GROWTH BEHAVIORS IN THE RANGE e^{r^α}

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Abstract. For every $\alpha \leq \beta$ in a left neighborhood $[\alpha_0, 1]$ of 1, a group $G(\alpha,\beta)$ is constructed, the growth function of which satisfies $\limsup \frac{\log\log b_{G(\alpha,\beta)}(r)}{\log r} = \alpha$ and $\liminf \frac{\log\log b_{G(\alpha,\beta)}(r)}{\log r} = \beta$. When $\alpha = \beta$, this provides an explicit uncountable collection of groups with growth functions strictly comparable. On the other hand, oscillation in the case $\alpha < \beta$ explains the existence of groups with non comparable growth functions. Some period exponents associated to the frequency of oscillation provide new group invariants.

1. Introduction

The growth function $b_{\Gamma,S}(r) = |S^r|$ of a group Γ with finite generating set S was introduced by Milnor [Mil] in relation with Riemannian geometry. The class $b_\Gamma(r)$ of $b_{\Gamma,S}(r)$ under the equivalence relation associated to the order $f(r) \leq g(Cr)$ for some C (written $f \precsim g$) is independant of the generating set S, so that $b_\Gamma(r)$ is a group invariant.

For many groups, e.g. those containing a free semigroup, the growth function is exponential. However, the growth function of a nilpotent group Γ is polynomial $b_\Gamma(r) \approx r^{d(\Gamma)}$ where $d(\Gamma) = \sum k.\text{rank}(\Gamma_k/\Gamma_{k+1})$ is the algebraic degree of nilpotency of $\Gamma = \Gamma_1$ associated to the filtration $\Gamma_{k+1} = [\Gamma_k, \Gamma]$ ([Bas], [Gui], [Wol]). Conversely, Gromov proved that polynomial growth implies virtual nilpotency ([Gro], see also [Kle] and [ST] for an explicit version applying to finite groups). This implies in particular that polynomial growth functions are indexed by integers $d(\Gamma)$ and any two are always comparable for $\precsim$.

In the eighties, Grigorchuk has shown some groups have intermediate growth, i.e. faster than polynomial and slower than exponential. In [Gri1], he considers a family indexed by a Cantor set $\{0,1,2\}^{\mathbb{N}}$ of groups G_ω acting on a binary rooted tree. Many of these groups satisfy growth inequalities of the form $e^{r^\alpha} \precsim b_{G_\omega}(r) \precsim e^{r^\beta}$ for exponents $\frac{1}{2} \leq \alpha < \beta < 1$. On the other hand, for some sequences ω, the growth of G_ω is "close to" e^r. Grigorchuk also proved the existence of uncountable antichains of growth functions (i.e. collections of pairwise non comparable such functions).

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Recently, Bartholdi and Erschler have computed the intermediate growth functions of some groups related to the group $G_{(012)^\infty}$ (see [BE]). More precisely, for $\Gamma_0 = \mathbb{Z}/2\mathbb{Z} \wr_X G_{(012)^\infty}$ and $\Gamma_{k+1} = \Gamma_k \wr_X G_{(012)^\infty}$, there are explicit exponents $\alpha_k < 1$ accumulating to 1, such that their growth functions satisfy $b_{\Gamma_k}(r) \approx e^{r^{\alpha_k}}$.

The purpose of the present article is to draw a panorama of growth behaviors in the range e^{r^α}. The two main points are that on the one hand there is a neighborhood of 1 in which any α is the growth exponent of some group, raising an explicit uncountable family of groups for which the growth functions are strictly comparable, and on the other hand, there are groups the growth function of which oscillates between two distinct exponents $\alpha < \beta$, which explains non comparison phenomena. More precisely:

Theorem 1.1. *Let $\eta \approx 0.8105$ be the real root of $X^3 + X^2 + X - 2$ and $\alpha_0 = \frac{\log 2}{\log 2 - \log \eta} \approx 0.7674$. Then for any $\alpha_0 \leq \alpha \leq \beta \leq 1$, there exists a group $G(\alpha, \beta)$ such that:*

$$\liminf \frac{\log\log b_{G(\alpha,\beta)}(r)}{\log r} = \alpha \text{ and } \limsup \frac{\log\log b_{G(\alpha,\beta)}(r)}{\log r} = \beta.$$

In particular, there exists a group $G(\alpha)$ such that $\lim \frac{\log\log b_{G(\alpha)}(r)}{\log r} = \alpha$.

The groups $G(\alpha, \beta)$ will be explicitely described as $F \wr_X G_\omega$ for appropriate sequence $\omega = \omega(\alpha, \beta)$. Note that the group $G(\alpha_0)$ is precisely the group $\Gamma_0 = \mathbb{Z}/2\mathbb{Z} \wr_X G_{(012)^\infty}$ considered in [BE]. Also a better study of oscillation phenomena provides uncountable antichains of growth functions satisfying a uniform upper bound e^{r^β} for any $\beta > \alpha_0$.

In order to ease notation, adopt the following:

Definition 1.2. Given a finitely generated group G, the *upper logarithmic growth exponent* $\overline{\alpha}(G)$ and the *lower logarithmic growth exponent* $\underline{\alpha}(G)$ are real numbers in $[0, 1]$ defined as:

$$\overline{\alpha}(G) = \limsup \frac{\log\log b_G(r)}{\log r} \text{ and } \underline{\alpha}(G) = \liminf \frac{\log\log b_G(r)}{\log r}.$$

In case of equality, call *logarithmic growth exponent* the number $\alpha(G) = \overline{\alpha}(G) = \underline{\alpha}(G)$.

For submultiplicative functions, inequality $b(Cr) \leq b(r)^C$ implies:

$$\frac{\log\log b(Cr)}{\log r} \leq \frac{\log\log b(r)}{\log r} + \frac{\log C}{\log r},$$

so that the logarithmic growth exponents of groups are independent of the choice of a particular representative $b_{\Gamma,S}(r)$, i.e. the choice of generating set. Note that if $b_G(r) \simeq e^{r^\alpha}$, then $\alpha(G) = \alpha$ but the converse is not true, as shown by functions $e^{r^\alpha (\log r)^p}$ for any value of p. In particular, the growth functions of the groups studied here are not computed, but only their logarithmic growth exponents.

The article is structured as follows. Sections 2 and 3 are devoted to the description of the involved groups Γ_ω, and in particular the notion of activity of a representative word. Section 4 presents the three main tools of estimation for growth. The activity of words is studied in section 5 to derive precise growth estimates, i.e. construct groups with a given logarithmic growth exponent. Oscillation phenomena are studied in sections 7 and 8, which permits to explain the existence of antichains of growth functions. Some explicit estimates on the frequency of oscillation are given. A few comments and some questions conclude the article.

Note that close results have been obtained, but not yet published, by Bartholdi and Erschler.

2. The groups involved

2.1. **Definition.** Following Grigorchuk [Gri1], associate to each given sequence $\omega = \omega_0\omega_1\omega_2\dots$ in $\{0,1,2\}^{\mathbb{N}}$ a group G_ω of automorphism of a binary rooted tree T, generated by four elements $G_\omega = \langle a, b_\omega, c_\omega, d_\omega\rangle$, defined via the wreath product isomorphism:

$$Aut(T) \simeq Aut(T) \wr S_2 = (Aut(T) \times Aut(T)) \rtimes S_2, \tag{1}$$

where S_2 acts on the product by permuting components. The generator $a = (1,1)\varepsilon$, where ε is non-identity in S_2, is independent of ω and only acts at the root of T. The three other generators are defined recursively by:

$$b_\omega = (u^b(\omega_0), b_{\sigma\omega}), c_\omega = (u^c(\omega_0), c_{\sigma\omega}), d_\omega = (u^d(\omega_0), d_{\sigma\omega}), \tag{2}$$

where σ is the shift of sequence $\sigma\omega = \omega_1\omega_2\dots$ and:

$$u^b\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}a\\a\\id\end{pmatrix}, u^c\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}a\\id\\a\end{pmatrix}, u^d\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}id\\a\\a\end{pmatrix}. \tag{3}$$

The group G_ω is defined by the sequence ω which rules the embeddings $G_\omega \hookrightarrow G_{\sigma\omega} \wr S_2$. The following relations are easily checked:

$$a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = id. \tag{4}$$

In particular, the group generated by $b_\omega, c_\omega, d_\omega$ is a Klein group $V = S_2 \times S_2$ and each of the four generators has order 2 (unless ω is constant), so they generate G_ω as a quotient semigroup of $\Omega_\omega = \{a, b_\omega, c_\omega, d_\omega\}^*$, the free semigroup of words in the generators with concatenation as product. Also note that conjugating by a exchanges the components on the two subtrees, in particular:

$$ab_\omega a = (b_{\sigma\omega}, u^b(\omega_0)), ac_\omega a = (c_{\sigma\omega}, u^c(\omega_0)), ad_\omega a = (d_{\sigma\omega}, u^b(\omega_0)). \tag{5}$$

Now following [BE], let $\rho = 1^\infty \in \partial T$ be the rightmost geodesic ray out of the root of T. Note that $b_\omega, c_\omega, d_\omega$ fix ρ independently of ω. Denote $X = \rho G_\omega$ the right orbit of ρ under G_ω. The permutational wreath product of G_ω and another group F over X is the group:

$$\Gamma_\omega = F \wr_X G_\omega = (\Sigma_X F) \rtimes G_\omega,$$

where $\Sigma_X F$ is the group of finitely supported functions $\varphi : X \to F$, on which G_ω acts on the left by $(g.\varphi)(x) = \varphi(xg)$, and in particular the supports satisfy $supp(g.\varphi) = supp(\varphi)g^{-1}$. The elements are denoted φg for $\varphi \in \Sigma_X F$ and $g \in G_\omega$. The computation rule is $(\varphi_1 g_1)(\varphi_2 g_2) = (\varphi_1(g_1.\varphi_2))(g_1 g_2)$. Throughout the present article, assume the group F is finite.

As a generating set, use $S_\omega = \{a\} \sqcup \{\varphi_f v | v \in \{id_{G_\omega}, b_\omega, c_\omega, d_\omega\}, f \in F\}$. Note that $\rho v = \rho$, so $[\varphi_f, v] = id_{\Gamma_\omega}$ and the set $\{\varphi_f v\}$ generates a finite subgroup in Γ_ω, which is abstractly isomorphic to $F \times V$.

2.2. **A short history.** The groups G_ω are commensurable with some groups introduced by Aleshin -see [Ale], where automata techniques were used to solve Burnside's problem. The groups G_ω and especially $G_{(012)^\infty}$ have been widely studied under the impulse of Grigorchuk, especially since they provide the essentially only known exemples of groups of intermediate growth ([Bar1], [Bar2], [BS], [Bri], [Ers1], [Ers2], [Ers3], [Gri1], [Gri2], [MP], [Zuk]). In particular, the best known estimates on the growth of $G_{(012)^\infty}$ are:

Theorem 2.1.

$$e^{r^{0.5207}} \precsim b_{G_{(012)^\infty}}(r) \precsim e^{r^{\alpha_0}}.$$

The upper bound comes from [Bar1] (see also [MP]) and the lower bound from [Bri] (see also [Bar2], [Leo]). The estimation on the growth exponents of G_ω is tightly related to the contraction of the length of reduced words $w = (w_0, w_1)$ under the wreath product decomposition (1). If for all reduced words, $|w_0| + |w_1|$ is a large contraction of $|w|$, the upper growth exponent is small. If for *all* pairs of reduced words, $|w|$ is a small dilatation of $|w_0| + |w_1|$, the lower growth exponent is big. As it turns out, the study of dilatation of pair of words is delicate to handle, explaining the large gap between the upper and lower exponents of $G_{(012)^\infty}$.

In [BE], Bartholdi and Erschler have bypassed this problem, considering (among others) the group $F \wr_X G_{(012)^\infty}$, where F is any finite group, for which they prove:

Theorem 2.2. [BE]

$$b_{F \wr_X G_{(012)^\infty}}(r) \approx e^{r^{\alpha_0}}.$$

In short, if the upper estimates still apply, the use of permutational wreath product permits to obtain a good lower bound from small dilatation of *some* pairs of words. The techniques developed in [BE] are not restricted to the specific sequence $\omega = (012)^\infty$, and can provide a good understanding of growth of Γ_ω for rotating sequences ω, as explained below. The construction of an appropriate sequence $\omega(\alpha)$ or $\omega(\alpha, \beta)$ will be the key point to prove Theorem 1.1.

3. A description of the groups

This section aims at giving description of the group $\Gamma_\omega = F \wr_X G_\omega$.

Lemma 3.1. *The group $\Gamma_\omega = F \wr_X G_\omega$ embeds cannonically into the finite permutational wreath product $\Gamma_{\sigma\omega} \wr S_2$. More precisely, the application Φ:*

$$\begin{aligned} \Gamma_\omega &\hookrightarrow \Gamma_{\sigma\omega} \wr S_2 \\ a &\mapsto (1,1)a \\ v_\omega &\mapsto (u^v(\omega_0), v_{\sigma\omega}) \\ \varphi_f &\mapsto (1, \varphi_f) \end{aligned}$$

is an injective morphism of groups.

Any γ in Γ_ω is decomposed $\gamma = \varphi g$, with $g \in G_\omega$ and $\varphi : X \to F$. The classical embedding $G_\omega \hookrightarrow G_{\sigma\omega} \wr S_2$ provides a decomposition $g = (g_0, g_1)\sigma$. Also the boundary of the tree can be decomposed into two components $\partial T = \partial T_0 \sqcup \partial T_1$ with T_t the tree descended from the first level vertex t. In particular, the orbit X inherits this decomposition into $X = X_0 \sqcup X_1$. Set $\varphi_t = \varphi|_{X_t}$ the restriction of φ to the subset X_t of the orbit X. With these notations, the application Φ is given by:

$$\Phi(\gamma) = (\varphi_0 g_0, \varphi_1 g_1)\sigma \in \Gamma_\omega \wr S_2.$$

In order to prove the lemma, it is sufficient to check that $\Phi(\gamma\gamma') = \Phi(\gamma)\Phi(\gamma')$.

Proof. On the one hand, $\gamma\gamma' = \varphi g \varphi' g' = \varphi(g.\varphi') g g' = \psi g g'$, with $\psi = \varphi(g.\varphi')$. As above set $\psi_t = \psi|_{X_t}$, and as $gg' = (g_0 g'_{\sigma(0)}, g_1 g'_{\sigma(1)})\sigma\sigma'$, the embedding is:

$$\Phi(\gamma\gamma') = (\psi_0 g_0 g'_{\sigma(0)}, \psi_1 g_1 g'_{\sigma(1)})\sigma\sigma'.$$

On the other hand:

$$\begin{aligned}\Phi(\gamma)\Phi(\gamma') &= (\varphi_0 g_0, \varphi_1 g_1)\sigma(\varphi'_0 g'_0, \varphi'_1 g'_1)\sigma' \\ &= (\varphi_0 g_0 \varphi'_{\sigma(0)} g'_{\sigma(0)}, \varphi_1 g_1 \varphi'_{\sigma(1)} g'_{\sigma(1)})\sigma\sigma' \\ &= (\varphi_0 (g_0.\varphi'_{\sigma(0)}) g_0 g'_{\sigma(0)}, \varphi_1 (g_1.\varphi'_{\sigma(1)}) g_1 g'_{\sigma(1)})\sigma\sigma'\end{aligned}$$

There remains to check $\psi_t = \varphi_t(g_t.\varphi'_{\sigma(t)})$, and indeed for any $y \in X_t \simeq X$:

$$\begin{aligned}\psi_t(y) &= \psi(ty) = (\varphi(g.\varphi'))(ty) = \varphi(ty)((g.\varphi')(ty)) = \varphi(ty)\varphi'(ty.g) \\ &= \varphi(ty)\varphi'(\sigma(t)(y.g_t)) = \varphi_t(y)\varphi'_{\sigma(t)}(y.g_t) = \varphi_t(y)(g_t.\varphi'_{\sigma(t)})(y).\end{aligned}$$

□

The embedding $\psi : \Gamma_\omega \hookrightarrow \Gamma_{\sigma\omega} \wr S_2$ can also be used at the word level. Let us describe the *rewriting process* of a given word of the form $w = a^{i_1} k_1 a k_2 \dots a k_r a^{i_2}$, for $k_i = \varphi_{f_i} v_i$ in $\{\varphi_f v | v \in \{id, b_\omega, c_\omega, d_\omega\}, f \in F\}$, which is said *pre-reduced.* Note that any reduced representative word in Γ_ω has this form.

Any such word can be rewritten $w = k_1^a k_2 k_3^a k_4 \dots k_r a^{i_3}$ or $w = k_1 k_2^a \dots k_r a^{i_4}$, where $i_j \in \{0, 1\}$. Note also that $k = \varphi_f v_\omega = (u^v(\omega_0), \varphi_f v_{\sigma\omega}) = (u^v(\omega_0), k)$ and $k^a = (k, u^v(\omega_0))$ and remind $u^v(\omega_0) \in \{id, a\}$. This permits to rewrite $w = (w_0, w_1)\sigma(w)$ via the wreath product embedding, and w_0, w_1 appear as products of the type $w_0 = a^{\varepsilon_1} k_2 a^{\varepsilon_3} k_4 \dots k_r$ and $w_1 = k_1 a^{\varepsilon_2} \dots a^{\varepsilon_r}$ for $\varepsilon_j \in \{0, 1\}$. Now reduce w_0, w_1 to obtain pre-reduced words in $S_{\sigma\omega}$, by using the rule $k_i a^0 k_{i+1} = k_i k_{i+1} = \varphi_{f_i} v_i \varphi_{f_{i+1}} v_{i+1} = \varphi_{(f_i f_{i+1})}(v_i v_{i+1})$.

The rewritting process associates to w this representation $w = (w_0, w_1)\sigma(w)$ where $\sigma(w)$ is the image of w in the quotient group S_2 acting at the root.

The process can be iterated, which provides for any level p a representation $w = (w_1, \dots, w_{2^p})\sigma_p(w)$ with w_i pre-reduced words in $S_{\sigma^p\omega}$ and $\sigma_p(w) \in Aut(T_2(p)) = S_2 \wr \dots \wr S_2$ with p factors describes the action of w on the subtree $T_2(p)$ consisting of the first p levels.

Definition 3.2. Given a pre-reduced word w in S_ω, define $T(w)$, called *minimal tree* of w, to be the minimal regular rooted subtree of T such that for any leaf z in $\partial T(w)$, one has $|w_z|_{pr} \leq 1$ for the word w_z obtained by iterated rewritting process,

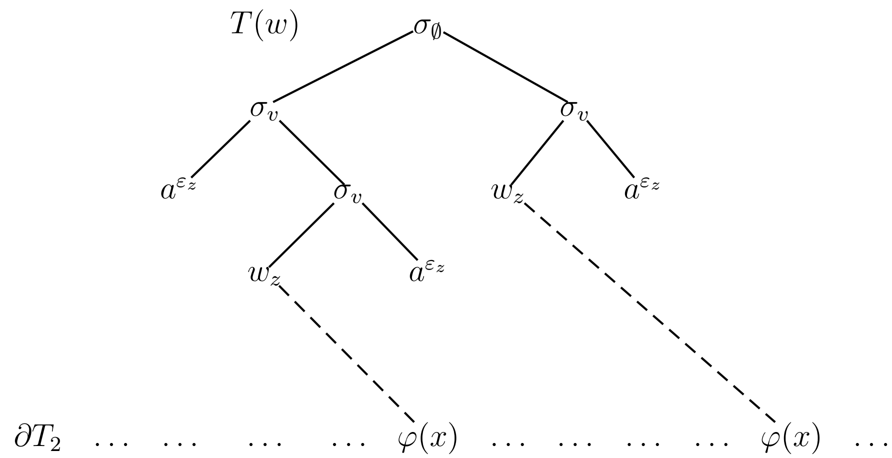

FIGURE 1. Description of the action of a word w via the minimal tree $T(w)$

where $|w|_{pr}$ is the number of factors $k_i = \varphi_{f_i} v_i$ in a pre-reduced word w_z. Remind that a subtree T is rooted if it contains the root and regular if any vertex in T either has its two descendants in T or none of them. Note that the leaves of $\partial T(w)$ have depth at most $\log_2 |w|$ because w_0, w_1 have length $\leq \frac{|w|+1}{2}$.

The tree $T(w)$ allows a nice description of the action of a word w in Γ_ω on T. Indeed, the group element $\gamma =_{\Gamma_\omega} w$ is described by the following data. First the minimal tree $T(w)$, secondly the permutations $\sigma_v \in S_2$ describing the action at vertex v in the interior of $T(w)$ and third the short words $w_z = a^{\varepsilon_z}\varphi_{f_z} v_z a^{\delta_z}$ for $z \in \partial T(w)$. The latter can be refined in the tree action $a^{\varepsilon_z} v_z a^{\delta_z}$ as an automorphism of T_z the subtree issued from the vertex z and the boundary function $\varphi(x) = id_F$ for all $x \in \partial T_z \setminus \{z\varepsilon_z(1)\rho\}$ and $\varphi(z\varepsilon_z(1)\rho) = f_z$.

Call $z \in \partial T(w)$ an *active leaf* if $|w_z|_{pr} = 1$, an *inactive leaf* if $|w_z|_{pr} = 0$, denote $S(w)$ the set of active leaves of w, and $s(w) = \#S(w)$ its size. Mind that if z is inactive then $w_z = a^{\varepsilon_z} \in S_2$ is just a permutation. Note also that regarding the rules of rewritting process $|\varphi_{id_F}|_{pr} = 1$ so that an active leaf does not necessarily act on the tree, nor its boundary (see figure 1).

However, it appears from the description above that the support of $\varphi : X \to F$ associated to w is included in $\{z\varepsilon_z(1)\rho | z \text{ is an active leaf}\}$. Call this set the *support a priori* of φ, denoted $supp^{ap}(\varphi)$. Note that for $w = a^{i_1}\varphi_{f_1} v_1 a \dots a\varphi_{f_r} v_r a^{i_2}$, if i_1, i_2, v_j are kept fixed and $(f_1, \dots, f_r)$ are taking all possible values, then any function with support included in $supp^{ap}(\varphi)$ can be obtained. In particular, the support a priori of the function φ for the word w depends only on the image in the quotient $\Gamma_\omega \to G_\omega$, $w \mapsto g = a^{i_1} v_1 a \dots a v_r a^{i_2}$.

Remark 3.3. In order to clarify the notion of support a priori, let us introduce a notion associated to the word combinatorics of the rewritting process of a fixed word w. For z an active leaf of $T(w)$, the rewritting process provides f_z as a product of terms $f_i^{z'}$ in $w_{z'}$ (where z' is the first ascendant of z), which are themselves products of terms $f_j^{z''}$ in $w_{z''}$, etc. so eventually f_z is a product of terms $(f_j)_{j \in J(z)}$ for a subset $J(z) \subset \{1, \dots, r\}$. Note that in this situation: $\bigsqcup_{z \in S(w)} J(z) = \{1, \dots, r\}$.

More generally, if y is a vertex of T, the rewritting process of w provides $w_y = a^{i_1^y}\varphi_{f_1^y}v_1^y a\dots a\varphi_{f_{r^y}^y}v_y^y r a^{i_2^y}$, and each factor f_i^y is obtained as an ordered product:

$$f_i^y = \prod_{j\in I(y,i)} f_j^{y'}, \tag{6}$$

where y' is the first ascendant of y, and $\sqcup I(y,i) = \{1,\dots,r_{y'}\}$ where the disjoint union runs over all direct descendants y of y' and $i\in\{1,\dots,r_y\}$.

Now the graph with vertex set $(f_i^y)_{y\in T, i\in\{1,\dots,r_y\}}$ and edges pairs of elements appearing on different sides of all possible products (6) is a forest, called the *ascendance forest* of w. It describes the combinatorics of the rewritting process of the word w. It depends only on $g = a^{i_1}v_1a\dots av_ra^{i_2}$. Precisely, this graph is a finite union of trees rooted in f_z for each $z\in S(w)$ and with respective sets of leaves $\{f_j | j\in J(z)\}$. The ordered product $f_z = \prod_{j\in J(z)} f_j$ shows that indeed, the function φ can take any value at the point $z\varepsilon_z(1)\rho$.

Proposition 3.4. *(Activity of a pre-reduced word) The activity $s(w)$ of a pre-reduced word $w = a^{i_1}\varphi_{f_1}v_1a\dots a\varphi_{f_r}v_ra^{i_2}$ in Γ_ω, S_ω, which counts equivalently*

1. *the size of the set $S(w)$ of active leaves in the minimal tree $T(w)$,*
2. *the number of components (i.e. trees) in the ascendance forest of w,*
3. *the size of the support a priori $supp^{ap}(\varphi)$,*
4. *the size of the inverted orbit $\mathcal{O}(g^{-1})$ of the word g^{-1} in the sense of* [BE]*,*

depends only on the word $\underline{w} = a^{i_1}v_1a\dots av_ra^{i_2}$ in G_ω and satisfies under rewritting process $w = (w_0, w_1)\sigma(w)$, with w_0, w_1 in $S_{\sigma\omega}$:

$$s(w) = s(w_0) + s(w_1).$$

Also there exists a constant C depending only on $\#F$ such that:

$$\#\{\gamma\in\Gamma_\omega | \exists w =_{\Gamma_\omega} \gamma, s(w)\le s\} \le C^s.$$

Proof. The equivalence of (1), (2) and (3), as well as the behavior of activity function under rewritting process follow from the descriptions above. Proceed by induction on r to show equivalence with (4). If $w = a^{i_1}\varphi_{f_1}v_1a\dots a\varphi_{f_r}v_ra^{i_2} =_{F\wr\Gamma} \varphi g$ then $w\varphi_{f_n}v_n = \varphi(g.\varphi_{f_n})gv_n$. The point $g^{-1}(1^\infty)$ is added to the support a priori of φ. This shows $supp^{ap}(\varphi) = \{(a^{i_1}v_1\dots v_ka^{i_2})^{-1}(1^\infty) | k\le n\} = \mathcal{O}(g^{-1})$. Mind that the inverse appears as a difference with [BE] notations, replacing gf by φg for elements of $F\wr G$. Then $\varphi g = (g.f)g$ and $g^{-1}supp(f) = supp(\varphi)$.

There remains only to show that the number of elements described grows at most exponentially fast with $s(w)$. First check that $2s(w)\ge \#\partial T(w)$ when $s(w)\ge 1$, by induction on $s(w)$. If $|w|_{pr} = 1$, then $T(w)$ is just the root of T. Now if $s(w)\ge 2$, then $s(w_0), s(w_1)\ge 1$ by pre-reduction of w, so that induction ensures $2s(w_t)\ge \#\partial T(w_t)$, and the result follows from $\#\partial T(w_0) + \#\partial T(w_1)\ge \#\partial T(w)$ by construction of minimal trees. Now if $s(w)\le s$, the minimal tree $T(w)$ has size $\le 2s$. There is 4^{2s} possibilities for $T(w)$ (Catalan numbers), and then $2^{\#\text{interior}(T(w))}\le 2^{2s}$ choices for the interior permutations σ_v for interior vertices v and finally $(2^2.4.\#F)^{\#\partial T(w)}\le C^{2s}$ choices for the boundary short words $a^{\varepsilon_z}\varphi_{f_z}v_za^{\delta_z}$. □

Corollary 3.5. *The relation between word activity and growth function is two-fold:*

(1) $b_{\Gamma_\omega}(r) \geq \#F^{s(w)}$ *for any* $|w| \leq r$.
(2) $b_{\Gamma_\omega}(r) \leq C^{\max\{s(w) | r \geq |w|\}}$.

In particular, word activity governs the growth function.

Proof. Point (1) is clear from remark 3.3 and point (2) from proposition 3.4. $\square$

4. Technics of estimation

4.1. **Growth Lemma.** The following lemma is used to estimate upper bounds on the growth of activity hence on the growth of groups. It improves on previous versions such as the Growth Theorem in [MP] and Lemma 4.3 in [BE] by keeping track of the constants in terms of the bound on the sequence $p(r)$ of variable depth of recursion.

Lemma 4.1. *Given* η *and a parameter* $\lambda \in [0,1]$, *set* $\alpha = \frac{\log(2)}{\log(2)-\lambda\log(\eta)}$, *so that* α *satisfies* $2 = \left(\frac{2}{\eta^\lambda}\right)^\alpha$.

Let $\Delta : \mathbb{N} \to \mathbb{N}$ *be a function such that for any* r *there exists* $q(r) \leq p(r)$ *and* $l_1, \dots, l_{2^{p(r)}}$ *integers such that, for a constant* C:

(1) $l_1 + \dots + l_{2^{p(r)}} \leq \eta^{q(r)} r + 2^p C$,
(2) $\Delta(r) \leq \Delta(l_1) + \dots + \Delta(l_{2^{p(r)}})$,
(3) $\frac{q(r)}{p(r)} \geq \lambda$.

Suppose moreover that $p(r) \leq P$ *is bounded. Then* $\Delta(r) \leq L r^\alpha$ *for some constant* $L = L(C,P)$.

Assume given a trivial bound $\Delta(r) \leq Kr$. *Then* L *can be chosen* $L = A^P$ *for* A *depending only on* C *and* K.

Proof. Choose R_0 big, to be determined later. Choose $R \geq R_0$ large enough so that the function:

$$\Delta^{**}(r) = \begin{cases} \left(r - r^{\frac{1}{2}}\right)^\alpha & \text{if } r \geq R, \\ 1 + \frac{r}{R}(\Delta^{**}(R) - 1) & \text{if } r \leq R, \end{cases}$$

is concave (it is also non decreasing). Choose M large enough so that for all $r \geq M$, $\frac{\eta^{q(r)}}{2^{p(r)}} r \geq R$, and let $L \geq 1$ be large enough so that $\Delta^*(r) = L\Delta^{**}(r) \geq \Delta(r)$ for all $r \leq M$. Let $r > M$, there exists $p(r), q(r), l_i$, with $\Delta(l_i) \leq \Delta^*(l_i)$ by induction, and using successively (2), induction, concavity of Δ^*, (1) and the choice of M:

$$\begin{aligned}
\Delta(r) &\leq \Delta(l_1) + \cdots + \Delta(l_{2^p}) \\
&\leq \Delta^*(l_1) + \cdots + \Delta^*(l_{2^p}) \\
&\leq 2^p \Delta^* \left(\frac{1}{2^p}(l_1 + \cdots + l_{2^p})\right) \\
&\leq 2^p \Delta^* \left(\frac{1}{2^p}(\eta^q r + 2^p C)\right) \\
&= 2^p L \left(\left(\frac{\eta^{\frac{q}{p}}}{2}\right)^p r + C - \left(\frac{\eta^q}{2^p} r + C\right)^{\frac{1}{2}}\right)^\alpha, \\
&= L \left(\left(2^{\frac{1}{\alpha}} \frac{\eta^{\frac{q}{p}}}{2}\right)^p r + 2^{\frac{p}{\alpha}} C - 2^{\frac{p}{\alpha}} \left(\frac{\eta^q}{2^p} r + C\right)^{\frac{1}{2}}\right)^\alpha,
\end{aligned}$$

Now $\frac{q}{p} \geq \lambda$ ensures $\left(2^{\frac{1}{\alpha}} \frac{\eta^{\frac{q}{p}}}{2}\right)^p \leq 1$, so:

$$\begin{aligned}
\Delta(r) &\leq L \left(r + 2^{\frac{p}{\alpha}} C - 2^{\frac{p}{\alpha}} \left(\frac{\eta^q}{2^p} r + C\right)^{\frac{1}{2}}\right)^\alpha \\
&\leq L \left(r - r^{\frac{1}{2}}\right)^\alpha = \Delta^*(r).
\end{aligned}$$

The last inequality holds when r is big enough so that:

$$2^{\frac{p}{\alpha}} \left(\frac{\eta^q}{2^p} r + C\right)^{\frac{1}{2}} - 2^{\frac{p}{\alpha}} C \geq r^{\frac{1}{2}}.$$

Observe that $\frac{2}{\eta^\lambda} \left(\frac{\eta^{\frac{q}{p}}}{2}\right)^{\frac{1}{2}} \geq \sqrt{2\eta} > 1$ so the latter is true when:

$$(\sqrt{2\eta})^{\frac{p}{2}} r^{\frac{1}{2}} \geq r^{\frac{1}{2}} + 2^{\frac{p}{\alpha}} C,$$

which holds when $r \geq a_0^P = R_0 = \frac{2^{\frac{2p}{\alpha}} C^2}{(\sqrt{2\eta}^{\frac{p}{2}} - 1)^2}$ with a constant a_0 depending only on C. For P big, $R = R_0$ and so $M = \left(\frac{2}{\eta}\right)^P R_0 = \left(\frac{2}{\eta} a_0\right)^P$. It is sufficient to take $L\Delta^{**}(M) \geq KM$ so $L \geq K \left(\frac{2}{\eta} a_0\right)^P$. □

4.2. **Localization.** The asymptotic behavior of the growth of Γ_ω depends on the asymptotic of ω. On the other hand, the description of a ball of a given radius in Γ_ω requires only some first terms of ω. The following lemma of localization is helpful to study growth of groups Γ_ω for non periodic sequences ω.

Lemma 4.2. *Suppose that the sequence ω is not asymptotically constant, then the ball $B_{\Gamma_\omega}(r)$ of radius r for the word norm with respect to the generating set $S_\omega = \{a\} \sqcup \{\varphi_f v | v \in \{id_{G_\omega}, b_\omega, c_\omega, d_\omega\}, f \in F\}$ depends only on $\omega_0\omega_1 \ldots \omega_k$ for $k = \log_2(r)$.*

The Cayley graph $Cay(\Gamma, S)$ of a group Γ with generating set S is the colored graph with vertices γ in Γ and edges $(\gamma, \gamma s)$ of color s in S. The ball $B_\Gamma(r)$ of radius r is the subgraph obtained by restriction to vertices and ends of edges such that $|\gamma| \leq r$ for the word norm for S.

Proof. The ball $B_\omega(1)$ of Γ_ω for the generating set S_ω is independent of ω among sequences that are not constant, it consists of the Cayley graph $Cay(F \times V, F \times V)$ together with an edge from the neutral element leading to the vertex a. By proposition 3.1, the ball $B_{\Gamma_\omega}(r)$ can be described using $B_{\Gamma_{\sigma\omega}}(\frac{r+1}{2})$ and the wreath product recursion (2), i.e. ω_0. Indeed, an element γ admits a reduced representative word $w = a^{i_1}k_1ak_2 \dots ak_{r/2}a^{i_2}$ and so $\gamma = (\gamma_0, \gamma_1)\varepsilon^s$ with $|\gamma_0|, |\gamma_1| \leq \frac{r+1}{2}$ by rewritting process. By iteration, $B_{\Gamma_\omega}(r)$ is described by $B_{\Gamma_{\sigma^k\omega}}(\frac{r}{2^k}+1)$ and $\omega_0 \dots \omega_k$. □

Remark 4.3. When $\omega = 0^\infty$ is constant, the generator d_ω acts trivially on the rooted tree T, hence is identity, so that the Klein group V degenerates into a group S_2, and $G_\omega = \langle a, b_{0^\infty} | a^2 = b^2 = id \rangle = D_\infty$ is dihedral infinite. However, the whole sequence ω is required to obtain this information. The group $\tilde{G}_{0^\infty}$ obtained by "finite information" (concretely as a limit group of $G_{0^k(012)^\infty}$ for instance) is in fact the group $\tilde{G}_{0^\infty} \simeq S_2 \wr_X G_{0^\infty} = \langle d_{0^\infty} \rangle \wr_X \langle a, b_{0^\infty} \rangle$, which is metabelien of exponential growth. It played a crucial role in Grigorchuk's construction of antichains of growth functions, cf. section 6 in [Gri1].

4.3. **Asymptotic growth.** Opposed to localization, the asymptotic behavior of the growth depends only on the asymptotic of ω.

Proposition 4.4. *For generating sets $S_\omega = \{a\} \sqcup \{\varphi_f v | v \in \{id, b_\omega, c_\omega, d_\omega\}, f \in F\}$, the growth function of $\Gamma_\omega = F \wr_X G_\omega$ satisfies for all r:*

$$b_{\Gamma_{\sigma\omega}}(\frac{r-1}{2}) \leq b_{\Gamma_\omega}(r) \leq 2 b_{\Gamma_{\sigma\omega}}(\frac{r+1}{2})^2.$$

Also by iteration:

$$b_{\Gamma_{\sigma^k\omega}}(\frac{r}{2^k}-1) \leq b_{\Gamma_\omega}(r) \leq 2^{2^k} b_{\Gamma_{\sigma^k\omega}}(\frac{r}{2^k}+1)^{2^k}.$$

Proof. Let $\gamma = \varphi g$ belong to $B_{\Gamma_\omega}(r)$. It admits a minimal representative word $w =_{\Gamma_\omega} \gamma$ of length r, which is uniquely described after rewritting process as $w = (w_0, w_1)\sigma(w)$ with $|w_0|, |w_1| \leq \frac{r+1}{2}$. Conclude that γ is determined by two elements γ_0, γ_1 in $B_{\Gamma_{\sigma\omega}}(\frac{r+1}{2})$ and a permutation $\sigma(w)$ in S_2, which proves the upper bound.

Suppose $\omega_0 \neq 1$ and let γ_0 belong to $B_{\Gamma_{\sigma\omega}}(\frac{r-1}{2})$. It admits a minimal representative word $w_0 = a^{i_1}k_1ak_2a \dots ak_la^{i_2}$ of length $\leq \frac{r-1}{2}$. Set $w = b_\omega \bar{k}_1^a b_\omega \bar{k}_2^a \dots b_\omega \bar{k}_l^a b_\omega^{i_2}$ if $i_1 = 1$ and $w = \bar{k}_1 b_\omega^a \bar{k}_2 b_\omega^a \dots b_\omega^a \bar{k}_l b_\omega^{i_2 a}$ if $i_1 = 0$ of length $\leq r$, where $\bar{k}_j = \varphi_{f_j} v_\omega^j$ for $k_j = \varphi_{f_j} v_{\sigma\omega}^j$. Proposition 3.1 and relations (2) from section 2.1 guarantee that $w = (w_0, w_1)\sigma(w)$ for some $w_1, \sigma(w)$. Now if $w =_{\Gamma_\omega} w'$, then $w_0 =_{\Gamma_{\sigma\omega}} w'_0$, so that $B_{\Gamma_{\sigma\omega}}(\frac{r-1}{2})$ injects into $B_{\Gamma_\omega}(r)$. (Note that when $\omega_0 = 1$, the same computation works if b_ω is replaced by d_ω.) □

5. Activity and growth

5.1. **Activity of some words and lower bound on growth.** Proposition 4.7 in [BE] generalizes as:

Proposition 5.1. *Denote:*

$$A_0 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

There is $C > 0$ such that for any k, there is a word w_k in Γ_ω, S_ω such that $s(w_k) \geq 2^k$ and $|w_k| \leq C||A_{\omega_0} \dots A_{\omega_k}||$.

Proof. Consider the subsemigroup $\Omega'_\omega = \{ab_\omega, ac_\omega, ad_\omega\}^* \subset \Omega_\omega$, and define the pull back substitution $\zeta : \Omega'_{\sigma\omega} \to \Omega'_\omega$ by:

$$\begin{array}{llll} \zeta(ab_{\sigma\omega}) = ab_\omega ab_\omega & \zeta(ac_{\sigma\omega}) = ac_\omega ac_\omega & \zeta(ad_{\sigma\omega}) = ab_\omega ad_\omega ac_\omega & \text{if } \omega_0 = 0, \\ \zeta(ab_{\sigma\omega}) = ab_\omega ab_\omega & \zeta(ac_{\sigma\omega}) = ad_\omega ac_\omega ab_\omega & \zeta(ad_{\sigma\omega}) = ad_\omega ad_\omega & \text{if } \omega_0 = 1, \\ \zeta(ab_{\sigma\omega}) = ac_\omega ab_\omega ad_\omega & \zeta(ac_{\sigma\omega}) = ac_\omega ac_\omega & \zeta(ad_{\sigma\omega}) = ad_\omega ad_\omega & \text{if } \omega_0 = 2. \end{array}$$

Such a pull back substitution is designed so that $\zeta(au) = (ua, au)$ when au is apre-reduced word containing an even number of v's (where $v = d$ if $\omega_0 = 0$, $v = c$ if $\omega_0 = 1$ and $v = b$ if $\omega_0 = 2$). Indeed, the following relations hold (take $\omega_0 = 0$, similar otherwise):

$$\begin{array}{ll} \zeta(ab) = abab = (ba, ab), & baba = (ab, ba), \\ \zeta(ac) = acac = (ca, ac), & caca = (ac, ca), \\ \zeta(ad) = abadac = (d, ada)a, & badaca = (ada, d)a. \end{array}$$

The pull back of $v_{\sigma\omega}$ furnishes v_ω on both components of the wreath product. The a's behave conveniently under the parity condition.

Given a word $w_0 = au_0$ in $\Omega'_{\sigma^k\omega}$, define by induction $\zeta(au_{k-1}) = au_k = w_k \in \Omega'_\omega$. The initial word u_0 can be chosen among the generators $\{b_{\sigma^k\omega}, c_{\sigma^k\omega}, d_{\sigma^k\omega}\}$ so that $\zeta(au_0) = avav$ for another generator v of $G_{\sigma^{k-1}\omega}$, so that $\zeta(au_k)$ always has an even number of v's, and the inverted orbit of au_k can be studied by induction via:

$$\zeta(au_{k-1}) = au_k = (u_{k-1}a, au_{k-1}) \text{ and } u_k a = (au_{k-1}, u_{k-1}a).$$

Proposition 3.4 now ensures that:

$$s(au_k) \geq s(au_{k-1}) + s(u_{k-1}a) \text{ and } s(u_k a) \geq s(u_{k-1}a) + s(au_{k-1}),$$

which is integrated in $s(au_k) \geq 2^k$.

To estimate the length of $w_k = \zeta(w_{k-1})$, it is sufficient to count the numbers $|w|_{b_\omega}, |w|_{c_\omega}, |w|_{d_\omega}$ of generators $b_\omega, c_\omega, d_\omega$ appearing in w, since the total length is controlled by $|w| \leq 2(|w|_{b_\omega} + |w|_{c_\omega} + |w|_{d_\omega})$. The construction of the pull back substitution ζ provides the relations:

$$A_{\omega_0} \begin{pmatrix} |w_{k-1}|_{b_{\sigma\omega}} \\ |w_{k-1}|_{c_{\sigma\omega}} \\ |w_{k-1}|_{d_{\sigma\omega}} \end{pmatrix} = \begin{pmatrix} |w_k|_{b_\omega} \\ |w_k|_{c_\omega} \\ |w_k|_{d_\omega} \end{pmatrix},$$

for the matrices A_0, A_1, A_2, and eventually by induction:

$$A_{\omega_0} A_{\omega_1} \dots A_{\omega_k} \begin{pmatrix} |w_0|_{b_{\sigma^k \omega}} \\ |w_0|_{c_{\sigma^k \omega}} \\ |w_0|_{d_{\sigma^k \omega}} \end{pmatrix} = \begin{pmatrix} |w_k|_{b_\omega} \\ |w_k|_{c_\omega} \\ |w_k|_{d_\omega} \end{pmatrix},$$

so that $|w_k| \leq C||A_{\omega_0} \dots A_{\omega_k}||$. □

The matrices A_0, A_1, A_2 are cyclic conjugates $A_1 = CA_0C^{-1}$ and $A_2 = C^{-1}A_0C = CA_1C^{-1}$, so that $A_0^{k_1} A_1^{k_2} A_2^{k_3} \dots = A_0^{k_1} C A_0^{k_2} C A_0^{k_3} \dots$ with

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, A_0 C = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrix A_0C has caracteristic polynomial $X^3 - X^2 - 2X - 4$ with positive real root $\frac{2}{\eta}$, and two complex conjugate roots of smaller absolute value, hence spectral radius $\rho(A_0C) = \frac{2}{\eta}$. (Remind that η is the positive root of $X^3 + X^2 + X - 2$.)

Examples 5.2. (1) For $\omega = (012)^\infty$, the spectral radius theorem gives:

$$||A_{\omega_0} \dots A_{\omega_k}|| \leq ||(A_0C)^{k+1}|| \leq C\rho(A_0C)^k = C\left(\frac{2}{\eta}\right)^k.$$

(2) For other periodic sequences, similar bounds are obtained, as for instance $\omega = (001122)^\infty$, then:

$$||A_{\omega_0} \dots A_{\omega_k}|| \leq ||(A_0^2C)^{\frac{k+1}{2}}|| \leq C\rho(A_0^2C)^{\frac{k}{2}},$$

where the spectral radius $\rho(A_0^2) \approx 5.63$ is the positive root of $X^3 - 3X^2 - 12X - 16$.

Such estimates for periodic sequences are not usually sharp enough. The following lemma is useful for the present purpose:

Lemma 5.3. *Let $\omega = 0^{m_1}(012)^{n_1}0^{m_2}(012)^{n_2}\dots$, with $m_i, n_i \to \infty$. There exists a constant C, such that for every $\varepsilon > 0$ and $k = \sum_{i=1}^j m_i + 3n_i$ big enough:*

$$||A_{\omega_0} \dots A_{\omega_k}|| \leq C^{\varepsilon k} \rho(A_0)^{\sum m_i} \rho(A_0C)^{3\sum n_i} = C^{\varepsilon k} 2^{\sum m_i} \left(\frac{2}{\eta}\right)^{3\sum n_i}.$$

Proof. By the spectral radius theorem, there exists C such that $||A_0^m|| \leq C\rho(A_0)^m$ and $||(A_0C)^{3n}|| \leq C\rho(A_0C)^{3n}$, so:

$$\begin{aligned} ||A_{\omega_0} \dots A_{\omega_k}|| &\leq ||A_0^{m_1}(A_0C)^{3n_1} A_0^{m_2}(A_0C)^{3n_2} \dots || \\ &\leq ||A_0^{m_1}||.||(A_0C)^{3n_1}||.||A_0^{m_2}||.||(A_0C)^{3n_2}|| \dots \\ &\leq C^j \rho(A_0)^{\sum_{i=1}^j m_i} \rho(A_0C)^{3\sum_{i=1}^j n_i}, \end{aligned}$$

where $j = o(k)$ since $m_i, n_i \to \infty$. □

Note that if m_i, n_i are of the order $\log i$, then $j \approx \frac{k}{\log k}$, and if m_i, n_i are of the order i^θ, then $j \approx k^{\frac{1}{\theta+1}}$.

5.2. **Activity of all words and upper bound on growth.** Say a sequence $\omega = \omega_0\omega_1\omega_2\dots$ in $\{0,1,2\}^{\mathbb{N}}$ is *rotating* if $\omega_{i+1} \in \{\omega_i, \omega_i+1\}$ mod 3 for all i. Remind that η is the positive root of X^3+X^2+X-2. Adapting [Bar1] to rotating sequences ω, define a length on G_ω by assigning weights to the generating set $\langle a, b_\omega, c_\omega, d_\omega\rangle$. Set $||a|| = 1-\eta^3$ and:

$$\begin{array}{llll} \text{if } \omega_0 = 0, & ||b_\omega|| = \eta^3, & ||c_\omega|| = 1-\eta^2, & ||d_\omega|| = 1-\eta, \\ \text{if } \omega_0 = 1, & ||b_\omega|| = 1-\eta^2, & ||c_\omega|| = 1-\eta, & ||d_\omega|| = \eta^3, \\ \text{if } \omega_0 = 2, & ||b_\omega|| = 1-\eta, & ||c_\omega|| = \eta^3, & ||d_\omega|| = 1-\eta^2. \end{array}$$

This defines a length on G_ω for which the minimal representative words are pre-reduced (η is chosen so that this is the case, see lemma 4.1 in [Bar1]), which is obviously equivalent to the usual word length $\frac{1}{C}|w| \le ||w|| \le C|w|$, and designed so that if $\omega_1 = \omega_0+1$, then:

$$\begin{aligned} \varepsilon_b(\omega_0)||a|| + ||b_{\sigma\omega}|| &= \eta(||a|| + ||b_\omega||), \\ \varepsilon_c(\omega_0)||a|| + ||c_{\sigma\omega}|| &= \eta(||a|| + ||c_\omega||), \\ \varepsilon_d(\omega_0)||a|| + ||d_{\sigma\omega}|| &= \eta(||a|| + ||d_\omega||), \end{aligned}$$

where: $\varepsilon_b\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \varepsilon_c\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}, \varepsilon_d\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$, and if $\omega_0 = \omega_1$, then the factor η on the right-hand sides disappears.

This length on G_ω can be extended to a function on the set of words in the generating set $\langle S_\omega\rangle$ of Γ_ω by $||\varphi_f v|| = ||v||$ if $v \in \{b_\omega, c_\omega, d_\omega\}$ and $||\varphi_f id|| = 0$. Note that even though $||w||$ is not a length on Γ_ω it is still equivalent to the length of pre-reduced words, i.e. $\frac{1}{C}|w| \le ||w|| \le C|w|$, because if $w = a^{i_1}\varphi_{f_1}v_1 a\varphi_{f_2}v_2\dots a\varphi_{f_r}v_r a^{i_2}$, then $||w|| = ||\underline{w}||$ for $\underline{w} = a^{i_1}v_1 a v_2 \dots a v_r a^{i_2}$, which is bilipschitz equivalent to r.

The following statement generalizes Lemma 4.2 in [BE].

Lemma 5.4. *Let w be a pre-reduced word of Γ_ω, S_ω with rewritting process giving $w = (w_0, w_1)\varepsilon^s$, then:*

$$||w_0|| + ||w_1|| \le \eta^{q(\omega_0,\omega_1)}||w|| + C,$$

where $C = \eta||a||$, $q(\omega_0,\omega_1) = 0$ if $\omega_1 = \omega_0$, $q(\omega_0,\omega_1) = 1$ if $\omega_1 = \omega_0+1$ and the left-handside lengths are in $\Gamma_{\sigma\omega}$, the right-hand side one in Γ_ω.

Proof. The inequalities for $\underline{w} = (\underline{w}_0, \underline{w}_1)\sigma(w)$ in G_ω and $G_{\sigma\omega}$ are obvious by construction of the length $||.||$, i.e. by choice of η. They still apply to pre-reduced words in Γ_ω and $\Gamma_{\sigma\omega}$. ☐

In order to estimate the growth function from above, the word activity function

$$s_\omega(r) = \max\{s(w) | w \in (\Gamma_\omega, S_\omega), |w| \le r\}$$

will be usefull. However, it is smoother to estimate first the bilipschitz equivalent auxiliary

$$\Delta_\omega(r) = \max\{s(w) | w \text{ is pre-reduced}, ||w|| \le r\}.$$

Fact 5.5. *For any r, there exists l_0, l_1 integers such that:*

(1) $\Delta_\omega(r) \le \Delta_{\sigma\omega}(l_0) + \Delta_{\sigma\omega}(l_1)$, *and*
(2) $l_0 + l_1 \le \eta^{q(\omega_0,\omega_1)} r + C$.

By induction, there exists $l_1, \dots, l_{2^p}$ *integers such that:*

(1) $\Delta_\omega(r) \le \Delta_{\sigma^p\omega}(l_1) + \cdots + \Delta_{\sigma^p\omega}(l_{2^p})$, *and*
(2) $l_1 + \cdots + l_{2^p} \le \eta^{q(\omega_0,\dots,\omega_p)} r + 2^{p+1}C$, *where* $q(\omega_0, \dots, \omega_p)$ *is the number of* i *such that* $\omega_{i+1} = \omega_i + 1 \mod 2$.

Proof. The maximum is realized for a certain word w, for which the rewritting process furnishes $w = (w_0, w_1)\sigma(w)$ with $l_0 = ||w_0||$ and $l_1 = ||w_1||$ such that $l_0 + l_1 \le \eta^{q(\omega_0,\omega_1)}||w|| + C$ by lemma 5.4. Thus:

$$\Delta_\omega(r) = s(w) = s(w_0) + s(w_1) \le \Delta_{\sigma\omega}(l_0) + \Delta_{\sigma\omega}(l_1).$$

□

Proposition 5.6. *Suppose* ω *is such that for all* i, *there exists* $p(i) \le P$ *such that* $q(\omega_i, \dots, \omega_{i+p(i)}) = q(i)$ *and* $\frac{q(i)}{p(i)} \ge \lambda$, *then:*

$$\log b_{\Gamma_\omega}(r) \le A^P r^\alpha, \text{ for } \alpha = \frac{\log(2)}{\log(2) - \lambda \log(\eta)}.$$

In particular, if ω is p-periodic and $q(\omega_0, \dots, \omega_p) = q$, then $\log b_{\Gamma_\omega}(r) \le Lr^\alpha$ for $\alpha = \frac{\log(2)}{\log(2) - \frac{q}{p}\log(\eta)}$.

Proof. Set $\Delta(r) = \sup\{\Delta_{\sigma^p\omega}(r) | p \in \mathbb{N}\}$. Fact 5.5 provides the existence of $l_1 + \cdots + l_{2^p} \le \eta^{q(\omega_0,\dots,\omega_p)} r + 2^p C$ such that $\Delta(r) \le \Delta(l_1) + \cdots + \Delta(l_{2^p})$, and so by lemma 4.1 there is a constant A such that $\Delta(r) \le A^P r^\alpha$, so that $s_\omega(r) \le A^P r^\alpha$ (mind that there is a trivial bound $\Delta(r) \le Kr$ because the activity is bounded by the word length). Now corollary 3.5 shows $\log b_{\Gamma_\omega}(r) \le A^P r^\alpha$. □

6. Precise growth estimates

The particular case of theorem 1.1 can now be derived. Recall that α_0 is such that $2 = \left(\frac{2}{\eta}\right)^{\alpha_0}$.

Theorem 6.1. *For any* $\alpha \in [\alpha_0, 1]$, *there exists a sequence* $\omega(\alpha)$ *such that* $\alpha(\Gamma_{\omega(\alpha)}) = \alpha$, *i.e.*

$$\lim \frac{\log\log b_{\Gamma_{\omega(\alpha)}}(r)}{\log r} = \alpha.$$

Proof. Given α, take λ in $[0, 1]$ such that $2 = (\frac{2}{\eta^\lambda})^\alpha$. Consider a sequence of the form $\omega = 0^{m_1}(012)^{n_1}0^{m_2}(012)^{n_2}\dots$. Denote the ith period $p_i = m_i + 3n_i$ and $q_i = 3n_i$ the number of steps of rotation of ω, and assume both tend to infinity. Suppose moreover that $\frac{q_i}{p_i} \ge \lambda$ for each i and $\frac{q_i}{p_i} \to \lambda$, so that $\sum_{i=1}^j p_i = k_j$ and $\sum_{i=1}^j q_i = \lambda k_j + o(k_j)$.

Lemma 4.2 of localization allows to use proposition 5.6 for P depending on the scale r. Indeed, $b_{\Gamma_\omega}(r)$ depends only on $\omega_0, \dots, \omega_k$ for $k = \log_2(r)$, for which $p_i \leq P(k)$, so that:

$$\log b_{\Gamma_\omega}(r) \leq A^{P(k)} r^\alpha \leq r^{\alpha+\varepsilon},$$

as soon as $P(k) \leq \varepsilon \log_A(r)$. In particular, if ω is chosen such that $P(k) = o(\log(r)) = o(k)$, the required upper bound holds: $\overline{\alpha}(\Gamma_\omega) \leq \alpha$.

Concerning lower estimates, the word w_{k_j} introduced in proposition 5.1 has length bounded by (lemma 5.3):

$$|w_{k_j}| \leq C^{\varepsilon' k_j} \frac{2^{\sum_{i=1}^j p_i}}{\eta^{\sum_{i=1}^j q_i}} \leq C^{2\varepsilon' k_j} \left(\frac{2}{\eta^\lambda}\right)^{k_j} \leq \left(\frac{2}{\eta^\lambda}\right)^{k_j(1+\varepsilon)}.$$

Now lemma 3.5 ensures, for $r_j = \left(\frac{2}{\eta^\lambda}\right)^{k_j(1+\varepsilon)}$:

$$r_j^{\frac{\alpha}{1+\varepsilon}} = 2^{k_j} \leq s(w_{k_j}) \leq \log b_{\Gamma_\omega}(|w_{k_j}|) \leq \log b_{\Gamma_\omega}\left(\left(\frac{2}{\eta^\lambda}\right)^{(1+\varepsilon)k_j}\right) = \log b_{\Gamma_\omega}(r_j).$$

Interpolating for $r_j \leq r \leq r_{j+1}$ gives:

$$\log b(r) \geq \log b(r_j) \geq r_j^{\frac{\alpha}{1+\varepsilon}} = r_{j+1}^{\frac{\alpha}{1+\varepsilon}} \left(\frac{2}{\eta^\lambda}\right)^{-\alpha p_j} \geq r^{\frac{\alpha}{1+\varepsilon}} \left(\frac{2}{\eta^\lambda}\right)^{-\alpha p_j} \geq r^{\alpha - 2\varepsilon},$$

where the last inequality holds for large r since $p_{j+1} \leq P(k) = o(k) = o(\log r)$. As ε is arbitrary, $\underline{\alpha}(\Gamma_\omega) \geq \alpha$ for any such sequence ω. $\square$

Remark 6.2. Obviously, the computation of the exact growth exponent $\alpha(G) = \alpha$ does not imply that $b_G(r) \simeq e^{r^\alpha}$. The precise estimates obtained with the proof above are (for $r_j \leq r \leq r_{j+1}$):

$$C^{-(j+p_j+e(j))} r^\alpha \leq \log b_\omega(r) \leq r^\alpha A^{p_{j+1}},$$

where $e(j) = (\sum_{i=1}^j q_i) - \lambda k_j = o(j)$ is the error on the rationnal approximation of λ by greater values. Taking p_i of the order $\log i$, and thus j of the order $\frac{\log r}{\log\log r}$, one obtains for some A:

$$r^{\alpha - \frac{A}{\log\log r}} \leq \log b_\omega(r) \leq r^{\alpha + \frac{A \log\log r}{\log r}},$$

and taking p_i of the order i^θ for $0 < \theta < 1$, thus j of order $(\log r)^{\frac{1}{\theta+1}}$, one obtains:

$$r^{\alpha - A(\log r)^{-\frac{\theta}{\theta+1}}} \leq \log b_\omega(r) \leq r^{\alpha + A(\log r)^{-\frac{1}{\theta+1}}}.$$

7. Oscillation phenomena

7.1. **Groups with oscillating logarithmic growth exponents.** The oscillation of logarithmic exponents of growth function is the phenomenon that underlies the construction of antichains of growth function in section 7 of [Gri1] and of "fast intermediate" growth in [Ers3]. It was studied for its own interest in the second chapter of [Bri]. Theorem 6.1 allows a better understanding.

Theorem 7.1. *For any $\alpha \leq \beta \in [\alpha_0, 1]$, there exists a sequence $\omega(\alpha,\beta)$ such that $\underline{\alpha}(\Gamma_{\omega(\alpha,\beta)}) = \alpha$ and $\overline{\alpha}(\Gamma_{\omega(\alpha,\beta)}) = \beta$, i.e.*

$$\liminf \frac{\log\log b_{\Gamma_{\omega(\alpha,\beta)}}(r)}{\log r} = \alpha \text{ and } \limsup \frac{\log\log b_{\Gamma_{\omega(\alpha,\beta)}}(r)}{\log r} = \beta.$$

To ease notations, $b_{\Gamma_\omega}(r) = b_\omega(r)$ from now on.

Proof. Take $\omega(\alpha,\beta) = \omega(\alpha)_{|0\ldots m_1}\omega(\beta)_{|m_1+1\ldots n_1}\omega(\alpha)_{|n_1+1\ldots m_2}\omega(\beta)_{|m_2+1\ldots n_2}\ldots$ for some sequences m_i, n_i tending to infinity. Such a choice ensures that:

$$\alpha \leq \underline{\alpha}(\Gamma_{\omega(\alpha,\beta)}) \text{ and } \overline{\alpha}(\Gamma_{\omega(\alpha,\beta)}) \leq \beta.$$

If m_i, n_i tend to infinity sufficiently fast, these inequalities become equalities. Indeed, take $\varepsilon_i \to 0$, and construct r_i, r_i' such that:

$$\frac{\log\log b(r_i)}{\log r_i} \leq \alpha + \varepsilon_i \text{ and } \frac{\log\log b(r_i')}{\log r_i'} \geq \beta - \varepsilon_i.$$

By localization 4.2, left inequality holds for all $\omega_{|0\ldots m_i} = \omega(\alpha,\beta)_{|0\ldots m_i}$ and right inequality for all $\omega_{|0\ldots n_i} = \omega(\alpha,\beta)_{|0\ldots n_i}$ with $\log_2 r_i = m_i$ and $\log_2 r_i' = n_i$.

Assume by induction that m_j, n_j are constructed for $j \leq i$ and construct $m_{i+1} = \log r_{i+1}$. Take $\omega' = \omega(\alpha,\beta)_{|0\ldots n_i}\omega(\alpha)_{|n_i+1\ldots}$. By proposition 4.4 on asymptotic growth:

$$b_{\omega'}(r) \leq 2^{2^{n_i}} b_{\sigma^{n_i}\omega'}(\frac{r}{2^{n_i}}+1)^{2^{n_i}} = 2^{2^{n_i}} b_{\sigma^{n_i}\omega(\alpha)}(\frac{r}{2^{n_i}}+1)^{2^{n_i}} \leq 2^{2^{n_i}} b_{\omega(\alpha)}(r+2^{n_i+1})^{2^{n_i}},$$

so that:

$$\frac{\log\log b_{\omega'}(r)}{\log r} \leq \frac{\log\log b_{\omega(\alpha)}(r+2^{n_i+1}) + n_i\log 2}{\log r} \simeq \frac{\log\log b_{\omega(\alpha)}(r)}{\log r} \longrightarrow_{r\to\infty} \alpha,$$

and there exists r_{i+1} as required. Set $m_{i+1} = \log_2(r_{i+1})$.

Now construct $n_{i+1} = \log_2(r_{i+1}')$. Take $\omega'' = \omega(\alpha,\beta)_{|0\ldots m_{i+1}}\omega(\beta)_{|m_{i+1}+1\ldots}$. Again proposition 4.4:

$$b_{\omega''}(r) \geq b_{\sigma^{m_{i+1}}\omega''}(\frac{r}{2^{m_{i+1}}}-1) = b_{\sigma^{m_{i+1}}\omega(\beta)}(\frac{r}{2^{m_{i+1}}}-1) \geq \frac{1}{2} b_{\omega(\beta)}(r-2^{m_{i+1}+1})^{\frac{1}{2^{m_{i+1}}}},$$

so that:

$$\frac{\log\log b_{\omega_{i+1}'}(r)}{\log r} \geq \frac{\log\log b_{\omega(\beta)}(r-2^{m_{i+1}+1}) - m_{i+1}\log 2}{\log r} \longrightarrow_{r\to\infty} \beta,$$

and there exists r_{i+1}' and $n_{i+1} = \log_2 r_{i+1}'$. □

7.2. **Antichains of growth functions.** The following result is a slight improvement of Theorem 7.2 in [Gri1], which shows the existence of antichains of intermediate growth functions accumulating to e^r.

Theorem 7.2. *For any $\alpha_0 \leq \alpha < \beta \leq 1$, there exists uncountably many groups Γ_ω with pairwise non comparable growth functions (such a collection of groups is called an antichain) satisfying $\underline{\alpha}(\Gamma_\omega) = \alpha$ and $\overline{\alpha}(\Gamma_\omega) = \beta$.*

Moreover, if $\beta < \beta' \leq 1$, such an antichain can be chosen so that $b_\omega(r) \leq Ce^{r^{\beta'}}$ for a constant C depending only on β, β' and not on ω.

Lemma 7.3. *Given $\alpha_0 \leq \alpha < \beta \leq 1$, there exists an application ω from the set $\mathcal{F}(\mathbb{N}, \{\alpha, \beta\})$ of functions $f : \mathbb{N} \to \{\alpha, \beta\}$ to the Cantor space of infinite sequences $\{0, 1, 2\}^{\mathbb{N}}$, and there exists sequences $r_i \to \infty$ and $\frac{\beta-\alpha}{2} > \varepsilon_i \to 0$ such that:*

(1) $\underline{\alpha}(\Gamma_{\omega(f)}) = \alpha$ *and* $\overline{\alpha}(\Gamma_{\omega(f)}) = \beta$,
(2) $\frac{\log\log b_{\omega(f)}(r_i)}{\log r_i} \geq \beta - \varepsilon_i$ *if* $f(i) = \beta$,
(3) $\frac{\log\log b_{\omega(f)}(r_i)}{\log r_i} \leq \alpha + \varepsilon_i$ *if* $f(i) = \alpha$.

Proof of theorem 7.2. There are uncountably many functions $\xi : \mathbb{N} \times \mathbb{N} \to \{\alpha, \beta\}$ such that $\xi(x, y) = \alpha$ implies $\xi(x, y+1) = \beta$ and $\xi(x, y) = \beta$ implies $\xi(x, y+1) = \alpha$. Any bijection $\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, provides an injection $\xi \mapsto f_\xi$ by $f_\xi(i) = \xi \circ \varphi(i)$. Now given $\xi_1 \neq \xi_2$, if $f_{\xi_1}(i) < f_{\xi_2}(i)$, there exists $j > i$ such that $f_{\xi_2}(j) < f_{\xi_1}(j)$. Lemma 7.3 ensures that $b_{\omega(f_{\xi_1})}(r)$ and $b_{\omega(f_{\xi_2})}(r)$ are not comparable. □

Proof of lemma 7.3. The proof of this lemma is a variation on the proof of theorem 7.1. Pick:

$$\omega(f) = \omega(f(0))_{|0\ldots m_0}\omega(f(1))_{|m_0+1\ldots m_1}\omega(f(2))_{|m_1+1\ldots m_2}\cdots$$

for a sequence $m_i = \log_2(r_i)$ increasing sufficiently fast. Mind that this guarantees a uniform upper bound $\frac{\log\log b_{\omega(f)}(r)}{\log r} \leq \beta + \varepsilon = \beta'$ for any ε and r big enough (depending on ε).

Assume by induction m_j and r_j constructed for $j \leq i$ and consider:

$$\begin{aligned} \omega' &= \omega(f(0))_{|0\ldots m_0}\ldots\omega(f(i))_{|m_{i-1}+1\ldots m_i}\omega(\alpha)_{|m_i+1\ldots}, \\ \omega'' &= \omega(f(0))_{|0\ldots m_0}\ldots\omega(f(i))_{|m_{i-1}+1\ldots m_i}\omega(\beta)_{|m_i+1\ldots}. \end{aligned}$$

As above, proposition 4.4 on asymptotic growth provides:

$$\begin{aligned} b_{\omega'}(r) &\leq 2^{2^{m_i}} b_{\omega(\alpha)}(r + 2^{m_i+1})^{2^{m_i}}, \\ b_{\omega''}(r) &\geq \frac{1}{2} b_{\omega(\beta)}(r - 2^{m_i+1})^{\frac{1}{2^{m_i+1}}}, \end{aligned}$$

so that there exists r_{i+1}, independent of $(f(0), \ldots, f(i))$, such that:

$$\begin{aligned} \frac{\log\log b_{\omega'}(r_{i+1})}{\log r_{i+1}} &\leq \alpha + \varepsilon_i, \\ \frac{\log\log b_{\omega''}(r_{i+1})}{\log r_{i+1}} &\geq \beta - \varepsilon_i, \end{aligned}$$

and this is true for any sequence ω coinciding with ω', ω'' on the $m_{i+1} = \log_2 r_{i+1}$ first values. □

Remark 7.4. The idea behind the proof of theorem 7.1, is that the asymptotic behavior of the growth function $b_\omega(r)$ of the group Γ_ω depends only on the asymptotic of the defining sequence ω, whereas locally a ball of given radius depends only on some first terms of ω. This permits to produce scales at which the growth function is essentially e^{r^α} and others at which it is essentially e^{r^β}, thus explaining oscillation between this two behaviors. Of course, the process can be used to produce a variety of different behaviors at different scales, for instance scales S_i at which Γ_ω seems to have growth $e^{r^{\gamma_i}}$ for countably many $\alpha_i \in [\alpha_0, 1]$, intertwined with scales S_j at which Γ_ω seems to have growth oscillating between $e^{r^{\alpha_j}}$ and $e^{r^{\beta_j}}$. The only point is to allow enough "time" so that the behavior at scale S_i or S_j becomes visible, i.e. functions m_i, n_i in the proofs above increasing sufficiently fast.

8. Frequency of oscillations

This section aims at studying the frequency of oscillations for groups of the type Γ_ω. The main question is to maximize the frequency of oscillation between two given bounds, or equivalently to minimize the period.

8.1. **Group invariants associated to oscillation.** Given $\alpha < \beta$ and a Lipschitz function $b : \mathbb{N} \to \mathbb{N}$, define the *upper set* $U(\alpha, \beta)$ and *lower set* $L(\alpha, \beta)$ of b for α, β to be:

$$U(\alpha,\beta) = \{s \in \mathbb{N} | \frac{\log\log b(s)}{\log s} \geq \beta\} \text{ and } L(\alpha,\beta) = \{t \in \mathbb{N} | \frac{\log\log b(t)}{\log t} \leq \alpha\}.$$

Note that $\frac{\log\log b(s)}{\log s} \geq \beta$ is equivalent to $\log b(s) \geq s^\beta$ and $\frac{\log\log b(t)}{\log t} \leq \alpha$ is equivalent to $\log b(t) \leq t^\alpha$.

Property 8.1. *Let $\alpha < \beta$ and $b : \mathbb{N} \to \mathbb{N}$ be a Lipschitz function, then:*

(1) $L(\alpha,\beta) \sqcup U(\alpha,\beta) \subset \mathbb{N}$, *and the inclusion is strict if both upper and lower sets are infinite.*
(2) *Assume $\alpha' < \alpha < \beta < \beta'$ then:*

$$L(\alpha',\beta) \subset L(\alpha,\beta) \text{ and } U(\alpha,\beta') \subset U(\alpha,\beta),$$

and the inclusions are strict if both upper and lower sets are infinite.

Note that when $b(r) = b_\Gamma(r)$ is the growth function of a finitely generated group Γ such that $\underline{\alpha}(\Gamma) < \alpha < \beta < \overline{\alpha}(\Gamma)$, then both upper and lower sets are infinite.

Property (1) allows to decompose $U = \bigsqcup_{j=0}^\infty U_j$ and $L = \bigsqcup_{j=0}^\infty L_j$ such that:

(1) U_i, L_i are non empty,
(2) for any $s \in U_i$, then $s \geq \max \cup_{j\leq i-1} L_j$ and $s \leq \min \cup_{j\geq i} L_j$,
(3) for any $t \in L_i$, then $t \geq \max \cup_{j\leq i} U_j$ and $t \leq \min \cup_{j\geq i+1} U_j$.

Call this decomposition alternating (see figure 2).

In order to study oscillation, set $s_i = \min U_i$, $s'_i = \max U_i$, $t_i = \min L_i$ and $t'_i = \max L_i$. The *upper pseudo period function* u is the partially defined $s_{i+1} = u(t'_i)$ and the *lower pseudo period function* l is the partially defined $t_i = l(s'_i)$. In order to investigate how small these functions can be, define:

$$u_{\alpha,\beta} = \inf\{\nu | \exists i_o, \forall i \geq i_o, s_{i+1} \leq (t'_i)^\nu\} \text{ and } l_{\alpha,\beta} = \inf\{\lambda | \exists i_o, \forall i \geq i_o, t_i \leq (s'_i)^\lambda\}.$$

Equivalently:

$$u_{\alpha,\beta} = \limsup_{i\to\infty} \frac{\log s_{i+1}}{\log t'_i} \text{ and } l_{\alpha,\beta} = \limsup_{i\to\infty} \frac{\log t_i}{\log s'_i}.$$

The following fact provides estimates on the pseudo period functions that any growth function of infinite group must satisfy.

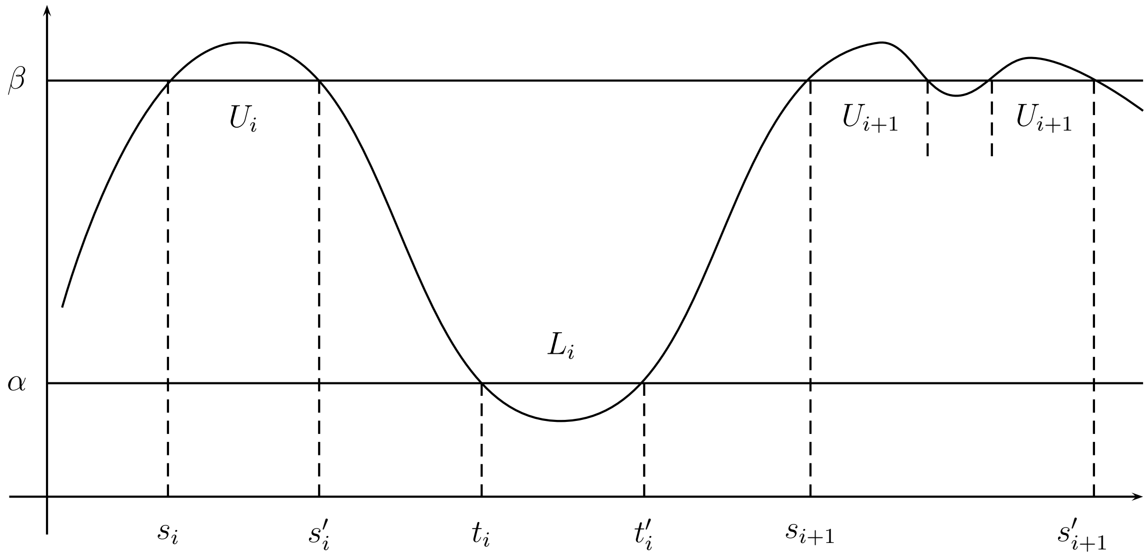

FIGURE 2. Upper and lower sets $U(\alpha,\beta)$ and $L(\alpha,\beta)$ seen by drawing the curve $f(r) = \frac{\log\log b(r)}{\log r}$.

Fact 8.2. *Consider $\alpha < \beta$ and a function $b : \mathbb{N} \to \mathbb{N}$, then:*

(1) *if $b(r)$ is submultiplicative, $u_{\alpha,\beta} \geq \frac{1-\alpha}{1-\beta} > 1$,*
(2) *if $b(r)$ is increasing, $l_{\alpha,\beta} \geq \frac{\beta}{\alpha} > 1$.*

Proof. Suppose $\log b(t) \leq t^\alpha$. Submultiplicativity implies $\log b(kt) \leq kt^\alpha$, so that $\log b(kt) \geq (kt)^\beta$ forces $kt^\alpha \geq (kt)^\beta$ hence $kt \geq t^{\frac{\beta-\alpha}{1-\beta}+1}$. Now suppose $\log b(s) \geq s^\beta$, then $\log b(t) \leq t^\alpha$ forces $t^\alpha \geq s^\beta$. □

By property (2), given $\alpha'' < \alpha' < \beta' < \beta''$, one has $u_{\alpha'',\beta''} \geq u_{\alpha',\beta'}$ and $l_{\alpha'',\beta''} \geq l_{\alpha',\beta'}$. This permits the:

Definition 8.3. The *upper pseudo period exponent* $u(\alpha,\beta)$ and the *lower pseudo period exponent* $l(\alpha,\beta)$ of a function $b(r)$ are:

$$u(\alpha,\beta) = \lim_{\substack{\alpha' \to \alpha^+ \\ \beta' \to \beta^-}} u_{\alpha',\beta'}, \text{ and } l(\alpha,\beta) = \lim_{\substack{\alpha' \to \alpha^+ \\ \beta' \to \beta^-}} l_{\alpha',\beta'}.$$

Remind notation $\alpha' \to \alpha^+$ (respectively $\beta' \to \beta^-$) for $\alpha' \to \alpha$ and $\alpha' > \alpha$ (respectively $\beta' \to \beta$ and $\beta' < \beta$).

This definition is appropriate because it permits to define $u(\underline{\alpha}(\Gamma), \overline{\alpha}(\Gamma))$ and $l(\underline{\alpha}(\Gamma), \overline{\alpha}(\Gamma))$ associated to the growth function $b_\Gamma(r)$ even though the upper and lower sets $U(\underline{\alpha}(\Gamma), \overline{\alpha}(\Gamma))$ and $L(\underline{\alpha}(\Gamma), \overline{\alpha}(\Gamma))$ may be empty. Also:

Proposition 8.4. *The upper and lower pseudo period exponents $u(\alpha,\beta)$ and $l(\alpha,\beta)$ are group invariants.*

Proof. In order to show the exponents are not perturbed by change of generating set, consider a function $b'(r)$ such that there exists C with $b(\frac{r}{C}) \leq b'(r) \leq b(Cr)$.

Then $L^{(b')}(\alpha',\beta') = \{t | \log b'(t) \leq t^{\alpha'}\} \subset \{t | \log b(\frac{t}{C}) \leq t^{\alpha'}\} = C\{x | \log b(x) \leq C^{\alpha'} x^{\alpha'}\}$. But given any $\alpha'' > \alpha'$ and x large enough, one has $C^{\alpha'}x^{\alpha'} \leq x^{\alpha''}$, so that if x large enough belongs to $L^{(b')}(\alpha',\beta')$, then x belongs to $CL^{(b)}(\alpha'',\beta'')$. Similarly, for any $\beta'' < \beta'$, large enough y that belong to $U^{(b')}(\alpha',\beta')$ also belong to $\frac{1}{C}U^{(b)}(\alpha'',\beta'')$.

This permits to deduce that there is a j such that:

$$\frac{\log s_{i+1}^{(b')}(\alpha',\beta')}{\log t_i^{'(b')}(\alpha',\beta')} \geq \frac{\log s_{j+1}^{(b)}(\alpha'',\beta'') - \log C}{\log t_j^{'(b)}(\alpha'',\beta'') + \log C}$$

so that $u^{(b')}_{\alpha',\beta'} \geq u^{(b)}_{\alpha'',\beta''}$ for any $\alpha' < \alpha'' < \beta'' < \beta'$, which implies $u^{(b')}(\alpha,\beta) \geq u^{(b)}(\alpha,\beta)$, and equality holds by symetry. Similar proof for $l(\alpha,\beta)$. □

Remark 8.5. Given $\alpha < \beta$, one can similarly define the pseudo period exponent of oscillations for a function $b(r)$, by $p(\alpha,\beta) = \lim p_{\alpha',\beta'}$ for $p_{\alpha',\beta'} = \limsup \frac{\log s_{i+1}}{\log s_i}$, and it is a group invariant for $b_\Gamma(r)$. However, it is not true a priori that replacing s_i by t_i, t'_i or s'_i would provide the same exponent.

8.2. **Estimates on pseudo-period.** Theorem 6.1 shows that for any $\gamma \in [\alpha_0, 1]$ there exists a group $\Gamma_{\omega(\gamma)}$ such that:

$$\frac{1}{C_\varepsilon} r^{\gamma-\varepsilon} \leq \log b_{\omega(\gamma)}(r) \leq C_\varepsilon r^{\gamma+\varepsilon},$$

where $\varepsilon > 0$ is arbitrary and C_ε depends only on ε.

Suppose that $\log b_\omega(t) = t^\alpha$ for some t. This fact depends only on $(\omega_i)_{i=0}^m$ for $m = \log_2 t$ by localization. Now consider the group $\Gamma_{\omega'}$ for the sequence $\omega' = \omega_0 \ldots \omega_m \omega(\gamma)|_{m+1\ldots}$, for some $\gamma \geq \beta > \alpha$. By proposition 4.4 on asymptotic growth, one has:

$$\log b_{\omega'}(s) \geq \frac{1}{2^m} \log b_{\omega(\gamma)}(s - 2^{m+1} - \log 2) \geq \frac{1}{C_\varepsilon t}(s-2t)^{\gamma-\varepsilon},$$

so that for any $\beta' < \beta$ and ε small enough:

$$\min\{s | \log b_{\omega'}(s) \geq s^{\beta'}\} \leq C_\varepsilon t^{\frac{1}{\gamma-\varepsilon-\beta'}} + o(t^{\frac{1}{\gamma-\varepsilon-\beta'}}).$$

Conversely suppose that $\log b_\omega(s) = s^\beta$ for some s, which depends only on $(\omega_i)_{i=0}^n$ for $n = \log_2 s$, and consider the group $\Gamma_{\omega''}$ for the sequence $\omega'' = \omega_0 \ldots \omega_n \omega(\delta)|_{n+1\ldots}$ for some $\delta \leq \alpha < \beta$. As above, one has:

$$\log b_{\omega''}(t) \leq 2^n(\log b_{\omega(\delta)}(t + 2^{n+1}) + \log 2) \leq C_\varepsilon s(t+2s)^{\delta+\varepsilon},$$

so that for any $\alpha < \alpha'$ and ε small enough:

$$\min\{t | \log b_{\omega''}(t) \leq \tau^{\alpha'}\} \leq C_\varepsilon s^{\frac{1}{\alpha'-\delta-\varepsilon}} + o(s^{\frac{1}{\alpha'-\delta-\varepsilon}}).$$

These two observations show the following (passing to the limits $\alpha' \to \alpha$, $\beta' \to \beta$ and $\varepsilon \to 0$):

Proposition 8.6. *Given $\alpha_0 \leq \delta \leq \alpha < \beta \leq \gamma \leq 1$, there exists a sequence*

$$\omega(\alpha,\beta,\gamma,\delta) = \omega(\delta)|_{0\ldots m_1}\omega(\gamma)|_{m_1+1\ldots n_1}\omega(\delta)|_{n_1+1\ldots m_2}\omega(\gamma)|_{m_2+1\ldots n_2}\cdots$$

such that the group $\Gamma_{\omega(\alpha,\beta,\gamma,\delta)}$ *satisfies:*

$$u(\alpha,\beta) \le \frac{1}{\gamma-\beta} \quad \textit{and} \quad l(\alpha,\beta) \le \frac{1}{\alpha-\delta}.$$

The choice of $\omega(\alpha,\beta,\gamma,\delta)$ guarantees $\underline{\alpha}(\Gamma_{\omega(\alpha,\beta,\gamma,\delta)}) \ge \delta$ and $\overline{\alpha}(\Gamma_{\omega(\alpha,\beta,\gamma,\delta)}) \le \gamma$, but these are probably strict inequalities.

Note that the construction of $\omega(\alpha,\beta)$ in the proof of theorem 7.1 is a particular instance of the above proposition with $\gamma=\beta$ and $\delta=\alpha$. In this case, the upper and lower pseudo period exponents are (a priori) infinite.

On the other hand, in order to minimize the upper and lower pseudo period exponents for a fixed oscillation magnitude $\alpha<\beta$, taking $\gamma=1$ and $\delta=\alpha_0$ gives upper bounds (the lower bounds are trivial from fact 8.2):

$$\frac{1-\alpha}{1-\beta} \le u(\alpha,\beta) \le \frac{1}{1-\beta} \quad \text{and} \quad \frac{\beta}{\alpha} \le l(\alpha,\beta) \le \frac{1}{\alpha-\alpha_0}.$$

Since the estimates above are done for any t in $L(\alpha,\beta)$ and s in $U(\alpha,\beta)$, they provide an upper bound for (any choice of definition in remark 8.5) pseudo period:

$$\frac{\beta(1-\alpha)}{\alpha(1-\beta)} \le p(\alpha,\beta) \le \frac{1}{(1-\beta)(\alpha-\alpha_0)}.$$

9. Comments and questions

9.1. **Precise growth estimates.** Theorem 6.1 provides the existence of many groups with precise logarithmic growth exponents. However, it is not clear how much their growth functions are regular. Indeed, the sequence $\omega(\alpha)$ used to define $\Gamma_{\omega(\alpha)}$ has the form $\omega(\alpha)=0^{m_1}(012)^{n_1}0^{m_2}(012)^{n_2}\dots$ for some sequences m_i, n_i tending to infinity (this permits to use lemma 5.3 to estimate the norm of a large product of matrices). It is likely that the growth function $b_{\omega(\alpha)}(r)$ oscillates around e^{r^α} with oscillations unseen by the logarithmic growth exponents.

It would be interesting to produce more regular growth functions, and in particular to know for which exponents α there is a group with precise growth function $b_\Gamma(r) \approx e^{r^\alpha}$. Two directions seem interesting.

On the one hand, periodic sequences ω should be studied further. The technics developped here provide some interesting estimates, as for instance (example 5.2 (2) and proposition 5.6):

$$r^{0.8019} \le \log b_{(001122)^\infty}(r) \le r^{0.8684}.$$

However, the specific norm defined in paragraph 5.2, which is very well suited for the sequence $\omega=(012)^\infty$, does not seem to be sufficient in general. Maybe considering other norms (making full use of a given period, for instance (001122)) would provide better upper estimates.

On the other hand, growth functions of random sequences ω would be interesting to compute. A natural model is given by rotating sequences $\omega_{i+1} \in \{\omega_i, \omega_i+1\}$ with probability p and $1-p$ respectively. The study of random product of matrices could give nice lower bounds via proposition 5.1, but on the other hand, an appropriate version of the growth lemma 4.1 is needed.

Concerning the space of groups $(\Gamma_\omega)_{\omega\in\Omega}$, it seems that the growth function is minimal for the sequence $(012)^\infty$, but it is not proved. Also the growth function of the group G_ω might be quite different from that of Γ_ω for a given sequence ω. For instance $\Gamma_{(01)^\infty}$ has exponential growth, whereas $G_{(01)^\infty}$ has growth essentially $e^{\frac{r}{\log r}}$ (see [Ers1]).

9.2. **Oscillations.** Groups with oscillating growth function appear by considering different sequences ω at different scales, i.e. with highly non periodic sequences ω. Is it true that oscillation does not occur if ω is periodic? In other terms, does the sequence $\frac{\log\log b_\Gamma(r)}{\log r}$ converge for $\Gamma = \Gamma_\omega$ with periodic ω, or for Γ an automata group?

The questions of amplitude and frequency naturally arise with the notion of oscillation. Theorem 7.1 provides a good description of amplitude (oscillation between any two bounds $\alpha_0 \leq \alpha < \beta \leq 1$), but is not satisfying regarding frequency, as the upper and lower pseudo period exponents seem to be infinite. Conversely, in proposition 8.6 the frequency is evaluated, but the exact amplitude is not known, though one can believe the lower and upper logarithmic growth exponents of $\Gamma_{\omega(\alpha,\beta,\gamma,\delta)}$ are exactly α and β.

Submultiplicativity and increasing nature of the growth functions $b_\Gamma(r)$ impose easy lower bounds on the pseudo period (fact 8.2). It is far from clear, especially concerning the lower pseudo period, that these bounds are optimal. In other terms, what are the values of the following functions of (α, β)?

$$\begin{aligned} u_\infty(\alpha,\beta) &= \inf\{u_\Gamma(\alpha,\beta) | \Gamma \text{ is a finitely generated group}\}, \\ l_\infty(\alpha,\beta) &= \inf\{l_\Gamma(\alpha,\beta) | \Gamma \text{ is a finitely generated group}\}, \\ p_\infty(\alpha,\beta) &= \inf\{p_\Gamma(\alpha,\beta) | \Gamma \text{ is a finitely generated group}\}. \end{aligned}$$

Concerning the notion of period, is it true that the four definitions of pseudo period exponents from remark 8.5 coincide for maximal frequency?

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