geodesic hypersurfaces in locally irreducible Einstein manifolds with special holonomy.*

Proof. Let U be an open neighborhood of M in $\bar{M}$ in which the geodesic reflection r in M is defined. Clearly, it suffices to prove the theorem in case $U = \bar{M}$. Since $r(p) = p$ for all points $p \in M$, we obtain an involutive Lie group homomorphism $\tau : \mathrm{SO}(T_p\bar{M}) \to \mathrm{SO}(T_p\bar{M})$ which is given by $\tau(g) = d_p r \circ g \circ d_p r$. Since $d_p r$ is the linear reflection in T_pM and the normal vector N_p spans the whole normal space at p, the connected component of the fixed point group under τ is equal to $\mathrm{SO}(T_pM)$. Further, let G denote the restricted holonomy group of $\bar{M}$ at the point $p \in M$. Then for every closed, null-homotopic curve $\alpha : [0, 1] \to \bar{M}$ the curve $r \circ \alpha$ is again closed and null-homotopic. If g denotes the parallel displacement along α, then the parallel displacement along $r \circ \alpha$ is given by $\tau(g)$ (because r is an isometry of $\bar{M}$). We obtain that $\tau(g) \in G$ for all $g \in G$. Let H denote the subgroup of G which is fixed under τ and H_0

be its connected component. Set $\tilde{H} := H \cap \mathrm{SO}(T_pM)$, then $H_0 \subset \tilde{H} \subset H$, hence $(G, \tilde{H})$ is a Riemannian symmetric pair in the sense of [9, Ch. IV, §3]. In particular, any G-invariant metric makes $G/\tilde{H}$ a Riemannian symmetric space. Moreover, there is a natural injective map $\iota : G/\tilde{H} \to \mathrm{S}^n$ which is given by $[g] \mapsto g(N_p)$, where S^n is considered as the Euclidian sphere of $T_p\bar{M}$. We claim that ι is a totally geodesic map, i.e. ι maps geodesics of $G/\tilde{H}$ into geodesics of S^n:

Let $\mathfrak{p} := \{x \wedge N_p \,|\, x \in T_pM\}$ be the Cartan complement of $\mathfrak{k} := \mathfrak{so}(T_pM)$ in $\mathfrak{so}(T_p\bar{M})$. Then $\mathfrak{so}(T_p\bar{M}) = \mathfrak{k} \oplus \mathfrak{p}$ with $d_p\tau(A) = A$ for all $A \in \mathfrak{k}$ and $d_p\tau(A) = -A$ for all $A \in \mathfrak{p}$. Let $\mathfrak{g}$ denote the Lie algebra of G, then the Cartan decomposition of $\mathfrak{g}$ is given by $(\mathfrak{k} \cap \mathfrak{g}) \oplus (\mathfrak{p} \cap \mathfrak{g})$. Let γ be a geodesic of $G/\tilde{H}$ through the origin $\tilde{H}$. Then there exists some $A \in \mathfrak{p} \cap \mathfrak{g}$ such that $\gamma(t) = [\exp(tA)]$ (cf. [2]). Therefore, $\iota(\gamma(t)) = \exp(tA)N_p$, which is a geodesic line of S^n.

This shows that the dimension k of the symmetric space $G/\tilde{H}$ is less or equal n and the orbit GN_p is a k-dimensional totally geodesic submanifold of S^n, i.e. GN_p is a standard Euclidian sphere $\mathrm{S}^k \subset \mathrm{S}^n$. Then the linear subspace of $T_p\bar{M}$ which is spanned by GN_p is G-invariant and hence $k = n$, since G acts irreducibly on $T_p\bar{M}$. It follows that $\dim(\mathfrak{p} \cap \mathfrak{g}) = \dim(G/\tilde{H}) = n = \dim(\mathfrak{p})$ and thus $\mathfrak{p} \subset \mathfrak{g}$, therefore $\mathfrak{so}(T_p\bar{M}) = [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p} \subset \mathfrak{g}$. We obtain that actually $\mathfrak{g} = \mathfrak{so}(T_p\bar{M})$. Switching from Lie algebras to Lie groups, we conclude that the connected component of G is equal to $\mathrm{SO}(T_p\bar{M})$. The result now follows.

$\square$

This also proves Theorem 1.2, since all ambient manifolds in question are Einstein with special holonomy.

$\square$

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