# Injectivity and flatness of semitopological modules

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July 12, 2011

**Abstract**

The spaces $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{E}'$ over $\mathbb{R}^n$ are known to be flat modules over $\mathbf{A} = \mathbb{C}[\partial_1, ..., \partial_n]$, whereas their duals $\mathcal{D}'$, $\mathcal{S}'$ and $\mathcal{E}$ are known to be injective modules over the same ring. Let $\mathbf{A}$ be a Noetherian $\mathbf{k}$-algebra ($\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$). The above observation leads us to study in this paper the link existing between the flatness of an $\mathbf{A}$-module $E$ which is a locally convex topological $\mathbf{k}$-vector space and the injectivity of its dual. We show that, for dual pairs $(E, E')$ which are $(\mathcal{K})$ over $\mathbf{A}$–a notion which is explained in the paper–, injectivity of $E'$ is a stronger condition than flatness of $E$. A preprint of this paper (dated September 2009) has been quoted and discussed by Shankar [12].

## 1 Introduction

Consider the spaces $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{E}'$ over $\mathbb{R}^n$, as well as their duals $\mathcal{D}'$, $\mathcal{S}'$ and $\mathcal{E}$. Ehrenpreis [5], Malgrange [8], [9] and Palamodov [10] proved that $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{E}'$ are flat modules over $\mathbf{A} = \mathbb{C}[\partial_1, ..., \partial_n]$ whereas $\mathcal{D}'$, $\mathcal{S}'$ and $\mathcal{E}$ are injective over $\mathbf{A}$. If $F$ is any of these modules, all maps $F \to F : x \mapsto a\,x$ ($a \in \mathbf{A}$) are continuous; using Pirkovskii' s terminology ([11], p. 5), this means that $F$ is *semitopological*. This observation leads to wonder whether there exists a link between the injectivity of a semitopological $\mathbf{A}$-module and the flatness of its dual. The existence of such a link is studied in this paper.

## 2 Preliminaries

**Notation 1** *In what follows,* $\mathbf{A}$ *is a Noetherian domain (not necessarily commutative) which is a* $\mathbf{k}$*-algebra (*$\mathbf{k} = \mathbb{R}$ *or* $\mathbb{C}$*).*

Let $E$, $E'$ be two $\mathbf{k}$-vector spaces. Assume that $E'$ is a left $\mathbf{A}$-module and that there exists a nondegenerate bilinear form $\langle -, - \rangle : E \times E' \to \mathbf{k}$. Then

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$E$ and $E'$ are locally convex topological vector spaces endowed with the weak topologies $\sigma(E, E')$ and $\sigma(E', E)$ defined by $\langle -, - \rangle$; the pair $(E, E')$ is called *dual* (with respect to the bilinear form $\langle -, - \rangle$).

Assume that the left $\mathbf{A}$-module $E'$ (written ${}_{\mathbf{A}}E'$) is semitopological for the topology $\sigma(E', E)$. Then the $\mathbf{k}$-vector space $E$ becomes a right $\mathbf{A}$-module (written $E_{\mathbf{A}}$), setting

$$\langle x\, a, x' \rangle = \langle x, a\, x' \rangle \tag{1}$$

for any $x \in E$, $x' \in E'$ and $a \in \mathbf{A}$, and it is obviously semitopological, i.e., all maps $E \to E : x \mapsto x\, a$ $(a \in \mathbf{A})$ are continuous. Conversely, one can likewise prove that if the right $\mathbf{A}$-module $E_{\mathbf{A}}$ is semitopological for the topology $\sigma(E, E')$, then ${}_{\mathbf{A}}E'$ is semitopological for the topology $\sigma(E', E)$. By (1), the transpose of the left multiplication by $a \in \mathbf{A}$, denoted by $a\bullet : E' \to E'$, is the right multiplication by $a$, denoted by $\bullet a : E \to E$.

**Notation 2** *In what follows, $(E, E')$ is a dual pair and $E_{\mathbf{A}}$ (or equivalently ${}_{\mathbf{A}}E'$) is a semitopological module.*

The duality bracket $\langle -, - \rangle$ is extended to an obvious way to $E^{1\times k} \times (E')^k$; then $\left(E^{1\times k}, (E')^k\right)$ is again a dual pair. Let $P \in \mathbf{A}^{q\times k}$; this matrix determines a continuous linear map $P\bullet : (E')^k \to (E')^q : x' \mapsto P\, x'$, the transpose of which is $\bullet P : E^{1\times q} \times E^{1\times k} : x \mapsto x\, P$.

**Example 3** *Let $E'$ be the space of distributions $\mathcal{D}'$, $\mathcal{S}'$ or $\mathcal{E}'$ over $\mathbb{R}^n$ and $E$ the associated space of test functions. From the above, the transpose of $\partial_i\bullet : E' \to E'$ is $\bullet\partial_i : \mathcal{E} \to \mathcal{E}$, and for any $T \in E'$, $\varphi \in E$, $\langle \varphi\, \partial_i, T \rangle = \langle \varphi, \partial_i\, T \rangle$. Since $\langle \varphi, \partial_i\, T \rangle = -\langle \partial_i\, \varphi, T \rangle$, one has $\varphi\, \partial_i = -\partial_i\, \varphi$ $(\varphi \in E)$, i.e., $\bullet\partial_i = -\partial_i\bullet$.*

Consider the following sequences where $P_1 \in \mathbf{A}^{k_1\times k_2}, P_2 \in \mathbf{A}^{k_2\times k_3}$:

$$\mathbf{A}^{1\times k_1} \xrightarrow{\bullet P_1} \mathbf{A}^{1\times k_2} \xrightarrow{\bullet P_2} \mathbf{A}^{1\times k_3}, \tag{2}$$

$$E^{1\times k_1} \xrightarrow{\bullet P_1} E^{1\times k_2} \xrightarrow{\bullet P_2} E^{1\times k_3}, \tag{3}$$

$$(E')^{k_3} \xrightarrow{P_2\bullet} (E')^{k_2} \xrightarrow{P_1\bullet} (E')^{k_1}. \tag{4}$$

The facts recalled below are classical:

**Lemma 4** *(i) The module $E_{\mathbf{A}}$ is flat if, and only if whenever (2) is exact, (3), deduced from (2) using the functor $E \bigotimes_{\mathbf{A}} -$, is again exact ([10], Part I, §I.3, Prop. 5).*
*(ii) The module ${}_{\mathbf{A}}E'$ is injective if, and only if whenever (2) is exact, (4), deduced from (2) using the functor $\operatorname{Hom}_{\mathbf{A}}(-, E')$, is again exact ([10], Part I, §I.3, Prop. 9).*
*(iii) For any matrix $P_2 \in \mathbf{A}^{k_2\times k_3}$, there exist a natural integer $k_1$ and a matrix $P_1 \in \mathbf{A}^{k_1\times k_2}$ such that (2) is exact. Conversely, given a matrix $P_1 \in \mathbf{A}^{k_1\times k_2}$, there exists a matrix $P_2 \in \mathbf{A}^{k_2\times k_3}$ such that (2) is exact if, and*

*only if* $\operatorname{coker}_{\mathbf{A}}(\bullet P_1) = \mathbf{A}^{1\times k_2}/\left(\mathbf{A}^{1\times k_1} P_2\right)$ *is torsion-free (see, e.g., [2], Lemma 2.15).*
*(iv) The following equalities hold ([1], §IV.6, Corol. 2 of Prop. 6):*

$$\begin{aligned} \ker_{E'}(P_1\bullet) &= (\operatorname{im}_E(\bullet P_1))^0, \\ \overline{\operatorname{im}_{E'}(P_2\bullet)} &= (\ker_E(\bullet P_2))^0 \end{aligned}$$

*where* $(.)^0$ *is the polar of* $(.)$.

Consider the sequence involving $2+n$ maps $\bullet P_i$ $(1 \le i \le 2+n)$

$$\mathbf{A}^{1\times k_1} \xrightarrow{\bullet P_1} \mathbf{A}^{1\times k_2} \xrightarrow{\bullet P_2} \mathbf{A}^{1\times k_3} \longrightarrow ... \xrightarrow{\bullet P_{2+n}} \mathbf{A}^{1\times k_{3+n}} \tag{5}$$

where $n \ge 0$.

**Definition 5** *The module* ${}_{\mathbf{A}}E'$ *is called* $n$*-injective if whenever* (5) *is exact,* (4) *is again exact.*

The following is obvious:

**Lemma 6** *(i) If the module* ${}_{\mathbf{A}}E'$ *is* $n$*-injective* $(n \ge 0)$*, then it is* $n'$*-injective for all integers* $n'$ *such that* $n' \ge n$.
*(ii) The module* ${}_{\mathbf{A}}E'$ *is* 0*-injective if, and only if it is injective.*

**Lemma 7** *(1) If* (3) *is exact, then* $\overline{\operatorname{im}_{E'}(P_2\bullet)} = \ker_{E'}(P_1\bullet)$.
*(2) If* (4) *is exact, then* $\overline{\operatorname{im}_E(\bullet P_1)} = \ker_E(\bullet P_2)$.

**Proof.** (1) If (3) is exact, then $\ker_E(\bullet P_2) = \operatorname{im}_E(\bullet P_1)$, therefore $(\ker_E(\bullet P_2))^0 = (\operatorname{im}_E(\bullet P_1))^0$ with $(\ker_E(\bullet P_2))^0 = \overline{\operatorname{im}_{E'}(P_2\bullet)}$ and $(\operatorname{im}_E(\bullet P_1))^0 = \ker_{E'}(P_1\bullet)$.

(2) If (4) is exact, then $\ker_{E'}(P_1\bullet) = \operatorname{im}_{E'}(P_2\bullet)$, therefore $(\operatorname{im}_E \bullet P_1)^0 = \operatorname{im}_{E'}(P_2\bullet)$, thus $(\operatorname{im}_E(\bullet P_1))^{00} = (\operatorname{im}_{E'}(P_2\bullet))^0 = \left(\overline{\operatorname{im}_{E'}(P_2\bullet)}\right)^0 = (\ker_E(\bullet P_2))^{00}$, and $\overline{\operatorname{im}_E(\bullet P_1)} = \ker_E(\bullet P_2)$ by the bipolar theorem since $\ker_E(\bullet P_2)$ is closed. ■

# 3 Injectivity vs. flatness

**Lemma and Definition 8** *(1) Let* $P \in \mathbf{A}^{k\times r}$*; Conditions (i)-(iv) below are equivalent:*
*(i)* $P\bullet : (E')^r \to (E')^k$ *is a strict morphism and so is also* $\bullet P : E^{1\times k} \to E^{1\times r}$*;*
*(ii)* $P\bullet : (E')^r \to (E')^k$ *is a strict morphism with closed image (in* $(E')^k$*);*
*(iii)* $\bullet P : E^{1\times k} \to E^{1\times r}$ *is a strict morphism with closed image (in* $E^{1\times r}$*);*
*(iv) both maps* $\bullet P : E^{1\times k} \to E^{1\times r}$ *and* $P\bullet : (E')^r \to (E')^k$ *have a closed image.*
*(2) The dual pair* $(E, E')$ *is said to be* Köthe *(or* $(\mathcal{K})$*, for short) over* $\mathbf{A}$ *if for any positive integers* $k, r$ *and any matrix* $P \in \mathbf{A}^{k\times r}$*, the following condition holds:* $\bullet P : E^{1\times k} \to E^{1\times r}$ *has a closed image if, and only if* $P\bullet : (E')^r \to (E')^k$ *has a closed image.*

**Proof.** (1): see, e.g., ([6], §32.3). ■

**Remark 9** *(1) The dual pair $(E, E')$ is not necessarily $(\mathcal{K})$ over* $\mathbf{A}$ *by ([1], §II.6, Remark 2 after Corol. 4 of Prop. 7); see, also, ([3], Prop. 2.3).*
*(2) Assume that E is a Fréchet space (e.g., $E = \mathcal{S}$), $E'$ is its dual and $\langle -, - \rangle$ is the canonical duality bracket. Then for any integer k, $E^{1\times k}$ is again a Fréchet space, and the dual pair $(E, E')$ is $(\mathcal{K})$ over* $\mathbf{A}$ *by ([1], §IV.4, Theorem 1).*
*(3) Whether the above holds when E is an arbitrary $(\mathcal{LF})$ space was mentioned in ([4], §15.10) as being an open question; to our knowledge, this question is still open today.*

**Lemma 10** *Let $P_1 \in \mathbf{A}^{k_1\times k_2}$.*
*(i) Assume that ${}_{\mathbf{A}}E'$ is injective. Then $\operatorname{im}_{E'}(P_1\bullet)$ is closed (or equivalently, $\bullet P_1 : E^{1\times k_1} \to E^{1\times k_2}$ is strict).*
*(ii) Assume that* $\operatorname{coker}_{\mathbf{A}}(\bullet P_1)$ *is torsion-free and $E_{\mathbf{A}}$ is flat. Then* $\operatorname{im}_E(\bullet P_1)$ *is closed (or equivalently, $P_1\bullet : (E')^{k_2} \to (E')^{k_1}$ is strict).*

**Proof.** (i): By Lemma 4(iii), there exists a matrix $P_0 \in \mathbf{A}^{k_0\times k_1}$ such that the sequence

$$\mathbf{A}^{1\times k_0} \xrightarrow{\bullet P_0} \mathbf{A}^{1\times k_1} \xrightarrow{\bullet P_1} \mathbf{A}^{1\times k_2}$$

is exact, and since ${}_{\mathbf{A}}E'$ is injective, the sequence

$$(E')^{k_2} \xrightarrow{P_1\bullet} (E')^{k_1} \xrightarrow{P_0\bullet} (E')^{k_0}$$

is exact. Therefore, $\operatorname{im}_{E'}(P_1\bullet) = \ker_{E'}(P_0\bullet)$, thus $\operatorname{im}_{E'}(P_1\bullet)$ is closed, and $\bullet P_1 : E^{1\times k_1} \to E^{1\times k_2}$ is strict by ([6], §32.3).

(ii): Since $\operatorname{coker}_{\mathbf{A}}(\bullet P_1)$ is torsion-free, by Lemma 4(iii) there exists $P_2 \in \mathbf{A}^{k_2\times k_3}$ such that the sequence (2) is exact. Since $E_{\mathbf{A}}$ is flat, the sequence (3) is exact. Therefore, $\operatorname{im}_E(\bullet P_1) = \ker_E(\bullet P_2)$ is closed, and $P_1\bullet : (E')^{k_2} \to (E')^{k_1}$ is strict by ([6], §32.3). ■

**Theorem 11** *Assume that the dual pair $(E, E')$ is $(\mathcal{K})$ over* $\mathbf{A}$*.*
*(1) If ${}_{\mathbf{A}}E'$ is injective, then $E_{\mathbf{A}}$ is flat.*
*(2) Conversely, if $E_{\mathbf{A}}$ is flat, then ${}_{\mathbf{A}}E'$ is* 1*-injective.*

**Proof.** (1) Assume that ${}_{\mathbf{A}}E'$ is injective and (2) is exact. Then (4) is exact, which implies that $\overline{\operatorname{im}_E(\bullet P_1)} = \ker_E(\bullet P_2)$ according to Lemma 7(2). By Lemma 10(i), $\operatorname{im}_{E'}(P_1\bullet)$ is closed. Since $(E, E')$ is $(\mathcal{K})$, $\operatorname{im}_E(\bullet P_1)$ is also closed. Hence $\operatorname{im}_E(\bullet P_1) = \ker_E(\bullet P_2)$, i.e., (3) is exact. This proves that $E_{\mathbf{A}}$ is flat.

(2) Assume $E_{\mathbf{A}}$ is flat and the sequence (5) is exact with $n = 1$. Then, the sequence

$$E^{1\times k_1} \xrightarrow{\bullet P_1} E^{1\times k_2} \xrightarrow{\bullet P_2} E^{1\times k_3} \xrightarrow{\bullet P_3} E^{1\times k_4}$$

is exact. By Lemma 7(1) we obtain

$$\overline{\operatorname{im}_{E'}(P_2\bullet)} = \ker_{E'}(P_1\bullet).$$

In addition, $\mathrm{im}_E(\bullet P_2) = \ker_E(\bullet P_3)$, thus $\mathrm{im}_E(\bullet P_2)$ is closed, and since $(E, E')$ is $(\mathcal{K})$, $\mathrm{im}_{E'}(P_2\bullet)$ is closed. This proves that $\mathrm{im}_{E'}(P_2\bullet) = \ker_{E'}(P_1\bullet)$, i.e., the sequence (4) is exact, and ${}_{\mathbf{A}}E'$ is 1-injective. ■

# 4 Concluding remarks

Consider a dual pair $(E, E')$ which is $(\mathcal{K})$ over the $\mathbf{k}$-algebra $\mathbf{A}$. As shown by Theorem 11, injectivity of ${}_{\mathbf{A}}E'$ implies flatness of $E_{\mathbf{A}}$. The converse does not hold, since flatness of $E_{\mathbf{A}}$ only implies 1-injectivity of ${}_{\mathbf{A}}E'$. For the sequence (5) to be exact with $n = 1$, $\mathrm{coker}_{\mathbf{A}}(\bullet P_1)$ must be torsion-free, therefore 1-injectivity is a weak property. To summarize, injectivity of ${}_{\mathbf{A}}E'$ is a stronger condition than flatness of the dual $E_{\mathbf{A}}$. A convenient characterization of dual pairs $(E, E')$ which are $(\mathcal{K})$ over the $\mathbf{k}$-algebra $\mathbf{A}$ is an interesting, probably difficult, and still open problem.

**Acknowledgement 12** *The author would like to thank Prof. Ulrich Oberst, whose comments were very helpful in improving the manuscript.*

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