ON p-COMPACT MAPPINGS AND p-APPROXIMATION

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Abstract. The notion of p-compact sets arises naturally from Grothendieck's characterization of compact sets as those contained in the convex hull of a norm null sequence. The definition, due to Sinha and Karn (2002), leads to the concepts of p-approximation property and p-compact operators, which form a ideal with its ideal norm κ_p. This paper examines the interaction between the p-approximation property and certain space of holomorphic functions, the p-compact analytic functions. In order to understand these functions we define a p-compact radius of convergence which allow us to give a characterization of the functions in the class. We show that p-compact holomorphic functions behave more like nuclear than compact maps. We use the ϵ-product of Schwartz, to characterize the p-approximation property of a Banach space in terms of p-compact homogeneous polynomials and in terms of p-compact holomorphic functions with range on the space. Finally, we show that p-compact holomorphic functions fit into the framework of holomorphy types which allows us to inspect the κ_p-approximation property. Our approach also allows us to solve several questions posed by Aron, Maestre and Rueda in [2].

arXiv:1107.1670v3 [math.FA] 4 Aug 2011

Introduction

In the Theory of Banach spaces (or more precisely, of infinite dimensional locally convex spaces), three concepts appear systematically related since the foundational articles by Grothendieck [21] and Schwartz [28]. We are referring to compact sets, compact operators and the approximation property. A Banach space E has the approximation property whenever the identity map can be uniformly approximated by finite rank operators on compact sets. Equivalently, if $E' \otimes E$, the subspace of finite rank operators, is dense in $\mathcal{L}_c(E;E)$, the space of continuous linear operators considered with the uniform convergence on compact sets. The other classical reformulation states that E has the approximation property if $F' \otimes E$ is dense in $\mathcal{K}(F;E)$, the space of compact operators, for all Banach spaces F. It was not until 1972 that Enflo provided us with the first example of a Banach space without the approximation property [19]. In the quest to a better understanding of this property and the delicate relationships inherit on different spaces of functions, important variants of the approximation property have emerged and were intensively studied. For the main developments on the subject we quote the comprehensive survey [9] and the references therein.

Key words and phrases. p-compact sets, holomorphic mappings, approximation properties.

This project was supported in part by UBACyT X218 and UBACyT X038.

Inspired in Grothendieck's result which characterize relatively compact sets as those contained in the convex hull of a norm null sequence of vectors of the space, Sinha and Karn [29] introduced the concept of relatively p-compact sets. Loosely speaking, these sets are determined by norm p-summable sequences. Associated to relatively p-compact sets we have naturally defined the notions of p-compact operators and the p-approximation property (see definitions below). Since relatively p-compact sets are, in particular, relatively compact, the p-approximation property can be seen as a way to weaken the approximation property. These three concepts were first studied by Sinha and Karn [29, 30] and, more recently, several authors continued the research on this subject [10, 12, 13, 14, 20].

This paper examines the interaction between the p-approximation property and the class of p-compact holomorphic functions. The connection between the approximation property and the space of holomorphic functions is not without precedent. The pioneer article on this topic is due to Aron and Schottenloher [5], who prove that a Banach space E has the approximation property if and only if $(\mathcal{H}(E), \tau_0)$, the space of the entire functions with the compact open topology, has the approximation property. Since then, many authors studied the approximation property for spaces of holomorphic functions in different contexts, see for instance [6, 7, 17, 18, 24]. Recently, Aron, Maestre and Rueda [2] prove that E has the p-approximation property if and only if $(\mathcal{H}(E), \tau_{0p})$ has the approximation property, here τ_{0p} denotes the topology of the uniform convergence on p-compact sets. The relation between the approximation property and holomorphic mappings was studied in detail in [5], where the class of compact holomorphic functions play a crucial role.

The article is organized as follows. In the first section we fix the notation and state some basic results on p-compact mappings. In Section 2 we study the behavior of p-compact homogeneous polynomials which can be considered as a polynomial Banach ideal with a natural norm denoted by κ_p. Following [10] we show that any p-compact homogeneous polynomial factors through a quotient of ℓ_1 and express the κ_p-norm in terms of an infimum of certain norms of all such possible factorizations. This result slightly improves [10, Theorem 3.1.]. We also show that the Aron-Berner extension preserves the class of p-compact polynomials with the same norm. Finally we show that there is an isometric relationship between the adjoint of p-compact polynomials and quasi p-nuclear operators, improving the analogous result for operators [14].

Section 3 is devoted to the study of p-compact holomorphic mappings. Since p-compact functions are compact, we pay special attention to the results obtained by Aron and Schottenloher [5], where the authors prove that any holomorphic function is compact if and only if each polynomial of its Taylor series expansion at 0 is compact, [5, Proposition 3.4]. Then, Aron, Maestre and Rueda [2, Proposition 3.5] show that each component of the Taylor series expansion of a p-compact holomorphic mapping has to be p-compact. We define a natural p-compact radius of convergence and, in Proposition 3.4, we give a characterization of this type of functions. Surprisingly, we found that p-compact holomorphic functions behave more like nuclear than compact mappings. We show this feature with two examples. Example 3.7 shows that Proposition 3.4 cannot be improved and also that it is possible to find an entire function whose polynomials at 0 are p-compact but the function fails to be p-compact at 0, which answers by the negative the question posed in [2, Problem 5.2]. In Example 3.8 we construct an entire function on ℓ_1 which is p-compact on the open unit ball, but it fails to be p-compact at the first element of the canonical basis of ℓ_1, giving an answer to [2, Problem 5.1].

We apply the results of Section 2 and 3 to study the p-approximation property in Section 4. We characterize the p-approximation property of a Banach space in terms of p-compact homogeneous polynomials with range on the space. Our proof requires the notion of the ϵ-product of Schwartz [28]. We also show that a Banach space E has the p-approximation property if and only if p-compact homogeneous polynomials with range on E can be uniformly approximated by finite rank polynomials. Then, we give the analogous result for p-compact holomorphic mappings endowed with the Nachbin topology, Proposition 4.7.

The final section is dedicated to the p-compact holomorphic mappings withing the framework of holomorphy types, concept introduced by Nachbin [25, 26] . This allows us to inspect the κ_p-approximation property introduced, in [13], in the spirit of [5, Theorem 4.1].

For general background on the approximation property and its different variants we refer the reader to [9, 22].

1. Preliminaries

Throughout this paper E and F will be Banach spaces. We denote by B_E the closed unit ball of E, by E' its topological dual, and by $\ell_p(E)$ the Banach space of the p-summable sequences of elements of E, endowed with the natural norm. Also, $c_0(E)$ denotes the space of null sequences of E endowed with the supremum norm. Following [29], we say that a subset $K \subset E$ is relatively p-compact, $1 \leq p \leq \infty$, if there exists a sequence $(x_n)_n \subset \ell_p(E)$ so that K is contained in the closure of $\{\sum \alpha_n x_n \colon (\alpha_n)_n \in B_{\ell_q}\}$, where B_{ℓ_q} denotes the closed unit ball of ℓ_q, with $\frac{1}{p} + \frac{1}{q} = 1$. We denote this set by p-co$\{x_n\}$ and its closure by $\overline{p\text{-co}}\{x_n\}$. Compact sets are considered ∞-compact sets, which corresponds to $q = 1$. When $p = 1$, the 1-convex hull is obtained by considering coefficients in B_{ℓ_∞} or, if necessary, with some extra work by coefficients in B_{c_0}, see [14, Remark 3.3].

Since the sequence $(x_n)_n$ in the definition of a relatively p-compact set K converges to zero, any p-compact set is compact. Such a sequence is not unique, then we may consider

$$m_p(K; E) = \inf\{\|(x_n)_n\|_p \colon K \subset p\text{-co}\{x_n\}\}$$

which measures the size of K as a p-compact set of E. For simplicity, along this work we write $m_p(K)$ instead of $m_p(K; E)$. When K is relatively p-compact and $(x_n)_n \subset \ell_p(E)$ is a sequence so that $K \subset p$-co$\{x_n\}$, any $x \in K$ has the form $x = \sum \alpha_n x_n$ with $(\alpha_n)_n$

some sequence in B_{ℓ_q}. By Hölder's inequality, we have that $\|x\| \leq \|(x_n)_n\|_{\ell_p(E)}$ and in consequence, $\|x\| \leq m_p(K)$, for all $x \in K$. We will use without any further mention the following equalities: $m_p(K) = m_p(\overline{K}) = m_p(\Gamma(K))$, where $\Gamma(K)$ denotes the absolutely convex hull of K, a relatively p-compact set.

The space of linear bounded operators from E to F will be denoted by $\mathcal{L}(E;F)$ and $E' \otimes F$ will denote its subspace of finite rank operators. As in [29], we say that an operator $T \in \mathcal{L}(E;F)$ is p-compact, $1 \leq p \leq \infty$, if $T(B_E)$ is a relatively p-compact set in F. The space of p-compact operators from E to F will be denoted by $K_p(E;F)$. If T belongs to $K_p(E;F)$, we define

$$\kappa_p(T) = \inf\big\{\|(y_n)_n\|_p \colon (y_n)_n \in \ell_p(F) \text{ and } T(B_E) \subset p\text{-co}\{y_n\}\big\},$$

where κ_∞ coincides with the supremum norm. It is easy to see that κ_p is a norm on $K_p(E;F)$ and following [27] (see also [14]) it is possible to show that the pair (K_p, κ_p) is a Banach operator ideal.

The Banach ideal of p-compact operators is associated by duality with the ideal of quasi-p-nuclear operators, introduced and studied by Persson and Pietsch [27]. A linear operator $T \in \mathcal{L}(E;F)$ is said to be quasi p-nuclear if $j_F T \colon E \to \ell_\infty(B_{F'})$ is a p-nuclear operator, where j_F is the natural isometric embedding from F into $\ell_\infty(B_{F'})$. It is known that an operator T is quasi p-nuclear if and only if there exists a sequence $(x_n')_n \subset \ell_p(E')$, such that

$$\|Tx\| \leq \Big(\sum_n |x_n'(x)|^p\Big)^{\frac{1}{p}},$$

for all $x \in E$ and the quasi p-nuclear norm of T is given by

$$\nu_p^Q(T) = \inf\{\|(x_n')_n\|_p \colon \|Tx\|^p \leq \sum_n |x_n'(x)|^p, \quad \forall x \in E\}.$$

The space of quasi p-nuclear operators from E to F will be denoted by $\mathcal{QN}_p(E;F)$. The duality relationship is as follows. Given $T \in \mathcal{L}(E;F)$, T is p-compact if and only if its adjoint is quasi p-nuclear. Also, T is is quasi p-nuclear if and only if its adjoint is p-compact, see [14, Corollary 3.4] and [14, Proposition 3.8].

A mapping $P \colon E \to F$ is an m-homogeneous polynomial if there exists a (unique) symmetric m-linear form $\overset{\vee}{P} \colon \underbrace{E \times \cdots \times E}_{m} \to F$ such that

$$P(x) = \overset{\vee}{P}(x, \ldots, x),$$

for all $x \in E$. The space of m-homogeneous continuous polynomials from E to F will be denoted by $\mathcal{P}(^m E;F)$, which is a Banach space considered with the supremum norm

$$\|P\| = \sup\{\|P(x)\| \colon x \in B_E\}.$$

Every $P \in \mathcal{P}(^mE, F)$ has associated two natural mappings: the *linearization* denoted by $L_P \in \mathcal{L}(\bigotimes_{\pi_s}^m E; F)$, where $\bigotimes_{\pi_s}^m E$ stands for the completion of the symmetric m-tensor product endowed with the symmetric projective norm. Also we have the polynomial $\overline{P} \in \mathcal{P}(^mE'', F'')$, which is the canonical extension of P from E to E'' obtained by weak-star density, known as the Aron-Berner extension of P [1]. While $\|L_P\| \leq \frac{m^m}{m!}\|P\|$, $\overline{P}$ is an isometric extension of P [11].

A mapping $f\colon E \to F$ is holomorphic at $x_0 \in E$ if there exists a sequence of polynomials $P_m f(x_0) \in \mathcal{P}(^mE, F)$ such that

$$f(x) = \sum_{m=0}^{\infty} P_m f(x_0)(x - x_0),$$

uniformly for all x in some neighborhood of x_0. We say that $\sum_{m=0}^{\infty} P_m f(x_0)$, is the Taylor series expansion of f at x_0 and that $P_m f(x_0)$ is the m-component of the series at x_0. A function is said to be holomorphic or entire if it is holomorphic at x for all $x \in E$. The space of entire functions from E to F will be denote by $\mathcal{H}(E; F)$.

We refer the reader to [16, 23] for general background on polynomials and holomorphic functions.

2. The p-compact polynomials

We want to understand the behavior of of p-compact holomorphic mappings. The definition, due to Aron, Maestre and Rueda [2] was introduced as a natural extension of p-compact operators to the non linear case. In [2] the authors show that for any p-compact holomorphic function each m-homogeneous polynomial of its Taylor series expansion must be p-compact. Motivated by this fact we devote this section to the study of polynomials.

Recall that $P \in \mathcal{P}(^mE; F)$ is said to be p-compact, $1 \leq p \leq \infty$, if there exists a sequence $(y_n)_n \in \ell_p(F)$, $(y_n)_n \in c_0$ if $p = \infty$, such that $P(B_E) \subset \{\sum_{n=1}^{\infty} \alpha_n y_n\colon (\alpha_n)_n \in B_{\ell_q}\}$, where $\frac{1}{p} + \frac{1}{q} = 1$. In particular, any p-compact polynomial is compact. We denote by $\mathcal{P}_{K_p}(^mE; F)$ the space of p-compact m-homogeneous polynomials and by $\mathcal{P}_K(^mE; F)$ the space of compact polynomials. Following [13], for $P \in \mathcal{P}_{K_p}(^mE; F)$ we may define

$$\kappa_p(P) = \inf\left\{\|(y_n)_n\|_p\colon (y_n)_n \in \ell_p(F) \textit{ and } P(B_E) \subset p\text{-co}\{y_n\}\right\}.$$

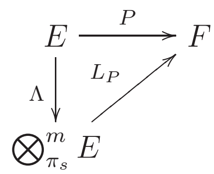
In other words, $\kappa_p(P) = \mathfrak{m}_p(P(B_E))$. It is easy to see that κ_p is, in fact, a norm satisfying that $\|P\| \leq \kappa_p(P)$, for any p-compact homogeneous polynomial P, and that $(\mathcal{P}_{K_p}(^mE; F), \kappa_p)$ is a polynomial Banach ideal. Furthermore, we will see that any p-compact m-homogeneous polynomial factors through a p-compact operator and a continuous m-homogeneous polynomial. Also, the κ_p-norm satisfies the natural infimum condition.

Lemma 2.1. *Let E and F be Banach spaces and let $P \in \mathcal{P}(^mE; F)$. The following statements are equivalent.*

(i) *P is p-compact.*
(ii) *$L_P\colon \bigotimes_{\pi_s}^m E \to F$, the linearization of P, is a p-compact operator.*

Moreover, we have $\kappa_p(P) = \kappa_p(L_P)$.

Proof. To show the equivalence, we appeal to the familiar diagram, where Λ is a norm one homogeneous polynomial ($\Lambda(x) = x^m$) and $P = L_P\Lambda$:



Note that the open unit ball of $\bigotimes_{\pi_s}^m E$ is the absolutely convex hull $\Gamma\{x^m\colon \|x\| < 1\}$. Then, we have that $P(B_E) \subset \Gamma(\{L_P(x^m)\colon \|x\| < 1\}) = \Gamma(P(B_E))$. Now, the equality $L_P(\Gamma\{x^m\colon \|x\| < 1\}) = \Gamma(P(B_E))$ shows that any sequence $(y_n)_n \in \ell_p(F)$ involved in the definition of $\kappa_p(P)$ is also involved in the definition of $\kappa_p(L_P)$ and vice versa, which completes the proof. □

Sinha and Karn [29, Theorem 3.2] show that a continuous operator is p-compact if and only if it factorizes via a quotient of ℓ_q and some sequence $y = (y_n)_n \in \ell_p(F)$. Their construction is as follows. Given $y = (y_n)_n \in \ell_p(F)$ there is a canonical continuous linear operator $\theta_y\colon \ell_q \to F$ associated to y such that on the unit basis of ℓ_q satisfies $\theta_y(e_n) = y_n$. Then, θ_y is extended by continuity and density. Associated to θ_y there is a natural injective operator $\widetilde{\theta_y}\colon \ell_q/\ker\theta_y \to F$ such that $\widetilde{\theta_y}[(\alpha_n)_n] = \theta_y((\alpha_n)_n)$ with $\|\widetilde{\theta_y}\| = \|\theta_y\| \le \|y\|_{\ell_p(F)}$. Now, if we start with T belonging to $K_p(E;F)$, there exists $y = (y_n)_n \in \ell_p(F)$ so that $T(B_E) \subset p\text{-co}\{y_n\}$ and, via θ_y, it is possible to define an operator $T_y\colon E \to \ell_q/\ker\theta_y$ by $T_y(x) = [(\alpha_n)_n]$ where $(\alpha_n)_n \in \ell_q$ is a sequence satisfying that $T(x) = \sum_{n=1}^\infty \alpha_n y_n$, which exists by the p-compactness of T. Note that the operator T_y is well defined, $\|T_y\| \le 1$ and T satisfies the factorization $T = \widetilde{\theta_y}T_y$, where $\widetilde{\theta_y}$ is p-compact and $\kappa_p(\widetilde{\theta_y}) = \|y\|_{\ell_p(F)}$. It is clear that, if an operator S admits such factorization, then S is p-compact.

Given a sequence $y = (y_n)_n \in \ell_p(F)$, Choi and Kim [10, Theorem 3.1] factorize the operator θ_y through ℓ_1 as the composition of a compact and a p-compact operator, concluding that any p-compact operator T factors through a quotient space of ℓ_1 as a composition of a compact mapping with a p-compact operator.

The behavior of p-compact polynomials is similar to that described for p-compact operators. In the proposition below a slight improvement of [10, Theorem 3.1] is obtained. We prove that the corresponding factorizations via a quotient of ℓ_1 suffice to characterize $\kappa_p(P)$ for P a p-compact polynomial and therefore, for a p-compact operator. In order to do so, we will use the following technical lemma. Although we believe it should be a known basic result, we have not found it mentioned in the literature as we need it. Thus, we also include a proof.

First, we fix some notation: for $\sigma \subset \mathbb{N}$ a finite set, we write $S_\sigma(r) = \sum_{n\in\sigma} r^n$. The following fact has a direct proof. Let $\sigma_1, \sigma_2, \dots$ be a disjoint sequence of finite sets such that its union, $\cup_n \sigma_n$, is infinite. Take $0 < r < t < 1$ and consider the sequence $\beta = (\beta_n)_n$ defined by $\beta_n = \frac{S_{\sigma_n}(r)}{S_{\sigma_n}(t)}$, for all n. Then, β belongs to B_{c_0}.

Lemma 2.2. *Let $1 \le p < \infty$. Given $(x_n)_n \in \ell_p$ and $\varepsilon > 0$, there exists $\beta = (\beta_n)_n \in B_{c_0}$ such that $(\frac{x_n}{\beta_n})_n \in \ell_p$ and $\|(\frac{x_n}{\beta_n})_n\|_p \le \|(x_n)_n\|_p(1+\varepsilon)$.*

Proof. It is enough to prove the lemma for $(x_n)_n \in \ell_1$ with $\|(x_n)_n\|_1 = 1$. Indeed, suppose the result holds for this case. Fix $(x_n)_n \in \ell_p$, a nonzero sequence and consider the sequence $(z_n)_n$ defined by $z_n = \frac{x_n^p}{\|(x_n)_n\|_p^p}$, which is a norm one element of ℓ_1. Given $\varepsilon > 0$, take $(\alpha_n)_n \in B_{c_0}$ such that $\|(\frac{z_n}{\alpha_n})_n\|_1 \le 1 + \varepsilon$ and the conclusion follows with $\beta = (\alpha_n^{1/p})_n$.

Now, suppose $(x_n)_n \in \ell_1$ and $\|(x_n)_n\|_1 = 1$, we also may assume that $x_n \neq 0$ for all n. Choose $\delta > 0$ such that $\frac{1+\delta}{1-\delta} < 1 + \varepsilon$. We construct β inductively.

Since $|x_1| < 1$, there exists $m_1 \in \mathbb{N}$ such that if $\sigma_1 = \{1, 2, \dots, m_1\}$, then $|x_1| < S_{\sigma_1}(\frac{1}{2})$. Let $n_1 \ge 2$ be the integer so that $\sum_{n<n_1} |x_n| < S_{\sigma_1}(\frac{1}{2})$ and $\sum_{n=1}^{n_1} |x_n| \ge S_{\sigma_1}(\frac{1}{2})$. Then, there exists $0 < t_{n_1} \le 1$ such that

$$\sum_{n<n_1} |x_n| + t_{n_1}|x_{n_1}| = S_{\sigma_1}(\tfrac{1}{2}) \quad \text{and} \quad (1 - t_{n_1})|x_{n_1}| + \sum_{n>n_1} |x_n| = \sum_{n>m_1} (\tfrac{1}{2})^n.$$

Now, since $(1 - t_{n_1})|x_{n_1}| + |x_{n_1+1}| < \sum_{n>m_1}(\frac{1}{2})^n$, there exists $m_2 > m_1$ such that if $\sigma_2 = \{m_1 + 1, \dots, m_2\}$ we have that $(1 - t_{n_1})|x_{n_1}| + |x_{n_1+1}| < S_{\sigma_2}(\frac{1}{2})$. Let $n_2 > n_1 + 1$ be the integer such that $(1-t_{n_1})|x_{n_1}| + \sum_{n_1+1}^{n_2-1} |x_n| < S_{\sigma_2}(\frac{1}{2})$ and $(1-t_{n_1})|x_{n_1}| + \sum_{n_1+1}^{n_2} |x_n| \ge S_{\sigma_2}(\frac{1}{2})$. Then, there exists $0 < t_{n_2} \le 1$ such that

$$(1-t_{n_1})|x_{n_1}| + \sum_{n_1<n<n_2} |x_n| \; + \; t_{n_2}|x_{n_2}| = S_{\sigma_2}(\tfrac{1}{2}) \quad \text{and} \quad (1-t_{n_2})|x_{n_2}| + \sum_{n>n_2} |x_n| = \sum_{n>m_2} (\tfrac{1}{2})^n.$$

Continuing this procedure, we can find $n_j, m_j \in \mathbb{N}$, $n_j + 1 < n_{j+1}, m_j < m_{j+1}$ $\forall j$, and $0 < t_{n_j} \le 1$ such that, if $\sigma_j = \{m_{j-1} + 1, \dots, m_j\}$, then we obtain

$$(1 - t_{n_{j-1}})|x_{n_{j-1}}| + \sum_{n_{j-1}<n<n_j} |x_n| \; + \; t_{n_j}|x_{n_j}| = S_{\sigma_j}(\tfrac{1}{2}), \quad \text{for all } j \ge 2.$$

Now, with $n_0 = 0$, choose β such that

$$\beta_n^{-1} = \begin{cases} t_{n_j} \dfrac{S_{\sigma_j}(\frac{1+\delta}{2})}{S_{\sigma_j}(\frac{1}{2})} + (1 - t_{n_j}) \dfrac{S_{\sigma_{j+1}}(\frac{1+\delta}{2})}{S_{\sigma_{j+1}}(\frac{1}{2})} & \text{if} \qquad n = n_j, \\[2ex] \dfrac{S_{\sigma_j}(\frac{1+\delta}{2})}{S_{\sigma_j}(\frac{1}{2})} & \text{if} \quad n_{j-1} < n < n_j. \end{cases}$$

Put $c_j = \frac{S_{\sigma_j}(\frac{1+\delta}{2})}{S_{\sigma_j}(\frac{1}{2})}$, as we mentioned above $|c_j| \geq 1$ and $c_j \to \infty$, then $(\beta_n)_n \in B_{c_0}$. Finally, we have

$$\begin{aligned}
\|(\tfrac{x_n}{\beta_n})_n\|_1 &= \sum_{n<n_1} \beta_n^{-1}|x_n| + \beta_{n_1}^{-1}|x_{n_1}| + \sum_{j\geq 2}\sum_{n_{j-1}<n<n_j} \beta_n^{-1}|x_n| + \sum_{j\geq 2}\beta_{n_j}^{-1}|x_{n_j}| \\
&= c_1 \sum_{n<n_1} |x_n| + t_{n_1}c_1|x_{n_1}| + (1-t_{n_1})c_2|x_{n_1}| \\
&\quad + \sum_{j\geq 2}\sum_{n_{j-1}<n<n_j} c_j|x_n| + \sum_{j\geq 2} t_{n_j}c_j|x_{n_j}| + (1-t_{n_j})c_{j+1}|x_{n_j}| \\
&= c_1\Big(\sum_{n<n_1}|x_n| + t_{n_1}|x_{n_1}|\Big) + \sum_{j\geq 2} c_j\Big[(1-t_{n_{j-1}})|x_{n_{j-1}}| + \sum_{n_{j-1}<n<n_j}|x_n| + t_{n_j}|x_{n_j}|\Big] \\
&= \textstyle\sum_{j=1}^{\infty} S_{\sigma_j}(\frac{1+\delta}{2}) = \frac{1+\delta}{1-\delta} < 1+\varepsilon.
\end{aligned}$$

Thus, the lemma is proved. □

Proposition 2.3. *Let E and F be Banach spaces, $1 \leq p < \infty$, and $P \in \mathcal{P}(^mE;F)$. The following statements are equivalent.*

(i) $P \in \mathcal{P}_{K_p}(^mE;F)$.

(ii) *There exist a sequence $(y_n)_n \in \ell_p(F)$, a polynomial $Q \in \mathcal{P}(^mE;\ell_q/\ker\theta_y)$ and an operator $T \in \mathcal{K}_p(\ell_q/\ker\theta_y;F)$ such that $P = T\circ Q$. In this case*

$$\kappa_p(P) = \inf\{\|Q\|\kappa_p(T)\},$$

where the infimum is taken over all the possible factorizations as above.

(iii) *There exist a closed subspace $M \subset \ell_1$, a polynomial $Q \in \mathcal{P}(^mE;\ell_q/\ker\theta_y)$ and operators $R \in K_p(\ell_q/\ker\theta_y;\ell_1/M)$ and $S \in \mathcal{K}(\ell_1/M;F)$ such that $P = SRQ$. In this case*

$$\kappa_p(P) = \inf\{\|Q\|\kappa_p(R)\|S\|\},$$

where the infimum is taken over all the possible factorizations as above.

Proof. It is clear that either (ii) or (iii) implies (i).

Now, assume that P is a p-compact polynomial. By Lemma 2.1, we have that $P = L_P\Lambda$, where L_P is a p-compact operator and Λ is a norm one polynomial. Applying [29, Theorem 3.2] to L_p we have that $L_P = \widetilde{\theta}_y(L_P)_y$ where $y = (y_n)_n \in \ell_p(F)$ is such that $L_P(B_{\otimes^m_{\pi_s}E}) \subseteq p\text{-co}\{y_n\}$, with $\kappa_p(\widetilde{\theta}_y) = \|y\|_{\ell_p(F)}$ and $\|(L_P)_y\| \leq 1$.

Then, $P = TQ$ for $T = \widetilde{\theta}_y$ and $Q = (L_P)_y\Lambda$, see the diagram below. Also, we have that whenever $P = TQ$ as in (ii), $\kappa_p(P) \leq \kappa_p(T)\|Q\|$. Again by Lemma 2.1, $\kappa_p(L_P) = \kappa_p(P)$ and, since $\|\Lambda\| = 1$, the infimum is attained.

To show that (i) implies (iii), we again consider the p-compact operator L_P and the factorization $L_P = \widetilde{\theta}_y(L_P)_y$. Now we modify the proof in [10, Theorem 3.1] to obtain the infimum equality. Fix $\varepsilon > 0$ and suppose $y = (y_n)_n$ is such that $\|(y_n)_n\|_{\ell_p(F)} \leq \kappa_p(L_P) + \varepsilon$. Using Lemma 2.2, choose $(\beta_n)_n \in B_{c_0}$ such that $(\frac{y_n}{\beta_n})_n$ belongs to $\ell_p(F)$ with $\|(\frac{y_n}{\beta_n})_n\|_{\ell_p} \leq \|(y_n)_n\|_{\ell_p(F)} + \varepsilon$.

For simplicity, we suppose that $y_n \neq 0$, for all n. Now, define the operator $s\colon \ell_1 \to F$ as $s((\gamma_n)_n) = \sum_n \gamma_n y_n \frac{\beta_n}{\|y_n\|}$ which satisfies $\|s\| \leq \|\beta\|_\infty \leq 1$, and consider the closed subspace $M = \ker s \subset \ell_1$. Then, we may set $R\colon \ell_q/\ker\theta_y \to \ell_1/M$ the linear operator such that $R([(\alpha_n)_n]) = [(\alpha_n \frac{\|y_n\|}{\beta_n})_n]$ and, with S the quotient map associated to s we get the following diagram:

$$\begin{array}{ccccc} E & \xrightarrow{\quad P \quad} & F & & \\ \Big\downarrow{\scriptstyle\Lambda} & \nearrow{\scriptstyle L_P} & \Big\uparrow{\scriptstyle\widetilde{\theta}_y} & \nwarrow{\scriptstyle S} & \\ \bigotimes^m_{\pi_s} E & \xrightarrow[(L_P)_y]{} & \ell_q/\ker\theta_y & \xrightarrow[R]{} & \ell_1/M \end{array}$$

Then, $P = SRQ$, with $Q = (L_P)_y\Lambda$ and $\|S\|, \|Q\| \leq 1$. Since $R(B_{\ell_q/\ker\theta_y}) \subset p\text{-co}\{[\frac{\|y_n\|}{\beta_n} e_n]\}$, by the choice of β, R is p-compact and $\kappa_p(R) \leq \|(\frac{\|y_n\|}{\beta_n})_n\|_p \leq \|(y_n)_n\|_{\ell_p(F)} + \varepsilon$.

Finally,

$$\kappa_p(P) \leq \|S\|\kappa_p(R)\|Q\| \leq \|(y_n)_n\|_{\ell_p(F)} + \varepsilon \leq \kappa_p(L_P) + 2\varepsilon.$$

By Lemma 2.1, the proof is complete. □

It was shown in [14], that an operator $T\colon E \to F$ is p-compact if and only if its bitranspose $T''\colon E'' \to F''$ is p-compact with $\kappa_p(T'') \leq \kappa_p(T)$. In [20], it is shown that, in fact, $\kappa_p(T'') = \kappa_p(T)$ regardless T'' is considered as an operator on F'' or, thanks to the Gantmacher theorem, as an operator on F. This result, allows us to show that the Aron-Berner extension is a κ_p-isometric extension which preserves the ideal of p-compact homogeneous polynomials. Recall that $\overline{P}$ denotes the Aron-Berner extension of P.

Proposition 2.4. *Let E and F be Banach spaces, $1 \leq p < \infty$, and $P \in \mathcal{P}(^mE; F)$. Then P is p-compact if and only if $\overline{P}$ is p-compact. Moreover, $\kappa_p(P) = \kappa_p(\overline{P})$.*

Proof. Clearly, P is p-compact whenever $\overline{P}$ is and also $\kappa_p(P) \leq \kappa_p(\overline{P})$. Now, suppose that P is p-compact. By Lemma 2.1, we can factorize P via its linearization as $P = L_P\Lambda$, with $\|\Lambda\| = 1$ and L_P a p-compact operator. Since $\overline{P} = L''_P\overline{\Lambda}$, by [14], $\overline{P}$ is p-compact with $\kappa_p(\overline{P}) \leq \kappa_p(L''_P)$. Now, applying [20] and Lemma 2.1, $\kappa_p(L''_P) = \kappa_p(L_P) = \kappa_p(P)$, which gives the reverse inequality. □

We finish this section by relating the transpose of p-compact polynomials with quasi p-nuclear operators. Given an homogeneous polynomial P its adjoint is defined as the linear operator $P'\colon F' \to \mathcal{P}(^mE)$ given by $P'(y') = y' \circ P$. In [20], it is shown that the transpose of a p-compact linear operator satisfies the equality $\kappa_p(T) = \nu_p^Q(T')$. Since $P' = L'_P$, where L_P is the linearization of P, using Lemma 2.1 we immediately have:

Corollary 2.5. *An homogeneous polynomial $P \in \mathcal{P}(^mE;F)$ is p-compact if and only if its transpose $P' \in \mathcal{L}(F';\mathcal{P}(^mE))$ is quasi p-nuclear, and $\kappa_p(P) = \nu_p^Q(P')$.*

When this manuscript was completed we learned that R. Aron and P. Rueda were also been working on p-compact polynomials [3]. They obtained Lemma 2.1 and a non isometric version of the corollary above.

3. The p-compact holomorphic mappings

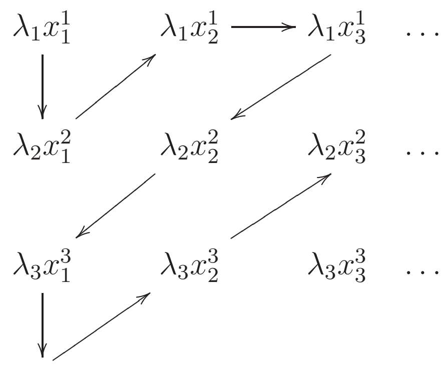
In this section we undertake a detailed study of p-compact holomorphic mappings, whose definition recovers the notion of compact holomorphic mappings for $p = \infty$, [2]. Recall that for E and F Banach spaces, $1 \leq p \leq \infty$, a holomorphic function $f\colon E \to F$ is said to be p-compact at x_0 if there is a neighborhood V_{x_0} of x_0, such that $f(V_{x_0}) \subset F$ is a relatively p-compact set. Also, $f \in \mathcal{H}(E;F)$ is said to be p-compact if it is p-compact at x for all $x \in E$. We denote by $\mathcal{H}_{K_p}(E;F)$ the space of p-compact entire functions and by $\mathcal{H}_K(E;F)$ the space of compact holomorphic mappings. For homogeneous polynomials, it is equivalent to be compact (p-compact) at some point of E or to be compact (p-compact) at every point of the space [2, 5]. The same property remains valid for compact holomorphic mappings [5, Proposition 3.4]. We will see that the situation is very different for p-compact holomorphic functions, $1 \leq p < \infty$. Furthermore, we will show that p-compact holomorphic mappings, $1 \leq p < \infty$, behave more like nuclear than compact holomorphic functions.

Having in mind that $(\mathcal{P}_{K_p}(^mE;F), \kappa_p)$ is a polynomial Banach ideal with $\kappa_p(P) = m_p(P(B_E))$, and that all polynomials in the Taylor series expansion of a p-compact holomorphic function at x_0 are p-compact [2, Proposition 3.5], we propose to connect the concepts of p-compact holomorphic mappings and the size of p-compact sets measured by m_p. We start with a simple but useful lemma.

Lemma 3.1. *Let E be a Banach space and consider $K_1, K_2, \ldots$ a sequence of relatively p-compact sets in E, $1 \leq p < \infty$. If $\sum_{j=1}^\infty m_p(K_j) < \infty$, then the series $\sum_{j=1}^\infty x_j$ is absolutely convergent for any choice of $x_j \in K_j$ and the set $K = \{\sum_{j=1}^\infty x_j \colon x_j \in K_j\}$ is relatively p-compact with $m_p(K) \leq \sum_{j=1}^\infty m_p(K_j) < \infty$.*

Proof. Note that K is well defined since for $x_j \in K_j$, $\|x_j\| \leq m_p(K_j)$, for all j and $\sum_{j=1}^\infty m_p(K_j) < \infty$.

First, suppose that $p > 1$ and fix $\varepsilon > 0$. For each $j \in \mathbb{N}$, we may assume that K_j is nonempty and we may choose $(x_n^j)_n \in \ell_p(E)$ such that $K_j \subset p\text{-co}\{x_n^j \colon n \in \mathbb{N}\}$ with $\|(x_n^j)_n\|_p \leq m_p(K_j)(1 + \frac{\varepsilon}{2^j} m_p(K_j)^{-1})^{1/p}$. Now, take $\lambda_j = m_p(K_j)^{-1/q}$, where $\frac{1}{p} + \frac{1}{q} = 1$ and define the sequence $(z_k)_k \subset E$ such that each term is of the form $\lambda_j x_n^j$, following the standard order:



Then

$$\begin{aligned}\sum_{k=1}^{\infty} \|z_k\|^p = & \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_j^p \|x_n^j\|^p \\ = & \sum_{j=1}^{\infty} \mathfrak{m}_p(K_j)^{-p/q} \|(x_n^j)_n\|_{\ell_p(E)}^p \\ \leq & \sum_{j=1}^{\infty} \mathfrak{m}_p(K_j)^{-p/q} \mathfrak{m}_p(K_j)^p (1 + \frac{\varepsilon}{2^j} \mathfrak{m}_p(K_j)^{-1}) \\ = & \sum_{j=1}^{\infty} \mathfrak{m}_p(K_j) + \varepsilon.\end{aligned}$$

Hence, $(z_k)_k$ belongs to $\ell_p(E)$ and $\|(z_k)_k\|_{\ell_p(E)} \leq (\sum_{j=1}^{\infty} \mathfrak{m}_p(K_j) + \varepsilon)^{1/p}$.

Now, if $K = \{\sum_{j=1}^{\infty} x_j : x_j \in K_j\}$, we claim that $K \subset (\sum_{j=1}^{\infty} \mathfrak{m}_p(K_j))^{1/q} p\text{-co}\{z_k\}$. Indeed, if $x \in K$, then $x = \sum_{j=1}^{\infty} x_j$ with $x_j \in K_j$. Fix $j \in \mathbb{N}$, there exists $(\alpha_n^j)_n \in B_{\ell_q}$ such that $x_j = \sum_{n=1}^{\infty} \alpha_n^j x_n^j$. Then, $x = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \alpha_n^j x_n^j$ and the series converges absolutely as the partial sums of $|\alpha_n^j| \|x_n^j\|$ are clearly convergent with the order given above. We may write $x = \sum \mathfrak{m}_p(K_j)^{1/q} \alpha_n^j \lambda_j x_n^j$ with

$$\sum |\mathfrak{m}_p(K_j)^{1/q} \alpha_n^j|^q = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} |\alpha_n^j|^q \mathfrak{m}_p(K_j) \leq \sum_{j=1}^{\infty} \mathfrak{m}_p(K_j).$$

Then $K \subset (\sum_{j=1}^{\infty} \mathfrak{m}_p(K_j))^{1/q} p\text{-co}\{z_k\}$. It follows that K is p-compact and

$$\mathfrak{m}_p(K) \leq (\sum_{j=1}^{\infty} \mathfrak{m}_p(K_j))^{1/q} \|(z_k)_k\|_{\ell_p(E)} \leq (\sum_{j=1}^{\infty} \mathfrak{m}_p(K_j))^{1/q} (\sum_{j=1}^{\infty} \mathfrak{m}_p(K_j) + \varepsilon)^{1/p}.$$

Letting $\varepsilon \to 0$, we conclude that $\mathfrak{m}_p(K) \leq \sum_{j=1}^{\infty} \mathfrak{m}_p(K_j)$.

With the usual modifications, the case $p = 1$ follows from the above construction considering $\lambda_j = 1$, for all j. □

Aron, Maestre and Rueda [2, Proposition 3.5] prove that if f is a p-compact holomorphic mapping at some $x_0 \in E$, every homogeneous polynomial of the Taylor series expansion of f at x_0 is p-compact. At the light of the existent characterization for compact holomorphic mappings [5], they also wonder if the converse is true [2, Problem 5.2]. To tackle this question we need to define the p-compact radius of convergence of a function f at $x_0 \in E$.

Definition 3.2. *Let E and F be Banach spaces, $f \in \mathcal{H}(E;F)$ and $x_0 \in E$. If $\sum_{m=0}^{\infty} P_m f(x_0)$ is the Taylor series expansion of f at x_0, we say that*

$$r_p(f, x_0) = 1/ \limsup \kappa_p(P_m f(x_0))^{1/m}$$

is the radius of p-compact convergence of f at x_0, for $1 \leq p < \infty$.

As usual, we understand that whenever $\limsup \kappa_p(P_m f(x_0))^{1/m} = 0$, the radius of p-compact convergence is infinite. Also, if $P_m f(x_0)$ fails to be p-compact for some m, f fails to be p-compact and $r_p(f, x_0) = 0$.

The following lemma is obtained by a slight modification of the generalized Cauchy formula given in the proof of [2, Proposition 3.5], which asserts that if $f \in \mathcal{H}(E;F)$ and $x_0 \in E$, fixed $\varepsilon > 0$ we have that $P_m f(x_0)(B_\varepsilon(0)) \subset \overline{\text{co}}\{f(B_\varepsilon(x_0))\}$, where $B_\varepsilon(x_0)$ stands for the open ball of center x_0 and radius ε. We state the result as it will be used in Section 4, also we are interested in measuring the m_p-size of $P_m f(x_0)(V)$ in terms of the m_p-size of $f(x_0 + V)$ for certain absolutely convex open sets $V \subset E$.

Lemma 3.3. *Let E and F be Banach spaces, let $x_0 \in E$ and let $V \subset E$ be an absolutely convex open set. Let $f \in \mathcal{H}(E;F)$ whose Taylor series expansion at x_0 is given by $\sum_{m=0}^{\infty} P_m f(x_0)$. Then*

(a) *$P_m f(x_0)(V) \subset \overline{\text{co}}\{f(x_0 + V)\}$, for all m.*
(b) *If $f(x_0 + V)$ is relatively p-compact then $m_p(P_m f(x_0)(V)) \leq m_p(f(x_0 + V))$, for all m.*

Now we are ready to give a characterization of p-compact functions in terms of the polynomials in its Taylor series expansion and the p-compact radius of convergence.

Proposition 3.4. *Let E and F be Banach spaces and let $f \in \mathcal{H}(E;F)$ whose Taylor series expansion at x_0 is given by $\sum_{m=0}^{\infty} P_m f(x_0)$. For $1 \leq p < \infty$, the following statements are equivalent.*

(i) *f is p-compact at x_0.*
(ii) *$P_m f(x_0) \in \mathcal{P}_{K_p}(^m E; F)$, for all m and $\limsup \kappa_p(P_m f(x_0))^{1/m} < \infty$.*

Proof. To prove that (i) implies (ii), take $\varepsilon > 0$ such that $f(B_\varepsilon(x_0))$ is relatively p-compact and $f(x) = \sum_{m=1}^{\infty} P_m f(x_0)(x - x_0)$, with uniform convergence in $B_\varepsilon(x_0)$. By [2, Proposition 3.5], $P_m f(x_0)(\varepsilon B_E) \subset \overline{\text{co}}\{f(B_\varepsilon(x_0))\}$ and $P_m f(x_0)$ is p-compact, for all m. Moreover, by the lemma above,

$$\kappa_p(P_m f(x_0)) = m_p(P_m f(x_0)(B_E)) = \frac{1}{\varepsilon^m} m_p(P_m f(x_0)(\varepsilon B_E)) \leq \frac{1}{\varepsilon^m} m_p(\overline{co}\{f(B_\varepsilon(x_0))\}).$$

It follows that $\limsup \kappa_p(P_m f(x_0))^{1/m} \leq \frac{1}{\varepsilon}$, as we wanted to prove.

Conversely, suppose that $\limsup \kappa_p(P_m f(x_0))^{1/m} = C > 0$ and choose $0 < \varepsilon < r_p(f, x_0)$ such that, for all $x \in B_\varepsilon(x_0)$, $f(x) = \sum_{m=1}^\infty P_m f(x_0)(x - x_0)$, with uniform convergence. Now we have

$$f(B_\varepsilon(x_0)) \subset \{\sum_{m=1}^\infty x_m \colon x_m \in P_m f(x_0)(\varepsilon B_E)\}.$$

By Lemma 3.1, we obtain the result if we prove that $\sum_{m=1}^\infty \mathfrak{m}_p(P_m f(x_0)(\varepsilon B_E)) < \infty$, which follows from the equality

$$\sum_{m=1}^\infty \mathfrak{m}_p(P_m f(x_0)(\varepsilon B_E)) = \sum_{m=1}^\infty \varepsilon^m \kappa_p(P_m f(x_0)),$$

and the choice of ε. □

Remark 3.5. Let f be a p-compact holomorphic mapping at x_0 and let $\sum_{m=0}^\infty P_m f(x_0)$ be its Taylor series expansion at x_0. Then, if $\varepsilon < r_p(f, x_0)$,

$$\mathfrak{m}_p(f(B_\varepsilon(x_0)) \leq \sum_{m=1}^\infty \mathfrak{m}_p(P_m f(x_0)(\varepsilon B_E)),$$

where the right hand series is convergent.

The p-compact radius has the following natural property.

Proposition 3.6. *Let E and F be Banach spaces, $1 \leq p < \infty$, and $f \in \mathcal{H}(E; F)$. Suppose that f is p-compact at x_0 with positive p-compact radius $r = r_p(f, x_0)$. Then f is p-compact for all $x \in B_r(x_0)$. Also, if f is p-compact at x_0 with infinite p-compact radius, then f is p-compact at x, for all $x \in E$.*

Proof. Without loss of generality, we can assume that $x_0 = 0$. For $r = r_p(f, 0)$, take $x \in E, \|x\| < r$. By [25, Proposition 1, p.26], there exists $\varepsilon > 0$ such that $f(y) = \sum_{m=1}^\infty P_m f(0)(y)$ converges uniformly for all $y \in B_\varepsilon(x)$. We also may assume that $\|x\| + \varepsilon < r$.

As in Proposition 3.4, we have that $f(B_\varepsilon(x)) \subset \{\sum_{m=1}^\infty x_m \colon x_m \in P_m f(0)(B_\varepsilon(x))\}$. Now, if we prove that $\sum_{m=1}^\infty \mathfrak{m}_p(P_m f(0)(B_\varepsilon(x))) < \infty$, the result will follow from Lemma 3.1. Indeed,

$$\begin{aligned}
\sum_{m=1}^\infty \mathfrak{m}_p(P_m f(0)(B_\varepsilon(x))) = & \sum_{m=1}^\infty (\|x\| + \varepsilon)^m \mathfrak{m}_p\Big(P_m f(0)(\tfrac{1}{\|x\|+\varepsilon} B_\varepsilon(x))\Big) \\
\leq & \sum_{m=1}^\infty (\|x\| + \varepsilon)^m \mathfrak{m}_p\big(P_m f(0)(B_E)\big) \\
= & \sum_{m=1}^\infty \Big((\|x\| + \varepsilon)\kappa_p(P_m f(0))^{1/m}\Big)^m.
\end{aligned}$$

Since $(\|x\| + \varepsilon)r^{-1} < 1$, the last series is convergent and the claim is proved. □

We recently learnt that R. Aron and P. Rueda defined, in the context of ideals of holomorphic functions [4], a radius of $\mathcal{I}$-boundedness which for p-compact holomorphic functions coincides with Definition 3.2. With the radius of $\mathcal{I}$-boundedness they obtained a partial version of Proposition 3.4.

Thanks to the Josefson-Nissenzweig theorem we have, for any Banach spaces E and F, a p-compact holomorphic mapping, $f \in \mathcal{H}_{K_p}(E;F)$, whose p-compact radius of convergence at the origin is finite. It is enough to consider a sequence $(x'_m)_m \subset E'$ with $\|x'_m\| = 1$ $\forall m \in \mathbb{N}$, and $(x'_m)_m$ point-wise convergent to 0. Then, $f(x) = \sum_{m=1}^{\infty} x'_m(x)^m$ belongs to $\mathcal{H}(E)$, is 1-compact (hence, p-compact for $p > 1$) and $r_p(f,0) = 1$ since $\kappa_p((x'_m)^m) = \|x'_m\| = 1$. The example can be modified to obtain a vector valued holomorphic function with similar properties.

There are two main questions related to p-compact holomorphic functions which where stated as Problem 5.1 and Problem 5.2 by Aron, Maestre and Rueda [2]. Both arise from properties that compact holomorphic functions satisfy. Recall that we may consider compact sets as ∞-compact sets and compact mappings as ∞-compact functions, where $\kappa_\infty(P) = \|P\|$, for any compact m-homogeneous polynomial P. Let us consider $f \in \mathcal{H}(E;F)$, by [5, Proposition 3.4] it is known that if f is compact at one point, say at the origin, then f is compact at x for all $x \in E$. Also, if $\sum_{m=0}^{\infty} P_m f(0)$ is the Taylor series expansion of f at 0, and for each m the homogeneous polynomial $P_m f(0)\colon E \to F$ is compact, then f is compact. With Example 3.7 we show that this later result is no longer true for $1 \leq p < \infty$. Note that $\limsup \|P_m\|^{1/m} < \infty$ is fulfilled by the Cauchy's inequalities whenever f is compact. Example 3.7 also shows that, in Proposition 3.4, the hypothesis $\limsup \kappa_p(P_m f(x_0))^{1/m} < \infty$ cannot be ruled out. For our purposes, we adapt [15, Example 10].

Example 3.7. *For every $1 \leq p < \infty$, there exists a holomorphic function $f \in \mathcal{H}(\ell_1;\ell_p)$ such that for all $m \in \mathbb{N}$, $P_m f(0)$ are p-compact, but f is not p-compact at* 0.

Furthermore, every polynomial $P_m f(y)$ in the Taylor series expansion of f at any $y \in \ell_1$ is 1*-compact, and therefore p-compact for all $1 \leq p < \infty$, but f is not p-compact at any y.*

Proof. Consider the partition of the natural numbers given by $\{\sigma_m\}_m$, where each σ_m is a finite set of $m!$ consecutive elements determined as follows:

$$\sigma_1 = \{1\}; \quad \sigma_2 = \{\underbrace{2,3}_{2!}\}; \quad \sigma_3 = \{\underbrace{4,5,6,7,8,9}_{3!}\}; \quad \sigma_4 = \{\underbrace{\ldots}_{4!}\}; \; \cdots$$

Let $(e_j)_j$ be the canonical basis of ℓ_p and denote by $(e'_j)_j$ the sequence of coordinate functionals on ℓ_1. Fixed $m \geq 1$, consider the polynomial $P_m \in \mathcal{P}(^m\ell_1;\ell_p)$, defined by

$$P_m(x) = (\frac{m^{m/2}}{m!})^{1/p} \sum_{j \in \sigma_m} e'_j(x)^m e_j.$$

Then

$$\|P_m\| = (\frac{m^{m/2}}{m!})^{1/p} \sup_{x\in B_{\ell_1}} \|\sum_{j\in\sigma_m} e_j'(x)^m e_j\|_{\ell_p} \leq (\frac{m^{m/2}}{m!})^{1/p} \sup_{x\in B_{\ell_1}} \|x\|_1^{1/p} = (\frac{m^{m/2}}{m!})^{1/p}.$$

First, note that P_m is p-compact since it is of finite rank. Now, as $\limsup \|P_m\|^{1/m} \leq \lim(\frac{m^{1/2}}{m!^{1/m}})^{1/p} = 0$, we may define f as the series $\sum_{m=1}^{\infty} P_m$, obtaining that $f \in \mathcal{H}(\ell_1; \ell_p)$.

In order to show that f fails to be p-compact at 0, by Proposition 3.4, it is enough to prove that $\limsup \kappa_p(P_m)^{1/m} = \infty$. Fix $m \in \mathbb{N}$ and take $(x_n)_n \in \ell_p(\ell_p)$, such that $P_m(B_{\ell_1}) \subset p\text{-co}\{x_n\}$. Each x_n may be written by $x_n = \sum_{k=1}^{\infty} x_k^n e_k$. For each $j \in \sigma_m$, there is a sequence $(\alpha_n^j)_n \in B_{\ell_q}$ such that

$$P_m(e_j) = (\frac{m^{m/2}}{m!})^{1/p} e_j = \sum_{n=1}^{\infty} \alpha_n^j x_n = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \alpha_n^j x_k^n e_k = \sum_{k=1}^{\infty}(\sum_{n=1}^{\infty} \alpha_n^j x_k^n) e_k.$$

Therefore, we have that $(\frac{m^{m/2}}{m!})^{1/p} = \sum_{n=1}^{\infty} \alpha_n^j x_j^n$, for each $j \in \sigma_m$. Then

$$\begin{aligned} m^{m/2} = \sum_{j\in\sigma_m} \big|(\frac{m^{m/2}}{m!})^{1/p}\big|^p = & \sum_{j\in\sigma_m} \big|\sum_{n=1}^{\infty} \alpha_n^j x_j^n\big|^p \\ \leq & \sum_{j\in\sigma_m} (\sum_{n=1}^{\infty} |\alpha_n^j x_j^n|)^p \\ \leq & \sum_{j\in\sigma_m} (\sum_{n=1}^{\infty} |\alpha_n^j|^q)^{p/q} \sum_{n=1}^{\infty} |x_j^n|^p \\ \leq & \sum_{j\in\sigma_m} \sum_{n=1}^{\infty} |x_j^n|^p \leq \|(x_n)_n\|_{\ell_p(\ell_p)}^p. \end{aligned}$$

We have shown that for any sequence $(x_n)_n \in \ell_p(\ell_p)$ such that $P_m(B_{\ell_1}) \subset p\text{-co}\{x_n\}$, the inequality $\|(x_n)_n\|_{\ell_p(\ell_p)} \geq m^{m/2p}$ holds. Then, $\kappa_p(P_m) \geq m^{m/2p}$ for all $m \in \mathbb{N}$. Hence, we conclude that $\limsup \kappa_p(P_m)^{1/m} = \infty$ and, by Proposition 3.4, f cannot be p-compact at 0, which proves the first statement of the example.

To show the second assertion, take any nonzero element $y \in \ell_1$ and fix $m_0 \in \mathbb{N}$. For all $x \in B_{\ell_1}$,

$$\begin{aligned} P_{m_0} f(y)(x) = & \sum_{m=m_0}^{\infty} \binom{m}{m_0} \overset{\vee}{P}_m(y^{m-m_0}, x^{m_0}) \\ = & \sum_{m=m_0}^{\infty} \binom{m}{m_0} \left(\frac{m^{m/2}}{m!}\right)^{1/p} \sum_{j\in\sigma_m} e_j'(y)^{m-m_0} e_j'(x)^{m_0} e_j. \end{aligned}$$

We claim that the sequence $\left(\binom{m}{m_0}\left(\frac{m^{m/2}}{m!}\right)^{1/p} e'_j(y)^{m-m_0}e_j\right)_{\substack{j\in\sigma_m\\ m>m_0}}$ belongs to $\ell_1(\ell_p)$. In fact,

$$\sum_{m>m_0}\binom{m}{m_0}\left(\frac{m^{m/2}}{m!}\right)^{1/p}\sum_{j\in\sigma_m}|e'_j(y)|^{m-m_0} \leq \sum_{m>m_0}\binom{m}{m_0}\left(\frac{m^{m/2}}{m!}\right)^{1/p}\|y\|_1^{m-m_0} < \infty.$$

Then, since $(e'_j(x)^m)_{\substack{j\in\sigma_m\\ m\geq m_0}}$ belongs to B_{c_0}, the set $P_{m_0}f(y)(B_{\ell_1})$ is included in the 1-convex hull of $\left\{\binom{m}{m_0}\left(\frac{m^{m/2}}{m!}\right)^{1/p} e'_j(y)^{m-m_0}e_j\colon m\geq m_0,\ j\in\sigma_m\right\}$, which proves that $P_{m_0}f(y)$ is 1-compact and, therefore, p-compact for every $1\leq p$, for any m_0.

To show that f is not p-compact at y, note that fixed m, it is enough to choose $j\in\sigma_m$, to obtain that $P_mf(y)(e_j)=\left(\frac{m^{m/2}}{m!}\right)^{1/p}e_j$. Now, we can proceed as in the first part of the example to show that $\limsup \kappa_p(P_mf(y))^{1/m}=\infty$. And, again by Proposition 3.4, we have that f cannot be p-compact at y. □

The following example gives a negative answer to [2, Problem 5.1]. We show an entire function which is p-compact at 0, but this property does not extend beyond the ball $B_{r_p(f,0)}(0)$. Example 3.8 proves, in addition, that Proposition 3.6 cannot be improved. We base our construction in [15, Example 11].

Example 3.8. *For every $1\leq p<\infty$, there exists a holomorphic function $f\in\mathcal{H}(\ell_1;\ell_p)$ such that f is p-compact at 0, with $\limsup\kappa_p(P_mf(0))^{1/m}=1$, but f is not p-compact at e_1.*

Proof. Consider $\{\sigma_m\}_m$, the partition of the natural numbers, as in Example 3.7. Let $(e_j)_j$ be the canonical basis of ℓ_p and denote $(e'_j)_j$ the sequence of coordinate functionals on ℓ_1.

Fixed $m\geq 2$, define $P_m\in\mathcal{P}(^m\ell_1;\ell_p)$, the m-homogeneous polynomial

$$P_m(x)=(\frac{1}{m!})^{1/p}e'_1(x)^{m-2}\sum_{j\in\sigma_m}e'_j(x)^2e_j.$$

Then

$$\begin{aligned}\|P_m\| = &\ (\frac{1}{m!})^{1/p}\sup_{x\in B_{\ell_1}}(\sum_{j\in\sigma_m}|e'_1(x)^{m-2}e'_j(x)^2|^p)^{1/p}\\ \leq &\ (\frac{1}{m!})^{1/p}\sup_{x\in B_{\ell_1}}(\sum_{j\in\sigma_m}|e'_j(x)|^{2p})^{1/p}\leq(\frac{1}{m!})^{1/p}.\end{aligned}$$

Since $\lim\|P_m\|^{1/m}\leq\lim(\frac{1}{m!})^{1/pm}=0$, we may define f as $f(x)=\sum_{m\geq 2}P_m(x)$, which belongs to $\mathcal{H}(\ell_1;\ell_p)$ and $\sum_{m\geq 2}P_m$ is its Taylor series expansion at 0.

Note that each P_m is p-compact, as it is of finite rank, for all $m\geq 2$. Moreover, when computing $\|P_m\|$, we showed that $\alpha(x)=\left(e'_1(x)^{m-2}e'_j(x)^2\right)_j\in B_{\ell_q}$ for all $x\in B_{\ell_1}$. Then $P_m(B_{\ell_1})\subset(\frac{1}{m!})^{1/p}p\text{-co}\{e_j\colon j\in\sigma_m\}$ and since $\|(e_j)_{j\in\sigma_m}\|_{\ell_p(\ell_p)}=(\sum_{j\in\sigma_m}1)^{1/p}=(m!)^{1/p}$, we have that $\kappa_p(P_m) \leq (\frac{1}{m!})^{1/p}(m!)^{1/p} = 1$. Then, $\limsup \kappa_p(P_m)^{1/m} \leq 1$ and, by Proposition 3.4, we have that f is p-compact at 0.

To show that $r_p(f, 0) = 1$, fix $m \geq 2$ and $\varepsilon > 0$. Take $x_j \in B_{\ell_1}$ such that $e_1'(x_j) = 1 - \varepsilon$, $e_j'(x_j) = \varepsilon$ and $e_k'(x_j) = 0$ for $j \in \sigma_m$ and $k \neq j$.

Now, fix any sequence $(y_n)_n \in \ell_p(\ell_p)$ such that $P_m(B_{\ell_1}) \subset p\text{-co}\{y_n\}$ and write $y_n = \sum_{k=1}^{\infty} y_k^n e_k$.

Then, for each $j \in \sigma_m$ there exists $(\alpha_n^j)_n \in B_{\ell_q}$ so that

$$P_m(x_j) = (\frac{1}{m!})^{1/p}(1-\varepsilon)^{m-2}\varepsilon^2 e_j = \sum_{n=1}^{\infty} \alpha_n^j y_n.$$

Thus, for each $j \in \sigma_m$, the equality $(\frac{1}{m!})^{1/p}(1-\varepsilon)^{m-2}\varepsilon^2 = \sum_{n=1}^{\infty} \alpha_n^j y_j^n$ holds.

In consequence

$$\begin{aligned}
((1-\varepsilon)^{m-2}\varepsilon^2)^p &= \sum_{j\in\sigma_m} \frac{1}{m!}((1-\varepsilon)^{m-2}\varepsilon^2)^p \\
&= \sum_{j\in\sigma_m} |(\frac{1}{m!})^{1/p}(1-\varepsilon)^{m-2}\varepsilon^2|^p \\
&= \sum_{j\in\sigma_m} |\sum_{n=1}^{\infty} \alpha_n^j y_j^n|^p \\
&\leq \sum_{j\in\sigma_m} (\sum_{n=1}^{\infty} |\alpha_n^j y_j^n|)^p \\
&\leq \sum_{j\in\sigma_m} \sum_{n=1}^{\infty} |y_j^n|^p \leq \|(y_n)_n\|_{\ell_p(\ell_p)}^p.
\end{aligned}$$

Finally, we get that $\kappa_p(P_m) \geq (1-\varepsilon)^{m-2}\varepsilon^2$ which implies that $\limsup \kappa_p(P_m)^{1/m} \geq 1 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain that $r_p(f, 0) = 1$.

Now, we want to prove that f is not p-compact at e_1. By Proposition 3.4, it is enough to show, for instance, that the 2-homogeneous polynomial $P_2 f(e_1)\colon \ell_1 \to \ell_p$ is not p-compact. We have

$$P_2 f(e_1)(x) = \sum_{m=2}^{\infty} \binom{m}{2} \overset{\vee}{P}_m(e_1^{m-2}, x^2) \tag{1}$$

where $\overset{\vee}{P}_m$ is the symmetric m-linear mapping associated to P_m.

By the definition of P_m we easily obtain a multilinear mapping $A_m \in \mathcal{L}(^m\ell_1; \ell_p)$ satisfying $P_m(x) = A_m(x, \dots, x)$, defined by

$$A_m(x_1, \dots, x_m) = (\frac{1}{m!})^{1/p} e_1'(x_1) \cdots e_1'(x_{m-2}) \sum_{j\in\sigma_m} e_j'(x_{m-1}) e_j'(x_m) e_j.$$

Let $\mathcal{S}_m$ be the symmetric group on $\{1,\dots,m\}$ and denote, for each $\xi\in\mathcal{S}_m$, the multilinear mapping A_m^ξ given by $A_m^\xi(x_1,\dots,x_m)=A_m(x_{\xi(1)},\dots,x_{\xi(m)})$. Then we have

$$\overset{\vee}{P}_m(e_1^{m-2},x^2)=\frac{1}{m!}\sum_{\xi\in S_m}A_m^\xi(e_1^{m-2},x^2).$$

Since $A_m(x_1,\dots,x_{m-2},e_1,x_{m-1})=A_m(x_1,\dots,x_{m-1},e_1)=0$, for all $x_1,\dots,x_{m-1}\in\ell_1$, and $A_m(e_1^{m-2},x^2)=\left(\frac{1}{m!}\right)^{1/p}\sum_{j\in\sigma_m}e_j'(x)^2e_j$, we obtain

$$\overset{\vee}{P}_m(e_1^{m-2},x^2)=\frac{1}{m!}2(m-2)!(\frac{1}{m!})^{1/p}\sum_{j\in\sigma_m}e_j'(x)^2e_j. \tag{2}$$

Combining (1) and (2) we get that

$$P_2f(e_1)(x)=\sum_{m\geq 2}(\frac{1}{m!})^{1/p}\sum_{j\in\sigma_m}e_j'(x)^2e_j.$$

Suppose that $P_2f(e_1)$ is p-compact. Hence, there exists a sequence $(y_n)_n\in\ell_p(\ell_p)$, $y_n=\sum_{k=1}^\infty y_k^ne_k$ such that $P_2f(e_1)(B_{\ell_1})\subset p\text{-co}\{y_n\}$. For each $j\in\sigma_m$, there exists $(\alpha_n^j)_n\in B_{\ell_q}$ such that $P_2f(e_1)(e_j)=(\frac{1}{m!})^{1/p}e_j=\sum_{n=1}^\infty\alpha_n^jy_n$. As in the Example 3.7, we conclude that $(\frac{1}{m!})^{1/p}=\sum_{n=1}^\infty\alpha_n^jy_j^n$, if $j\in\sigma_m$.

Hence

$$\begin{aligned}\sum_{m\geq 2}\sum_{j\in\sigma_m}((\frac{1}{m!})^{1/p})^p= &\sum_{m\geq 2}\sum_{j\in\sigma_m}|\sum_{n=1}^\infty\alpha_n^jy_j^n|^p\\ \leq &\sum_{m\geq 2}\sum_{j\in\sigma_m}(\sum_{n=1}^\infty|\alpha_n^j|^q)^{p/q}\sum_{n=1}^\infty|y_n^j|^p\\ \leq &\sum_{m\geq 2}\sum_{j\in\sigma_m}\sum_{n=1}^\infty|y_n^j|^p\\ \leq &\|(y_n)_n\|_{\ell_p(\ell_p)}^p<\infty,\end{aligned}$$

which is a contradiction since $\sum_{m\geq 2}\sum_{j\in\sigma_m}((\frac{1}{m!})^{1/p})^p$ is not convergent. Therefore, f cannot be p-compact at e_1, and the result is proved. $\square$

4. The p-approximation property and p-compact mappings

The concept of p-compact sets leads naturally to that of p-approximation property. A Banach space E has the p-approximation property if the identity can be uniformly approximated by finite rank operators on p-compact sets. Since p-compact sets are compact, every space with the approximation property has the p-approximation property. Then, this property can be seen as a way to weaken the classical approximation property.

The p-approximation property has been studied in [10, 12] related with p-compact linear operators and in [2] related with non linear mappings. The relation between the approximation property and compact holomorphic mappings was first addressed in [5]. Here, we are concern with the study of the p-approximation property and its relation with p-compact polynomials and holomorphic functions in the spirit of [2] and [5].

We start characterizing the notion of a homogeneous polynomial P being p-compact in terms of different conditions of continuity satisfied by P' the transpose of P. The first proposition gives an answer to [2, Problem 5.8] and should be compared with [5, Proposition 3.2].

Before going on, some words are needed on the topologies which we will use. We denote by $\mathcal{P}_c(^mE)$ the space $\mathcal{P}(^mE)$ considered with the uniform convergence on compact sets of E, if $m = 1$ we simply write E'_c. When compact sets are replaced by p-compact sets we use the notation $\mathcal{P}_{cp}(^mE)$ and E'_{cp}. By the Ascoli theorem, any set $L \subset \mathcal{P}_c(^mE)$ is relatively compact if and only if $\sup_{P \in L} \|P\|$ is finite. Also, if $L \subset \mathcal{P}_{cp}(^mE)$ is relatively compact we have that L is point-wise bounded and then, by the Principle of uniform boundedness, L is relatively compact in $\mathcal{P}_c(^mE)$. Now we have:

Proposition 4.1. *Let E and F be Banach spaces, $1 \leq p < \infty$, and $P \in \mathcal{P}(^mE; F)$. The following statements are equivalent.*

(i) $P \in \mathcal{P}_{\mathcal{K}_p}(E; F)$.
(ii) $P' \colon F'_{cp} \to \mathcal{P}(^mE)$ *is continuous.*
(iii) $P' \colon F'_{cp} \to \mathcal{P}_c(^mE)$ *is compact.*
(iv) $P' \colon F'_{cp} \to \mathcal{P}_{cq}(^mE)$ *is compact for any q, $1 \leq q < \infty$.*
(v) $P' \colon F'_{cp} \to \mathcal{P}_{cq}(^mE)$ *is compact for some q, $1 \leq q < \infty$.*

Proof. Suppose (i) holds, then $\overline{P(B_E)} = K$ is p-compact and its polar set K° is a neighborhood in F'_{cp}. For $y' \in K^\circ$ we have that $\|P'(y')\| = \sup_{x \in B_E} |y'(Px)| \leq 1$, and $P' \colon F'_{cp} \to \mathcal{P}(^mE)$ is continuous.

Now suppose (ii) holds, then there exists a p-compact set $K \subset F$ such that $P'(K^\circ)$ is equicontinuous in $\mathcal{P}(^mE)$. By the Ascoli theorem, $P'(K^\circ)$ is relatively compact in $\mathcal{P}_c(^mE)$ and $P' \colon F'_{cp} \to \mathcal{P}_c(^mE)$ is compact.

The continuity of the identity map $\mathcal{P}_c(^mE) \hookrightarrow \mathcal{P}_{cq}(^mE)$ gives that (iii) implies (iv), for all $1 \leq q < \infty$. Obviously, (iv) implies (v). To complete the proof, suppose (v) holds. Then, there exist an absolutely convex p-compact set $K \subset F$ and a compact set $L \subset \mathcal{P}_{cq}(^mE)$ such that $P'(K^\circ) \subset L$ and therefore, there exists $c > 0$ such that $\sup_{y' \in K^\circ} \|P'(y')\| \leq c$. Note that for any $x \in c^{-\frac{1}{n}} B_E$ and $y' \in K^\circ$ we have that $|P'(y')(x)| = |y'(Px)| \leq 1$. Then $P(x) \in K$, for all $x \in c^{-\frac{1}{n}} B_E$ and P is p-compact. □

Now, we characterize the p-approximation property on a Banach space in terms of the p-compact homogeneous polynomials with values on it. In order to do so we appeal to the notion of the ϵ-product introduced by Schwartz [28]. Recall that for E and F two locally convex spaces, $F \epsilon E$ is defined as the space of all linear continuous operators from E'_c to F, endowed with the topology of uniform convergence on all equicontinuous sets of E'. The space $F\epsilon E$ is also denoted by $\mathcal{L}_\epsilon(E'_c; F)$. In [5, Proposition 3.3] its shown, for all Banach spaces E and F, that $(\mathcal{P}(^mF), \|.\|)\epsilon E = \mathcal{L}_\epsilon(E'_c; (\mathcal{P}(^mF), \|.\|)) = (\mathcal{P}_K(^mF; E), \|.\|)$, where the isomorphism is given by the transposition $P \leftrightarrow P'$. As a consequence, it is proved that $\mathcal{P}(^mF)$ has the approximation property if and only if $\mathcal{P}(^mF) \otimes E$ is $\|.\|$-dense in $\mathcal{P}_K(^mF; E)$ for all Banach spaces E and all $m \in \mathbb{N}$. We have the following result.

Proposition 4.2. *Let E and F be Banach spaces. Then $(\mathcal{P}_{K_p}(^mF; E), \|.\|)$ is isometrically isomorphic to $\mathcal{L}_\epsilon(E'_{c_p}; (\mathcal{P}(^mF), \|.\|))$.*

As a consequence, E has the p-approximation property if and only if $\mathcal{P}(^mF)\otimes E$ is $\|.\|$-dense in $\mathcal{P}_{K_p}(^mF; E)$ for all Banach spaces F and all $m \in \mathbb{N}$.

Proof. Note that [(i) implies (ii)] of Proposition 4.1, says that the transposition operator maps a p-compact polynomial into a linear map in $\mathcal{L}_\epsilon(E'_{c_p}; \mathcal{P}(^mF), \|.\|))$. Now, take T an operator in $\mathcal{L}_\epsilon(E'_{c_p}; \mathcal{P}(^mF), \|.\|))$. Since the identity map $\iota \colon E'_c \to E'_{cp}$ is continuous, T belongs to $\mathcal{L}_\epsilon(E'_c; \mathcal{P}(^mF), \|.\|))$. By [5, Proposition 3.3], we have that $T = P'$ for some $P \in \mathcal{P}_K(^mF; E)$. In particular, $P' \colon E'_{cp} \to \mathcal{P}(^mF)$ is continuous and by [(ii) implies (i)] of Proposition 4.1, P is p-compact.

For the second statement, if E has the p-approximation property, $G \otimes E$ is dense in $\mathcal{L}_\epsilon(E'_{c_p}; G)$, for every locally convex space G, [20]. In particular we may consider $G = (\mathcal{P}(^mF), \|.\|)$. Conversely, with $m = 1$ we have that $F' \otimes E$ is $\|.\|$-dense in $\mathcal{K}_p(F; E)$ for every Banach space F. Now, an application of [12, Theorem 2.1] completes the proof. □

At the light of [5, Proposition 3.3], we expected to obtain a result of the type $\mathcal{P}(^mE)$ *has the p-approximation property if and only if $\mathcal{P}(^mE) \otimes F$ is $\|.\|$-dense in $\mathcal{P}_{K_p}(^mE; F)$ for all Banach spaces F and all $m \in \mathbb{N}$.* Unfortunately, our characterization is not as direct as we wanted and requires the following notion.

Definition 4.3. *Let E be a Banach space, $\mathcal{A}$ an operator ideal and α a norm on $\mathcal{A}$. We say that E has the $(\mathcal{A}, \alpha)$-approximation property if $F' \otimes E$ is α-dense in $\mathcal{A}(F, E)$, for all Banach spaces F.*

The relation between an ideal $\mathcal{A}$ with the ideal of those operators whose transpose belongs to $\mathcal{A}$ leads us to work with the ideal of quasi p-nuclear operators $\mathcal{QN}_p$.

Proposition 4.4. *Let E be a Banach space and fix $m \in \mathbb{N}$. Then,*

(a) *$\mathcal{P}(^mE)\otimes F$ is $\|.\|$-dense in $\mathcal{P}_{K_p}(^mE; F)$, for all Banach spaces F if and only if $\mathcal{P}(^mE)$ has the $(\mathcal{QN}_p, \|.\|)$-approximation property.*
(b) *$\mathcal{P}(^mE)$ has the p-approximation property if and only if $\mathcal{P}(^mE) \otimes F$ is $\|.\|$-dense in $\{P \in \mathcal{P}(E; F) \colon L_p \in \mathcal{QN}_p(\otimes^m_{\pi_s} E; F)\}$, for all Banach spaces F.*

Proof. The space $\mathcal{P}({}^mE)$, or equivalently $(\otimes^m_{\pi_s}E)'$, has the $(\mathcal{QN}_p, \|.\|)$-approximation property if and only if $(\otimes^m_{\pi_s}E)' \otimes F$ is $\|.\|$-dense in $\mathcal{K}_p(\otimes^m_{\pi_s}E; F)$ for all Banach spaces F, see [20]. In virtue of Lemma 2.1, it is equivalent to have that $\mathcal{P}({}^mE) \otimes F$ is $\|.\|$-dense in $\mathcal{P}_{\mathcal{K}_p}({}^mE; F)$. Then, statement (a) is proved. Note that (a) can be reformulated saying that $\mathcal{P}({}^mE)$ has the $(\mathcal{QN}_p, \|.\|)$-approximation property if and only if $\mathcal{P}({}^mE) \otimes F$ is $\|.\|$-dense in $\{P \in \mathcal{P}(E; F) \colon L_p \in \mathcal{K}_p(\otimes^m_{\pi_s}E; F)\}$, for all Banach spaces F.

For the proof of (b), we use that the p-approximation property corresponds to the $(\mathcal{A}, \|.\|)$-approximation property for the ideal $\mathcal{A} = \mathcal{K}_p$, of p-compact operators. The result follows proceeding as before if the ideal $\mathcal{K}_p$ and its dual ideal $\mathcal{QN}_p$ are interchanged. □

Now, we change our study to that of p-compact holomorphic mappings. Aron and Schottenloher described the space of compact holomorphic functions considered with τ_w, the Nachbin topology [25], via the ϵ-product. Namely, they show that $(\mathcal{H}_K(E; F), \tau_\omega) = \mathcal{L}_\varepsilon(F'_c; \mathcal{H}(E), \tau_\omega)$, where the isomorphism is given by the transposition map $f \mapsto f'$ [5, Theorem 4.1]. The authors use this equivalence to obtain, in presence of the approximation property, results on density similar to that of Proposition 4.2. Recall that $f' \colon F' \to \mathcal{H}(E)$ denotes the linear operator given by $f'(y') = y' \circ f$. With the next proposition we try to clarify the relationship between p-compact holomorphic mappings and the ϵ-product. The result obtained gives, somehow, a partial answer to [2, Problem 5.6].

Proposition 4.5. *Let E and F be Banach spaces. Then,*

(a) $(\mathcal{H}_{K_p}(E; F), \tau_\omega)$ *is topologically isomorphic to a subspace of* $\mathcal{L}_\epsilon(F'_{cp}; (\mathcal{H}(E), \tau_\omega))$.
(b) $\mathcal{L}_\epsilon(F'_{cp}; (\mathcal{H}(E), \tau_\omega))$ *is topologically isomorphic to a subspace of* $\{f \in \mathcal{H}(E; F) \colon P_m f(x) \in \mathcal{P}_{K_p}({}^mE; F),\ \forall x \in E,\ \forall m \in \mathbb{N}\}$, *considered with the Nachbin topology,* τ_ω.

Proof. To prove (a), fix f in $\mathcal{H}_{K_p}(E; F)$ and consider q any τ_ω-continuous seminorm on $\mathcal{H}(E)$. By [16, Proposition 3.47], we may consider only the seminorms such that, for $g \in \mathcal{H}(E)$,

$$q(g) = \sum_{m=0}^{\infty} \|P_m g(0)\|_{K + a_m B_E},$$

with $K \subset E$ an absolutely convex compact set and $(a_m)_m$ a sequence in c_0^+. There exists $V \subset E$, an open set such that $2K \subset V$ and $f(V) \subset F$ is p-compact. Fix $m_0 \in \mathbb{N}$ such that $2K + 2a_m B_E \subset V$, for all $m \geq m_0$. Now, choose $c > 0$ such that $c(2K + 2a_m B_E) \subset 2K + 2a_{m_0} B_E \subset V$, for all $m < m_0$. The polar set of $f(V)$, $f(V)^\circ$, is a neighborhood in F'_{cp}. By the Cauchy inequalities for entire functions, we have for all $y' \in f(V)^\circ$,

$$\begin{aligned}
q(f'(y')) = & \sum_{m=0}^{\infty} \|P_m\,(y'\circ f)\,(0)\|_{K+a_mB_E} \\
= & \sum_{m=0}^{\infty} \frac{1}{2^m}\|P_m\,(y'\circ f)\,(0)\|_{2K+2a_mB_E} \\
= & \sum_{m<m_0} \frac{1}{2^m}\|P_m\,(y'\circ f)\,(0)\|_{2K+2a_mB_E} + \sum_{m\geq m_0} \frac{1}{2^m}\|P_m\,(y'\circ f)\,(0)\|_{2K+2a_mB_E} \\
\leq & \sum_{m<m_0} \frac{1}{(2c)^m}\|P_m\,(y'\circ f)\,(0)\|_{c(2K+2a_mB_E)} + \sum_{m\geq m_0} \frac{1}{2^m}\|P_m\,(y'\circ f)\,(0)\|_{2K+2a_mB_E} \\
\leq & \sum_{m<m_0} \frac{1}{(2c)^m}\|y'\circ f\|_{c(2K+2a_mB_E)} + \sum_{m\geq m_0} \frac{1}{2^m}\|y'\circ f\|_{2K+2a_mB_E} \\
\leq & \sum_{m<m_0} \frac{1}{(2c)^m}\|y'\circ f\|_{V} + \sum_{m\geq m_0} \frac{1}{2^m}\|y'\circ f\|_{V} \\
\leq & \sum_{m<m_0} \frac{1}{(2c)^m} + \sum_{m\geq m_0} \frac{1}{2^m} < \infty.
\end{aligned}$$

Then $f' \in \mathcal{L}(F'_{cp}; (\mathcal{H}(E), \tau_\omega))$. Again, we use the continuity of the identity map $\iota\colon F'_c \to F'_{cp}$ now, [5, Theorem 4.1] implies the result.

To prove that (b) holds, take $T \in \mathcal{L}(F'_{cp}; (\mathcal{H}(E), \tau_\omega))$ which, in particular, is an operator in $\mathcal{L}(F'_c; (\mathcal{H}(E), \tau_\omega))$. By [5, Theorem 4.1], $T = f'$ for some $f \in \mathcal{H}_K(E; F)$. By virtue of Proposition 4.1, it is enough to show that $(P_m f(x))' : F'_{cp} \to (\mathcal{P}(^mE), \|.\|)$ is continuous, for each $m \in \mathbb{N}$. Consider $D^m_x \colon (\mathcal{H}(E), \tau_\omega) \to (\mathcal{P}(^mE), \|.\|)$ the continuous projection given by $D^m_x(g) = P_m g(x)$, for all $g \in \mathcal{H}(E)$. Note that $(P_m f(x))'$ and $D^m_x \circ f'$ coincide as linear operators. Hence, we obtain the result. □

Example 3.7 shows that there exists an entire function $f\colon \ell_1 \to \ell_p$, so that every homogeneous polynomial in its Taylor series expansion at y is q-compact for any $y \in \ell_1$, for all $1 \leq q < \infty$, but f fails to be q-compact at y, for every y and every $q \leq p$. However, we have the following result.

Lemma 4.6. *Let E and F be Banach spaces. Then,*

$$\mathcal{H}_{K_p}(E; F) \text{ is } \tau_\omega\text{-dense in } \{f \in \mathcal{H}(E; F)\colon P_m f(x) \in \mathcal{P}_{K_p}(^mE; F),\ \forall x \in E,\ \forall m \in \mathbb{N}\}.$$

Proof. Fix $f \in \mathcal{H}(E; F)$ so that $P_m f(x) \in \mathcal{P}_{K_p}(^mE; F)$ for all $x \in E$ and for all m. Let $\varepsilon > 0$ and let q be any τ_ω-continuous seminorm on $\mathcal{H}(E; F)$ of the form

$$q(g) = \sum_{m=0}^{\infty} \|P_m g(0)\|_{K+a_mB_E},$$

with $K \subset E$ absolutely convex and compact and $(a_m)_m \in c_0^+$. Consider $m_0 \in \mathbb{N}$ such that $\sum_{m\geq m_0} \|P_m f(0)\|_{K+a_mB_E} < \varepsilon$. Now, let $f_0 = \sum_{m<m_0} P_m f(0)$, which is p-compact. Note that $q(f - f_0) \leq \varepsilon$ and the lemma follows. □

Proposition 4.7. *Let E be a Banach space. Then, the following statements are equivalent.*

(i) *E has the p-approximation property.*
(ii) *$\mathcal{H}(F) \otimes E$ is τ_ω-dense in $\mathcal{H}_{K_p}(F;E)$ for all Banach spaces F.*

Proof. If E has the p-approximation property, $E \otimes G$ is dense in $\mathcal{L}_\epsilon(E'_{c_p}; G)$ for all locally convex space G [20], in particular if we consider $G = (\mathcal{H}(F), \tau_\omega)$. Applying Proposition 4.5 (a), we have the first assertion.

For the converse, put $\mathcal{H}_0 = \{f \in \mathcal{H}(F;E) \colon P_m f(x) \in \mathcal{P}_{K_p}(^m F; E),\ \forall x \in E,\ \forall m \in \mathbb{N}\}$. By Lemma 4.6, $\mathcal{H}(F) \otimes E$ is τ_ω-dense in $\mathcal{H}_0$. Now, take $T \in \mathcal{K}_p(F;E)$ and $\varepsilon > 0$. Since $T \in \mathcal{H}_0$ and $q(f) = \|P_1 f(0)\|$ is a τ_ω-continuous seminorm, there exists $g \in \mathcal{H}(F) \otimes E$ such that $q(T-g) \le \varepsilon$. But $q(T-g) = \|T - P_1 g(0)\|$ and since $P_1 g(0) \in F' \otimes E$, we have shown that $F' \otimes E$ is $\|.\|$-dense in $\mathcal{K}_p(F;E)$. By [12, Theorem 2.1], E has the p-approximation property. $\square$

5. Holomorphy types and topologies

In this section we show that p-compact holomorphic functions fit into the framework of holomorphy types. Our notation and terminology follow that given in [15]. Since, $\mathcal{P}_{K_p}(^m E; F)$ is a subspace of $\mathcal{P}(^m E; F)$ and $\mathcal{P}_{K_p}(^0 E; F) = F$, the first two conditions in the definition of a holomorphy type are fulfilled. Therefore, we only need to corroborate that the sequence $(\mathcal{P}_{K_p}(^m E; F), \kappa_p)_m$ satisfies the third condition. Indeed, this last condition will be also fulfilled if we show

$$\kappa_p(P_l(P)(a)) \le (2e)^m \kappa_p(P) \|a\|^{m-l}, \tag{3}$$

for every $P \in \mathcal{P}_{K_p}(^m E; F)$, for all $l = 1, \ldots, m$ and for all m, where $P_l(P)(a)$ denotes the l-component in the expansion of P at a.

A function $f \in \mathcal{H}(E;F)$ is said to be of holomorphic type κ_p, at a, if there exist $c_1, c_2 > 0$ such that each component of its Taylor series expansion, at a, is a p-compact polynomial satisfying that $\kappa_p(P_m f(a)) \le c_1 c_2^m$.

To give a simple proof of the fact that $(\mathcal{P}_{K_p}(^m E; F), \kappa_p)_m$ satisfy the inequalities given in (3) we use the following notation. Let $P \in \mathcal{P}(^m E; F)$ and fix $a \in E$, we denote by P_{a^l} the $(m-l)$-homogeneous polynomial defined as

$$P_{a^l}(x) := \overset{\vee}{P}(a^l, x^{m-l}),$$

for all $x \in E$ and $l < m$. Note that, for any $j < l < m$, we have that $P_{a^l} = (P_{a^{l-j}})_{a^j}$ and that $P_l(P)(a) = \binom{m}{m-l} P_{a^{m-l}}$. We appeal to the description of P_a given in [8, Corollary 1.8, b)]:

$$P_a(x) = \overset{\vee}{P}(a, x^{m-1}) = \frac{1}{m^2} \frac{1}{(m-1)^{m-1}} \sum_{j=1}^{m-1} P((m-1) r^j x + a). \tag{4}$$

where $r \in \mathbb{C}$ is such that $r^m = 1$ and $r^j \neq 1$ for $j < m$.

Theorem 5.1. *For any Banach spaces E and F, the sequence $(\mathcal{P}_{K_p}({}^mE;F), \kappa_p)_m$ is a holomorphy type from E to F.*

Proof. If $P \in \mathcal{P}_{K_p}({}^mE;F)$ by [2, Proposition 3.5] or Proposition 3.4 we have that $P_j(P)(a) \in \mathcal{P}_{K_p}({}^jE;F)$ for all $a \in E$, for all $j \leq m$. To prove the holomorphy type structure, we will show that $\kappa_p(P_j(P)(a)) \leq 2^m e^m \|a\|^{m-j}\kappa_p(P)$, for all $j \leq m$.

Fix $a \in E$. If we show that $\kappa_p(P_a) \leq e\|a\|\kappa_p(P)$ then the proof is complete using a generalized inductive reasoning. Indeed, suppose that for any p-compact homogeneous polynomial Q, of degree less than m, the inequality $\kappa_p(Q_a) \leq e\|a\|\kappa_p(Q)$ holds. Then, since $P_{a^l} = (P_{a^{l-1}})_a$ and $P_j(P)(a) = \binom{m}{m-j} P_{a^{m-j}}$, we obtain

$$\begin{aligned}
\kappa_p(P_j(P)(a)) &= \tbinom{m}{m-j}\kappa_p(P_{a^{m-j}}) = \tbinom{m}{m-j}\kappa_p((P_{a^{m-j-1}})_a) \\
&\leq \tbinom{m}{m-j} e\|a\|\kappa_p((P_{a^{m-j-1}})) \\
&\leq \tbinom{m}{m-j} e^{m-j}\|a\|^{m-j}\kappa_p(P) \\
&\leq 2^m e^m \|a\|^{m-j}\kappa_p(P).
\end{aligned}$$

Now, take $P \in \mathcal{P}_{K_p}({}^mE;F)$. Then

$$\kappa_p(P_a) = m_p(\overset{\vee}{P}(a, B_E^{m-1})) = \|a\| m_p(\overset{\vee}{P}(\tfrac{a}{\|a\|}, B_E^{m-1})). \tag{5}$$

Using (4) and Lemma 3.1 we have

$$\|a\| m_p(\overset{\vee}{P}(\frac{a}{\|a\|}, B_E^{m-1})) \leq \|a\| \tfrac{1}{m^2}\tfrac{1}{(m-1)^{m-1}} \sum_{j=1}^{m-1} m_p(P((m-1)r^j B_E + \tfrac{a}{\|a\|})). \tag{6}$$

Since $\sup\{\|x\| \colon\ x \in (m-1)r^j B_E + \frac{a}{\|a\|}\} = m$,

$$\begin{aligned}
\|a\| m_p(\overset{\vee}{P}(\tfrac{a}{\|a\|}, B_E^{m-1})) &\leq \tfrac{\|a\|}{m^2(m-1)^{m-1}} \sum_{j=1}^{m-1} m_p(P((m-1)r^j B_E + \tfrac{a}{\|a\|})) \\
&= \tfrac{\|a\|}{m^2(m-1)^{m-1}} \sum_{j=1}^{m-1} m^m m_p\big(P\big(\tfrac{1}{m}((m-1)r^j B_E + \tfrac{a}{\|a\|})\big)\big) \\
&\leq \|a\|(\tfrac{m}{m-1})^{m-1}\kappa_p(P) \leq e\|a\|\kappa_p(P).
\end{aligned} \tag{7}$$

Combining (5), (6) and (7) we get that $\kappa_p(P_a) \leq e\|a\|\kappa_p(P)$, as we wanted to show. ☐

As a consequence we have the following result.

Corollary 5.2. *Let f be a function in $\mathcal{H}(E;F)$, then $f \in \mathcal{H}_{K_p}(E;F)$ if and only if f is of κ_p-holomorphy type.*

Proof. It follows from Theorem 5.1 and [2, Proposition 3.5] or Proposition 3.4. ☐

Remark 5.3. Theorem 5.1 can be improved. Indeed, the same proof of Theorem 5.1 shows that the sequence $(\mathcal{P}_{K_p}(^m E;F))_m$ is a coherent sequence associated to the operator ideal $\mathcal{K}_p(E;F)$ (see [8] for definitions).

Since $\mathcal{H}_{K_p}(E,F)$ is a holomorphy type, following [26] we have a natural topology defined on $\mathcal{H}_{K_p}(E,F)$ denoted by τ_{ω,m_p}. This topology may be generated by different families of continuous seminorms. The original set of seminorms used to define τ_{ω,m_p} corresponds to the family of seminorms given below in Theorem 5.5, item (c). Our aim is to to characterize the κ_p-approximation property of a Banach space E in an analogous way to [5, Theorem 4.1]. In order to do so, we will give different descriptions of τ_{ω,m_p}. First, we need the following result.

Proposition 5.4. *Let E and F be Banach spaces. Then, $f \in \mathcal{H}_{K_p}(E;F)$ if and only if, for all m, $P_m f(0) \in \mathcal{P}_{K_p}(^m E;F)$ and for any absolutely convex compact set K, there exists $\varepsilon > 0$ such that $\sum_{n=0}^{\infty} m_p(P_m f(0)(K+\varepsilon B_E)) < \infty$.*

Proof. Take $f \in \mathcal{H}_{K_p}(E;F)$ and K an absolutely convex compact set. Then, $2K$ is also absolutely convex and compact. For each $x \in 2K$, there exist $\varepsilon_x > 0$ such that $f(x+\varepsilon_x B_E)$ is p-compact. Now, we choose $x_1,\dots,x_n \in 2K$ such that $K \subset \bigcup_{j=1}^n (x_j + \varepsilon_{x_j} B_E)$ and with $V = \bigcup_{j=1}^n (x_j + \varepsilon_{x_j} B_E)$ we have that $f(V)$ is p-compact. Let $d = \text{dist}(2K, \mathcal{C}V) > 0$, where $\mathcal{C}V$ denotes the complement of V. Let us consider $W = 2K + dB_E$, then W is an absolutely convex open set and $2K \subset W \subset V$. Then, applying Proposition 3.3 we have

$$\sum_{n=0}^{\infty} m_p(P_m f(0)(K + d/2B_E) = \sum_{n=0}^{\infty} (1/2)^m m_p(P_m f(0)(W)) \leq 2m_p(f(W)) < \infty,$$

which proves the first claim.

Conversely, let $f \in \mathcal{H}(E;F)$ satisfy the conditions in the proposition. We have to show that f is p-compact at x for any fixed $x \in E$. Consider the absolutely convex compact set K, given by $K = \{\lambda x \colon |\lambda| \leq 1\}$. Then, there exists $\varepsilon_1 > 0$ such that $\sum_{n=0}^{\infty} m_p(P_m f(0)(K + \varepsilon_1 B_E)) < \infty$. Since f is an entire function, by [25, Proposition 1, p.26], there exists $\varepsilon_2 > 0$ such that $f(y) = \sum_{m=1}^{\infty} P_m f(0)(y)$ uniformly for $y \in B_{\varepsilon_2}(x)$. Let $\varepsilon = \min\{\varepsilon_1;\varepsilon_2\}$, then $f(B_\varepsilon(x)) \subset \{\sum_{m=0}^{\infty} x_m \colon x_m \in P_m f(0)(B_\varepsilon(x))\}$.

Also

$$\sum_{m=0}^{\infty} m_p(P_m f(0)(B_\varepsilon(x)) \leq \sum_{m=0}^{\infty} m_p(P_m f(0)(K+\varepsilon_1 B_E)) < \infty.$$

Now, applying Lemma 3.1 we obtain that f is p-compact at x, and the proof is complete. □

The next characterization of the topology τ_{ω,m_p} associated to the holomorphy type $\mathcal{H}_{K_p}(E;F)$ follows that of [15] and [25].

Theorem 5.5. *Let E and F be Banach spaces and consider the space $\mathcal{H}_{K_p}(E;F)$. Any of the following families of seminorms generate the topology τ_{ω,m_p}.*

(a) *The seminorms p satisfying that there exists a compact set K such that for every open set $V \supset K$ there exists $C_V > 0$ so that*

$$p(f) \leq C_V \mathfrak{m}_p(f(V)) \quad \forall f \in \mathcal{H}_{K_p}(E;F).$$

In this case, we say that p is $\mathfrak{m}_p$-ported by compact sets.

(b) *The seminorms p satisfying that there exists an absolutely convex compact set K such that for every absolutely convex open set $V \supset K$ there exists $C_V > 0$ so that*

$$p(f) \leq C_V \mathfrak{m}_p(f(V)) \quad \forall f \in \mathcal{H}_{K_p}(E;F).$$

In this case, we say that p is AC-$\mathfrak{m}_p$-ported by absolutely convex compact sets.

(c) *The seminorms p satisfying that there exists an absolutely convex compact set K such that, for all $\varepsilon > 0$ exists $C(\varepsilon) > 0$ so that*

$$p(f) \leq C(\varepsilon) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in K} \kappa_p(P_m f(x)) \quad \forall f \in \mathcal{H}_{K_p}(E;F).$$

(d) *The seminorms p satisfying that there exists an absolutely convex compact set K such that, for all $\varepsilon > 0$ exists $C(\varepsilon) > 0$ so that*

$$p(f) \leq C(\varepsilon) \sum_{m=0}^{\infty} \mathfrak{m}_p(P_m f(0)(K + \varepsilon B_E) \quad \forall f \in \mathcal{H}_{K_p}(E;F).$$

(e) *The seminorms of the form*

$$p(f) = \sum_{m=0}^{\infty} \mathfrak{m}_p(P_m f(0)(K + a_m B_E)),$$

where K ranges over all the absolutely convex compact sets and $(a_m)_m \in c_0^+$.

Proof. First note that if f is p-compact and K is a compact set, there exists a open set $V \supset K$ such that $f(V)$ is p-compact. Then, seminorms in (a) and (b) are well defined on $\mathcal{H}_{K_p}(E;F)$. Also, in virtue of Proposition 5.4, seminorms in (d) and (e) are well defined. Standard arguments show that seminorms in (a) and (b) define the same topology.

Now we show that seminorms in (b) and (c) coincide. Let p be a seminorm and let K be an absolutely convex compact set satisfying the conditions in (c). Let $V \supset K$ be any absolutely convex open set and take $d = \text{dist}(K, \mathcal{C}V) > 0$. By Proposition 3.3, since $K + dB_E \subset V$, we get

$$\mathfrak{m}_p(P_m f(x)(dB_E)) \leq \mathfrak{m}_p(f(x + dB_E)) \leq \mathfrak{m}_p(f(V)),$$

for all $f \in \mathcal{H}_{K_p}(E;F)$. Thus,

$$d^m \sup_{x \in K} \kappa_p(P_m f(x)) \leq \mathfrak{m}_p(f(V)),$$

for each m. Hence

$$p(f) \leq C(\tfrac{d}{2}) \sum_{m=0}^{\infty} (\tfrac{d}{2})^m \sup_{x\in K} \kappa_p(P_m f(x)) \leq 2C(\tfrac{d}{2}) m_p(f(V)),$$

which shows that p is AC-m_p-ported by K.

Conversely, let p be a seminorm, let K be an absolutely convex compact set satisfying the conditions in (b). Fix $\varepsilon > 0$ and take $x_1, \ldots, x_n$ in K such that $K \subset V$ with $V = \bigcup_{j=1}^n B_\varepsilon(x_j)$. As we did before, we may find an absolutely convex open set W so that $K \subset W \subset V$. Let $f \in \mathcal{H}_{K_p}$, without loss of generality we may assume that $\varepsilon < r_p(f, x)$ for all $x \in K$. By Remark 3.5, we obtain

$$m_p(f(B_\varepsilon(x_j))) \leq \sum_{m=0}^{\infty} \varepsilon^m \kappa_p(P_m f(x_j)) \leq \sum_{m=0}^{\infty} \varepsilon^m \sup_{x\in K} \kappa_p(P_m f(x)).$$

As p is AC-m_p-ported by K, we have that $p(f) \leq C_W m_p(f(W)) \leq C_W m_p(f(V))$ and therefore

$$\begin{aligned} p(f) &\leq C_W \sum_{j=1}^{n} m_p(f(B_\varepsilon(x_j))) \\ &\leq C_W \sum_{j=1}^{n} \sum_{m=0}^{\infty} \varepsilon^m \sup_{x\in K} \kappa_p(P_m f(x)) \\ &= nC_W \sum_{m=0}^{\infty} \varepsilon^m \sup_{x\in K} \kappa_p(P_m f(x)). \end{aligned}$$

Thus p belongs to the family in (c). If $\varepsilon \geq r_p(f, x)$, then $\sum_{m\geq 0} \varepsilon^m \sup_{x\in K} \kappa_p(P_m f(x)) = \infty$ and the inequality follows.

By the proof of [15, Proposition 4], we have that seminorms in (d) and (e) generate the same topology. Finally, we show that seminorms in (d) and (b) are equivalent. The proof of Proposition 5.4 shows that seminorms in (d) are AC-m_p-ported by absolutely convex compact sets.

To conclude the proof, consider a seminorm p and an absolutely convex compact set K satisfying conditions in (b). We borrow some ideas of [16, Chapter 3]. For each m, let W_m be the absolutely convex open set defined by $W_m = K + (\frac{1}{2})^m B_E$. Since p is AC-m_p-ported by K, for each $m \in \mathbb{N}$, there exists a constant $C_m = C_{W_m}$ such that $p(f) \leq C_m m_p(f(W_m))$, every p-compact function f.

For $m = 1$, there exists $n_1 \in \mathbb{N}$, such that for all $n > n_1$, $C_1^{1/n} < 2$. Take $V_1 = 2W_1$. Now, if $n > n_1$ and $Q \in \mathcal{P}_{K_p}(^nE; F)$,

$$p(Q) \leq C_1 m_p(Q(W_1)) = m_p(Q(C_1^{1/n} W_1)) \leq m_p(Q(V_1)).$$

For $m = 2$, there exists $n_2 > n_1$ such that $C_2^{1/n} \leq 2$, for all $n > n_2$. Now, take $V_2 = 2W_2$ and, as before, we have for any $Q \in \mathcal{P}_{K_p}(^nE; F)$, with $n > n_2$,

$$p(Q) \leq C_2 m_p(Q(W_2)) = m_p(Q(C_2^{1/n} W_2)) \leq m_p(Q(V_2)).$$

Repeating this procedure we obtain a sequence of absolutely convex open sets V_j satisfying

$$\begin{aligned} p(f) \leq \sum_{m\geq 0} p(P_m f(0)) &= \sum_{m<n_1} p(P_m f(0)) + \sum_{j\geq 1} \sum_{n_j \leq m < n_{j+1}} p(P_m f(0)) \\ &\leq C_{V_1} \sum_{m<n_1} \mathfrak{m}_p(P_m f(0)(V_1)) + \sum_{j\geq 1} \sum_{n_j \leq m < n_{j+1}} \mathfrak{m}_p(P_m f(0)(V_j)) \\ &\leq C \left(\sum_{m<n_1} \mathfrak{m}_p(P_m f(0)(V_1)) + \sum_{j\geq 1} \sum_{n_j \leq m < n_{j+1}} \mathfrak{m}_p(P_m f(0)(V_j)) \right) \end{aligned}$$

where $C = \min\{1, C_{V_1}\}$ and the result follows since $V_j = 2K + (\frac{1}{2})^{j-1} B_E$ and the seminorm p is bounded above by a seminorm of the family of the form (e). Now, the proof is complete. □

We finish this section by inspecting the κ_p-approximation property introduced in [13]. We will show that p-compact homogeneous polynomials from F to E can be κ_p-approximated by polynomials in $\mathcal{P}(^m F) \otimes E$ whenever E has the κ_p-approximation property. We then obtain a similar result for p-compact holomorphic functions. What follows keeps the spirit of [5, Theorem 4.1]. Recall that a Banach space E has the κ_p-approximation property if for every Banach space F, $F' \otimes E$ is κ_p-dense in $\mathcal{K}_p(F; E)$.

Theorem 5.6. *Let E be a Banach space. The following statements are equivalent.*

(i) *E has the κ_p-approximation property.*
(ii) *For all $m \in \mathbb{N}$, $\mathcal{P}(^m F) \otimes E$ is κ_p-dense in $P_{K_p}(^m F, E)$, for every Banach space F.*
(iii) *$\mathcal{H}(F) \otimes E$ is $\tau_{\omega,\mathfrak{m}_p}$-dense in $\mathcal{H}_{K_p}(F; E)$ for all Banach spaces F.*

Proof. First, suppose that E has the κ_p-approximation property and fix $m \in \mathbb{N}$. Then, $(\bigotimes_{\pi_s}^m F)' \otimes E$ is κ_p-dense in $K_p(\bigotimes_{\pi_s}^m F; E)$ which, by Proposition 2.1, coincides with $(P_{K_p}(^m F, E), \kappa_p)$, via the isomorphism given by $P \mapsto L_P$. Thus, (ii) is satisfied.

Now, assume (ii) holds. Take $f \in \mathcal{H}_{K_p}(F, E)$, $\varepsilon > 0$. By Theorem 5.5, we may consider a $\tau_{\omega,\mathfrak{m}_p}$-continuous seminorm of the form $q(f) = \sum_{m=0}^{\infty} \mathfrak{m}_p(P_m f(0)(K + a_m B_F))$, where $K \subset F$ is an absolutely convex compact set and $(a_m)_m \in c_0^+$. Let $m_0 \in \mathbb{N}$ be such that $\sum_{m>m_0} \mathfrak{m}_p(P_m f(0)(K + a_m B_F)) \leq \frac{\varepsilon}{2}$ and let $C > 0$ be such that $\frac{1}{C}(K + a_m B_F) \subset B_F$, for all $m \leq m_0$. Given $\delta > 0$, to be chosen later, by hypothesis, we may find $Q_m \in \mathcal{P}(^m F) \otimes E$ such that $\kappa_p(P_m f(0) - Q_m) \leq \delta$, for all $m \leq m_0$. Define $g = \sum_{m=0}^{m_0} Q_m$, which belongs to $\mathcal{H}(F) \otimes E$, then

$$\begin{aligned} q(f-g) &= \sum_{m=0}^{m_0} \mathfrak{m}_p((P_m f(0) - Q_m)(K + a_m B_F)) + \sum_{m>m_0} \mathfrak{m}_p(P_m f(0)(K + a_m B_F)) \\ &\leq \sum_{m=0}^{m_0} C^m \kappa_p((P_m f(0) - Q_m)) + \frac{\varepsilon}{2}. \end{aligned}$$

Thus, $q(f-g) < \varepsilon$ for a suitable choice of δ, which proves (iii).

Finally, suppose we have (iii). Take $T \in K_p(F,E)$, $\varepsilon > 0$ and the seminorm on $\mathcal{H}_{K_p}(F;E)$ defined by $q(f) = \kappa_p(P_1 f(0))$. Since q is $\tau_{\omega,\mathfrak{m}_p}$-continuous, by assumption, there exist $f_1,\dots,f_n \in \mathcal{H}(F)$ and $x_1,\dots,x_n \in E$, such that $q(T - \sum_{j=1}^n f_j \otimes x_j) < \varepsilon$. In other words, $\kappa_p(T - \sum_{j=1}^n P_1 f_j(0) \otimes x_j) < \varepsilon$ which proves that $F' \otimes E$ is κ_p-dense in $K_p(F,E)$. Whence, the proof is complete. □

Acknowledgements. We wish to express our gratitude to Richard Aron for introducing us to the problem during his visit to Buenos Aires in 2007 and for many useful conversations thereafter. Also, we are grateful to Manuel Maestre for sharing with us the open problems in a preliminary version of [2]. Finally, we thank Nacho Zalduendo for the suggestions he made to consider holomorphy types and to Christopher Boyd for his careful reading and comments on the manuscript.

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