CONFORMAL PURE RADIATION WITH PARALLEL RAYS

THOMAS LEISTNER AND PAWEŁ NUROWSKI

Abstract. We define pure radiation metrics with parallel rays to be n-dimensional pseudo-Riemannian metrics that admit a parallel null line bundle K and whose Ricci tensor vanishes on vectors that are orthogonal to K. We give necessary conditions in terms of the Weyl, Cotton and Bach tensors for a pseudo-Riemannian metric to be conformal to a pure radiation metric with parallel rays. Then we derive conditions in terms of tractor calculus that are equivalent to the existence of a pure radiation metric with parallel rays in a conformal class. We also give an analogous result for n-dimensional pseudo-Riemannian pp-waves.

MSC: 53A30; 53B30; 53C29
Keywords: Pure radiation, parallel rays, pp–waves, conformal geometry, tractor calculus

1. Introduction

In general relativity, i.e. on a four dimensional Lorentzian manifold identified as space-time, an energy momentum-tensor T_{ab} satisfying the conditions

$$T_{ab} = \phi^2 K_a K_b \quad \text{with a null vector } K_a K^a, \tag{1}$$

is called an energy momentum tensor of a *pure radiation* or of a *null dust* (see [10] for an overview). Ignoring the sign in (1), this is equivalent to the existence of a null vector K^a such that

$$T_{ab} X^a = 0 \text{ for all } X^a \text{ with } K_a X^a = 0.$$

In case of a zero cosmological constant, this equation, via the Einstein field equations,

$$R_{ab} - \tfrac{1}{2} R g_{ab} = T_{ab},$$

becomes

$$R_{ab} X^a = 0 \text{ for all } X^a \text{ with } K_a X^a = 0. \tag{2}$$

Here by R_{ab} w denote the Ricci tensor of the metric.

Another condition that appears in general relativity is the existence of a null vector K^a that spans a parallel ray, i.e.

$$\nabla_a K^b = f_a K^b. \tag{3}$$

Such metrics belong to the so-called Kundt's class and have special Lorentzian holonomy. If in addition $\nabla_{[a} f_{b]} \equiv 0$, the vector K^a can be rescaled to a parallel null vector. A particularly interesting class of metric in Kundt's class are those for which we have a *parallel null vector* and the energy momentum tensor is of pure radiation. Such metrics are called pp-waves in general relativity.

Date: July 29, 2011.
This work was supported by the first author's Start-Up-Grant of the Faculty of Engineering, Computer and Mathematical Sciences of the University of Adelaide and by grants NN201 607540 and NN202 104838 of the Polish Ministry of Research and Higher Education.

In the present article we are interested in generalisations of pure radiation metrics with parallel rays to arbitrary dimension and signature and we will focus on their conformal properties. First, let us define the class of metrics we are interested in.

Definition 1. Let g be a pseudo-Riemannian metric on an n-dimensional manifold.

(i) The metric g is a *pure radiation metric* if there is a null vector field K^a such that

$$R_{ab} = \phi K_a K_b,$$

with a function ϕ and R_{ab} being the Ricci tensor of the metric.

(ii) The metric g has *parallel rays* if there is a null vector field K^a such that K^a spans a parallel null line bundle, i.e.

$$\nabla_a K^b = f_a K^b$$

(iii) If g satisfies both conditions (i) and (ii) for the same vector K^a it is called *pure radiation metric with parallel rays* or *aligned pure radiation metric.*

The property of a metric to be a pure radiation metric is equivalent to

$$R_{ab}X^a = 0 \text{ for all } X^a \text{ with } K_a X^a = 0. \tag{4}$$

Hence, pure radiation metrics have vanishing scalar curvature. This implies that the curvature R_{abcd} and the Weyl tensor C_{abcd} of a pure radiation metrics satisfy

$$R_{abcd}X^aY^b = C_{abcd}X^aY^b \tag{5}$$

for all X^a and Y^b orthogonal to K^a. Note also that that the Ricci tensor satisfies property (4) if and only if the Schouten satisfies property (4).

Note that for a metric with parallel rays spanned by the vector field K^a, i.e. with

$$\nabla_a K^b = f_a K^b, \tag{6}$$

the vector K^a can be rescaled to a parallel vector field if and only if

$$\nabla_{[a} f_{b]} = 0.$$

Furthermore, one can show that for a metric with parallel rays defined as in (ii) of Definition 1 we always find a null vector K^a spanning the rays, such that

$$\nabla_a K_b = \psi K_a K_b,$$

with a function ψ.

In the first part of the paper we will derive necessary conditions in terms of the Weyl, Cotton and Bach tensors for a pseudo-Riemannian metric to be *conformal*, i.e. locally conformally equivalent, to a pure radiation metric with parallel rays. Then, as a special class of such metrics, we study the pp–waves in arbitrary signature. In the main part of the paper we use the normal conformal tractor calculus of [2] in order to derive *equivalent* conditions for a conformal class to contain a pure radiation metric with parallel rays. In Theorem 2 we prove: *A conformal class contains a pure radiation metric with parallel rays if and only if the tractor bundle contains a sub-bundle of totally null 2-planes that is parallel with respect to the normal conformal tractor connection.* As a corollary we obtain a characterisation of conformal pp–waves in terms of the tractor connection.

2. Tensorial obstructions

In this section we derive tensorial obstructions for a metric to be conformal to an aligned pure radiation metric. Our conventions are as in [5] with the basic tensors derived from the pseudo-Riemannian metric g_{ab} defined as follows: the curvature tensor, $R_{ab}{}^{c}{}_{d}$ defined by $(\nabla_a\nabla_b - \nabla_b\nabla_a)X^c = R_{ab}{}^{c}{}_{d}X^d$ and its trace, the Ricci tensor, defined by $R_{ab} := R_{cb}{}^{c}{}_{d}$ and the scalar curvature, defined by$R := R_a{}^a$. Fundamental tensors in conformal geometry are the Schouten tensor

$$\mathsf{P}_{ab} := \tfrac{1}{n-2}\left(R_{ab} - \tfrac{R}{2(n-1)}g_{ab}\right),$$

the Weyl tensor

$$C_{abcd} := R_{abcd} - 2(g_{c[a}\mathsf{P}_{b]d} + g_{d[b}\mathsf{P}_{a]c}),$$

as well as the Cotton and the Bach tensor

$$\begin{aligned} A_{abc} &:= 2\nabla_{[b}\mathsf{P}_{c]a} \\ B_{ab} &:= \nabla^c A_{acb} + \mathsf{P}^{dc}C_{dacb}. \end{aligned}$$

The Cotton tensor has a cyclic symmetry and satisfies

$$\begin{aligned} (n-3)A_{abc} &= \nabla^d C_{dabc} \\ \nabla^a A_{abc} &= 0. \end{aligned}$$

Now we derive properties of these tensors when the metric is an aligned pure radiation metric.

Proposition 1. *Let g be a pure radiation metric with parallel rays spanned by K^a. Then the Weyl tensor C, the Cotton tensor A and the Bach tensor B of g satisfy*

$$C_{abcd}K^aX^b = 0 \tag{7}$$
$$A_{abc}X^a = 0 \tag{8}$$
$$B_{ab}X^a = 0 \tag{9}$$

for all X^a orthogonal to K^b, i.e. with $X^aK_a = 0$.

Proof. Pure radiation implies $R = 0$ and we get for the Schouten tensor P that

$$\mathsf{P}_{ab} = \phi K_aK_b, \tag{10}$$

for a function ϕ. For X^a and Y^b orthogonal to K^a equation (10) gives

$$C_{abcd}X^cY^d = R_{abcd}X^cY^d - 2\phi\left(g_{c[a}K_{b]}K_d + g_{d[b}K_{a]}K_c\right)X^cY^d = R_{abcd}X^cY^d.$$

Equation (6) gives

$$R_{abcd}K^d = \nabla_{[a}f_{b]}K_c$$

which implies $R_{abcd}X^cK^d = 0$ for all X^a orthogonal to K^a. Hence, $C_{abcd}X^cK^d = 0$.

Equation (8) follows immediately from the definition of A_{abc}, from $R_{ab}X^b = \mathsf{P}_{ab}X^b = 0$ and the fact that $K^\perp$ is parallel, i.e. $\nabla_aX^bK_b = 0$.

Finally, we prove (9). By the definition of the Bach tensor and from equations (8), (7) and (10) we get

$$B_{ab}X^a = \nabla^c A_{acb}X^a + \mathsf{P}^{cd}C_{acbd}X^a = \phi C_{acbd}X^aK^cK^d = 0$$

for all X^b orthogonal to K^b. By the symmetry of B, this proves (9). □

Now we will use this proposition in order to derive obstructions for a metric to be conformal to an aligned pure radiation metric. Under a conformal change of the metric, $\hat{g}_{ab} = \mathrm{e}^{2\Upsilon} g_{ab}$, the Levi-Civita changes as

$$\hat{\nabla}_a X_b \;=\; \nabla_a X_b - \Upsilon_a X_b - \Upsilon_b X_a + g_{ab}\Upsilon^d X_d, \tag{11}$$

with $\Upsilon_a = \nabla_a \Upsilon$. The Schouten tensor transforms as

$$\hat{\mathsf{P}}_{ab} \;=\; \mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2}\Upsilon_c \Upsilon^c g_{ab}. \tag{12}$$

Using this, we can show:

Theorem 1. *Let g_{ab} be a metric that is conformal to a pure radiation metric with parallel rays K^a. Then the Weyl tensor of g_{ab} satisfies*

$$C_{abcd} K^c X^d \;=\; 0\,, \quad \text{for all } X^d \text{ orthogonal to } K^a\text{, i.e. with } K_d X^d = 0. \tag{13}$$

Furthermore, there is a gradient field Υ^a such that

$$\left(A_{abc} + C_{cba}{}^d \Upsilon_d\right) X^a \;=\; 0 \tag{14}$$

$$\left(B_{ab} - (n-4) C_{acbd} \Upsilon^c \Upsilon^d\right) X^a Y^b \;=\; 0 \tag{15}$$

for all X^a and Y^b orthogonal to K^a.

Remark 1. In four dimensions and Lorentzian signature, condition (13) means that W is of Petrov type III or N.

Proof. Let $\hat{g} = \mathrm{e}^{2\Upsilon} g$ be a pure radiation metric with parallel rays spanned by K^a and with $\hat{R}_{ab} X^b = 0$ for all X^a orthogonal to K^a. Then equations (13) and (14) follow immediately from the proposition by the conformal invariance of the Weyl tensor and the conformal transformation formula for the Cotton tensor:

$$\hat{A}_{abc} = A_{abc} + C_{cba}{}^d \Upsilon_d.$$

In order to prove equation (15), we take the divergence of (14). Using the definition of the Bach tensor, the well known identity for the divergence of the Weyl tensor

$$\nabla^b C_{cbad} = (n-3) A_{cad}, \tag{16}$$

and the transformation formula (12) in order to eliminate the $\nabla^c \Upsilon_d$-terms, we get

$$\begin{aligned} 0 \;&=\; \left(\nabla^b A_{abc} + \nabla^b (C_{cba}{}^d \Upsilon_d)\right) X^a + \left(A_{abc} + C_{cba}{}^d \Upsilon_d\right) \nabla^b X^a \\ &=\; \left(B_{ac} - (n-3) A_{acd} \Upsilon^d + C_{cbad}\left(\Upsilon^b \Upsilon^d - \hat{\mathsf{P}}^{bd}\right)\right) X^a \\ &\quad + \left(A_{abc} + C_{cba}{}^d \Upsilon_d\right) \nabla^b X^a \end{aligned} \tag{17}$$

By equations (10) and (13) we get

$$C_{acbd} \hat{\mathsf{P}}^{cd} X^b = \phi\, C_{acbd} K^c K^d X^b = 0.$$

Since only $\hat{\nabla}^b X^a$ is orthogonal to K^a but not necessarily $\nabla^b X^a$ we compute using (11)

$$\begin{aligned} \left(A_{abc} + C_{cba}{}^d \Upsilon_d\right) \nabla^b X^a \;&=\; \left(A_{abc} + C_{cba}{}^d \Upsilon_d\right) \Upsilon^a X^b \\ &=\; -(A_{bca} + A_{cab}) \Upsilon^a X^b \\ &=\; (C_{acb}{}^d \Upsilon_d - A_{cab}) \Upsilon^a X^b \\ &=\; -(C_{bac}{}^d \Upsilon_d + A_{cab}) \Upsilon^a X^b \end{aligned}$$

since both, the Cotton and the Weyl tensor are trace free and satisfy the Bianchi identity. Hence, when contracting equation (18) with Y^c we get on one hand

$$\left(A_{abc} + C_{cba}{}^d \Upsilon_d\right) (\nabla^b X^a) Y^c = 0,$$

and on the other hand, by (14),

$$A_{acd}\Upsilon^d X^a Y^c = -C_{dcab}\Upsilon^d\Upsilon^b X^a Y^c = C_{cdab}\Upsilon^b\Upsilon^d X^a Y^c$$

for X^b and Y^a orthogonal to K^a. This yields equation (15). □

Finally, we return to metrics with parallel rays. From the transformation formula (11) it is obvious that the property of admitting parallel rays is not invariant under a general conformal change of the metric. The following proposition shows that a parallel rays cannot be improved to a parallel vector field under a conformal change.

Proposition 2. *Let g be a pseudo-Riemannian metric with parallel rays spanned by K^a which cannot be rescaled to a parallel vector. Then there is also no metric in the conformal class of g for which K^a can be rescaled to a parallel vector field.*

Proof. This statement follows from (11): Assume that K^a spans a parallel ray, i.e. $\nabla_a K_a = f_a K_a$ for $K_a = g_{ab}K^b$, such that it cannot be rescaled to a parallel vector field. This means that $\nabla_{[a} f_{b]} \neq 0$. For a conformally changed metric $\hat{g}_{ab} = \mathrm{e}^{2\Upsilon} g_{ab}$ the metric dual of K^a with respect to $\hat{g}$ is $\hat{K}_a = \mathrm{e}^{2\Upsilon} K_a$. Then, that the rescaled one-form $\varphi\hat{K}_a$, where φ is a function φ without zeros, is parallel, reads as

$$\begin{aligned} 0 &= \mathrm{e}^{2\Upsilon}\left(\nabla_a\varphi K_b + 2\varphi\Upsilon_a + \varphi\hat{\nabla}_a K_b\right) \\ &= \mathrm{e}^{2\Upsilon}\left(\nabla_a\varphi + \varphi\Upsilon_a + \varphi f_a\right)K_b + \mathrm{e}^{2\Upsilon}\varphi\left(g_{ab}\Upsilon^d K_d - \Upsilon_b K_a\right). \end{aligned}$$

Contracting this with arbitrary X^a and Y^b from $K^\perp$ shows that $\Upsilon^d K_d = 0$. Then contracting only with X^b gives $\Upsilon_b X^b = 0$ which shows that $\Upsilon_b = \psi K_b$, and therefore $\Upsilon_b K_a = \Upsilon_a K_b$ Hence, the above equation gives

$$0 = \nabla_a\varphi + \varphi f_a = \varphi(\nabla_a \log\varphi + f_a)$$

which contradicts $\nabla_{[a} f_{b]} \neq 0$. □

3. pp–waves as aligned pure radiation metrics

Now we will derive similar conditions for a more special class of aligned pure radiation metrics that can be considered as pp-waves in arbitrary signature. For a given null vector K^a, not necessarily parallel, we will consider the fundamental property of the curvature tensor R,

$$R_{abcd}X^a Y^b = 0 \text{ for all } X^a \text{ and } Y^b \text{ orthogonal to } K^a. \tag{18}$$

Then, in analogy with the Lorentzian case, we define:

Definition 2. An n-dimensional pseudo-Riemannian metric is a *pp–wave* if it admits a parallel null vector field for which property (18) is satisfied.

These metrics have a highly degenerated curvature tensor and hence a very special restricted holonomy group. By the restricted holonomy group we refer to the group of parallel transports along contractible loops. The restricted holonomy group is the connected component of the full holonomy group. In analogy to the Lorentzian case, if the signature of the metric is (p, q), one can show that these metrics have their restricted holonomy groups contained in the abelian normal subgroup $\mathbb{R}^{p-1,q-1}$ in the group $\mathrm{SO}^0(p-1, q-1) \ltimes \mathbb{R}^{p-1,q-1}$, where the latter is the is the stabiliser in $\mathrm{SO}^0(p, q)$ of a null vector. In particular, if we denote by $\mathcal{K}$ the line distribution given by K^a, then the vector bundle $S := \mathcal{K}^\perp/\mathcal{K}$ is flat with respect to the connection $\nabla^S_X[Y] := [\nabla_X Y]$ for Y a section of $\mathcal{K}^\perp$.

Again in analogy to the Lorentzian case [8], the conformal Fefferman-Graham ambient metric and the tensor obstructing its existence can be computed explicitly for pp-waves.

In the following we will describe how pp-waves relate to pure radiation metrics. First, we verify that pp–waves are pure radiation metrics with parallel rays spanned by the parallel null vector field:

Proposition 3. *Let g be a pseudo-Riemanian pp–wave metric with a parallel null vector K^a. Then g is a pure radiation metric, i.e. equations* (8) *and* (9) *are satisfied, and furthermore we have*

$$C_{abcd}K^a = 0 \tag{19}$$

$$C_{abcd}X^aY^b = 0 \tag{20}$$

$$A_{cab}X^aY^b = 0, \tag{21}$$

for all X^a and Y^b orthogonal to K^a.

Proof. Since K^a is parallel and because of property (18) we get $R_{ab}X^a = 0$ for all X^a orthogonal to K^a. Hence, $R_a{}^a = 0$ which implies $\mathsf{P}_{ab}X^a = 0$ and $R_{abcd}X^aY^b = C_{abcd}X^aY^b$ for all X^a and Y^a orthogonal to K^a. Then the required equation (21) immediately follows from the identity (16) for the divergence of the Weyl tensor. Finally, as K^a is parallel, it follows

$$\begin{aligned} 0 &= R_{abcd}K^a \\ &= C_{abcd}K^a + 2(g_{c[a}\mathsf{P}_{b]d} + g_{d[b}\mathsf{P}_{a]c})K^a \\ &= C_{abcd}K^a + 2K_c\mathsf{P}_{bd} - 2g_{cb}\mathsf{P}_{ad}K^a + 2g_{db}\mathsf{P}_{ac}K^a - 2K_d\mathsf{P}_{bc} \\ &= C_{abcd}K^a + 2K_c\mathsf{P}_{bd} - 2K_d\mathsf{P}_{bc}. \end{aligned}$$

which shows that $C_{abcd}K^aX^c = 0$ for all X^c orthogonal to K^a. But this implies that $C_{abcd}K^a = 0$. □

The following proposition shows that for pure radiation metrics with parallel rays the property (18) implies that the rays contains a parallel null vector. Recall (5) which implies that a pure radiation metric satisfies (18) for the curvature tensor if and only if it satisfies (20) for the Weyl tensor.

Proposition 4. *A pure radiation metric with parallel rays K^a that satisfies property* (18)*, or the equivalent property* (20)*, admits a parallel null vector field in direction of K^a and hence, is a pp–wave.*

Proof. Assume that $\nabla_a K^b = f_a K^b$ with a one-form f_a. Then, locally K^a can be rescaled to a parallel vector field if and only if f_a is closed, i.e. iff $\nabla_{[a}f_{b]} = 0$. Since

$$R_{abc}{}^dK^c = \nabla_{[a}f_{b]}K^d,$$

we have to show that $R_{abc}{}^dK^c = 0$. Now (18) implies that

$$0 \;=\; R_{abc}{}^dX^aY^bK^c \;=\; R_{abc}{}^dX^aK^b$$

for all X^a and Y^b orthogonal to K^a. Hence, the only possibly non-vanishing terms of $R_{abc}{}^dK^c$ could be $R_{abcd}L^aX^bK^cL^d$ with L^a a vector transversal to $K^\perp$ and $X^b \in K^\perp$. But since we have pure radiation, also these terms vanish

$$0 = R_{ab}X^aL^b = R_{cabd}L^cX^aL^bK^d.$$

Indeed, if $(K, L, E_1, \ldots, E_{n-2})$ is a basis with $K^\perp$ spanned by $(K, E_1, \ldots, E_{n-2})$, $L^\perp$ is spanned by $(L, E_1, \ldots, E_{n-2})$, and such that $g(K, L) = 1$ and $g(E_i, E_j) = \pm\delta_{ij}$, we have

$$0 = Ric(Y, L) = R(L, Y, K, L) + \sum_{i=1}^{n-2} \epsilon_i R(E_i, Y, E_i, L) = R(L, Y, K, L).$$

Here $Ric(.,.)$ denotes the Ricci tensor and $R(.,.,.,.)$ the $(4,0)$-curvature tensor. □

We also have the following proposition.

Proposition 5. *Let (M, g) be pseudo-Riemannian manifold of dimension $n > 2$ with parallel null vector field K^a, i.e. with $\nabla_a K^b = 0$, satisfying (20) for the Weyl tensor. Then g is a pure radiation metric aligned with K^a and hence a pp–wave.*

Proof. First we show that g has vanishing scalar curvature $R = 0$. Since K^a is parallel we have that $R_{ab}K^a = 0$ and $R_{abcd}K^a = 0$. Because of property (20), with the same notation and same basis $(K, L, E_1, \ldots, E_{n-2})$ as in the proof of previous proposition we get for the scalar curvature

$$\begin{aligned} R &= \sum_{i,j=1}^{n-2} \epsilon_i \epsilon_j R(E_i, E_j, E_i, E_j) \\ &= 2(n-3) \sum_{i=1}^{n-2} \mathsf{P} \epsilon_i (E_i, E_i) \\ &= 2(n-3) \left(\mathsf{tr}(\mathsf{P}) - \mathsf{P}(K, L) \right) \\ &= 2(n-3) \left(\frac{R}{2(n-1)} + \frac{R}{(n-1)(n-2)} g(K, L) \right) \\ &= \frac{n(n-3)}{(n-1)(n-2)} R, \end{aligned}$$

which shows that $R = 0$. Next, we verify that the metric is a pure radiation metric. For $X \in K^\perp$ and $V \in TM$, since K is parallel and the Weyl tensor satisfies (20), we get

$$\begin{aligned} Ric(X, V) &= \sum_{i=1}^{n-2} \epsilon_i R(E_i, X, E_i, V) \\ &= (n-2)\mathsf{P}(X, V) - \sum_{i=1}^{n-2} \epsilon_i (g(E_i, X)\mathsf{P}(E_i, V) + g(V, E_i)\mathsf{P}(X, E_i)), \end{aligned}$$

in which we have used for the last equality that $R = 0$. Hence we get

$$0 = \sum_{i=1}^{n-2} \epsilon_i (g(E_i, X)\mathsf{P}(E_i, V) + g(V, E_i)\mathsf{P}(X, E_i)) = 2\mathsf{P}(X, V) - g(K, V)\mathsf{P}(L, X),$$

which shows that $\mathsf{P}(X, V) = 0$. Hence, g is a pure radiation metric and thus, by (5), a pp-wave. □

From Theorem 1 we get the following obstruction for the existence of a pp–wave metric in a conformal class:

Corollary 1. *Let g be a metric that is conformal to a pp-wave metric with parallel null vector K^a. Then, in addition to properties (14) and (15), for the Weyl and Cotton tensors of g we have equations (19), (20) and (21).*

Proof. Let $\hat{g} = \mathrm{e}^{2\Upsilon} g$ be a pp–wave metric with parallel null vector K^a. First note that the rays spanned by K^a and its orthogonal complement are conformally invariantly defined. Both are parallel with respect to $\hat{\nabla}$ but not parallel for ∇.

By Proposition 3, we have that $\hat{C}_{abcd}K^a = 0$ and $\hat{C}_{abcd}X^aY^b = 0$ for all X^a and Y^b orthogonal to K^a. The conformal invariance of the Weyl tensor then implies equations (19) and (20).

In order to derive equation (21) we apply Theorem 1. The Bianchi identity for the Cotton and Weyl tensors together with (14) for X^a and Y^b orthogonal to K^a gives

$$\begin{aligned} 0 &= (A_{abc} + A_{bca} + A_{cab})X^bY^c \\ &= A_{abc}X^bX^c - C_{acb}{}^d\Upsilon_d X^bY^c - C_{bac}{}^d\Upsilon_d X^bY^c \\ &= A_{abc}X^bX^c + (C_{cba}{}^d + C_{bac}{}^d)\Upsilon_d X^bY^c - C_{bac}{}^d\Upsilon_d X^bY^c. \end{aligned}$$

Then (20) is $C_{bca}{}^d X^bY^c = 0$ and implies the required equation (21). □

4. Conformal standard tractors

An invariant way of describing the geometry of a conformal manifold $(M,[g])$ is provided by the *normal conformal Cartan connection* ω (refer to [6], [2], [4] or to the survey [3] for details of the following). For example, ω detects local Einstein metrics in the conformal class, which correspond to covariantly constant sections with respect to ω. Now we will establish a condition on ω that is equivalent to the existence of a local pure radiation metric in the conformal class.

The normal conformal Cartan conection is defined uniquely by the conformal class $[g]$ and defined by the following data: Let (p,q) be the signature of the conformal class and $P \subset \mathrm{SO}_0(p+1,q+1)$ the stabiliser in $\mathrm{SO}_0(p+1,q+1)$ of a null line I. Then ω is a Cartan connection with values in $\mathfrak{so}(p+1,q+1)$ on a principal P-bundle $\mathcal{P}$, the *Cartan bundle*. Furthermore, it satisfies a normalisation condition that makes it unique. As a Cartan connection, ω defines an invariant absolute parallelism and hence, gives no horizontal distribution in $T\mathcal{P}$. In order to use the usual principle fibre bundle formalism, one extends the Cartan bundle to a bundle $\overline{\mathcal{P}}$ with structure group $\mathrm{SO}^0(p+1,q+1)$ via $\overline{\mathcal{P}} = \mathcal{P} \times_P \mathrm{SO}^0(p+1,q+1)$ on which the Cartan connection ω extends to a principle fibre bundle connection $\overline{\omega}$.

By associating the standard representation $\mathbb{R}^{p+1,q+1}$ of $\mathrm{SO}(p+1,q+1)$ to $\overline{\mathcal{P}}$ we obtain a vector bundle $\mathcal{T}$ of rank $p+q+2$, called *standard tractor bundle*. $\mathcal{T}$ is equipped with the covariant derivative $\overline{\nabla}$ induced by $\overline{w}$. Since $\overline{\omega}$ is an $\mathfrak{so}(p+1,q+1)$-connection, there is an invariant metric h on $\mathcal{T}$. The triple $(\mathcal{T},\overline{\nabla},h)$ is called *normal conformal standard tractor bundle*.

Let $I \subset \mathbb{R}^{p+1,q+1}$ be the null line defining P, i.e. P is the stabiliser group of I. Then, corresponding to the filtration $I \subset I^\perp \subset \mathbb{R}^{p+1,q+1}$, there is a bundle filtration of the tractor bundle

$$\mathcal{I} \subset \mathcal{I}^\perp \subset \mathcal{T}$$

into a null line bundle and its orthogonal complement, w.r.t. to the tractor metric. Then the projection $p : \mathcal{P} \to \mathcal{P}^0$ of the Cartan bundle onto the bundle of conformal frames $\mathcal{P}^0$ defines the following projection

$$\begin{array}{rcccccc} \mathrm{pr}_{TM} \ : \ \mathcal{I}^\perp = \mathcal{P} \times_P I^\perp & \to & \mathcal{I}^\perp/\mathcal{I} & \simeq & TM & \simeq & \mathcal{P}^0 \times_{\mathrm{CO}_0(p,q)} \left(I^\perp/I\right) \\ {[\varphi, v]} & \mapsto & [\varphi,[v]] & \mapsto & [p(\varphi),[v]] & & \end{array}$$

in an invariant way. Via this projection, the tractor metric defines the conformal structure.

Every metric g in the conformal class $[g]$ induces a splitting

$$(\mathcal{T},h) \ \simeq \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}},$$

where each $\underline{\mathbb{R}}$ denotes the trivial line bundle $M \times \mathbb{R}$. Changing the metric in the conformal class, $\hat{g}_{ab} = \mathrm{e}^{2\Upsilon} g_{ab}$, this splitting transforms via

$$\begin{pmatrix} \hat{\rho} \\ \hat{X}^a \\ \hat{\sigma} \end{pmatrix} = \begin{pmatrix} \rho - \Upsilon_a X^a - \frac{1}{2}\sigma \Upsilon^b \Upsilon_b \\ X^a + \sigma\ \Upsilon_a \\ \sigma \end{pmatrix}. \tag{22}$$

In every such splitting the line bundle $\mathcal{I}$ is spanned by

$$I^A := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

where capital indices range in $0, 1, \ldots, n, n+1$, with n the dimension of M. The bundle metric h of signature $(p+1, q+1)$ on $\mathcal{T}$ is given by

$$h\left(\begin{pmatrix} \rho \\ X^a \\ \sigma \end{pmatrix}, \begin{pmatrix} \alpha \\ Y^b \\ \beta \end{pmatrix}\right) = \sigma\alpha + \rho\beta + g_{ab} X^a Y^b.$$

Hence, $\mathcal{I}^\perp$ is given by tractors of the form

$$\begin{pmatrix} \rho \\ X^a \\ 0 \end{pmatrix}$$

with tangent vectors X^a, and that the projection pr_{TM} is invariantly given as

$$\mathrm{pr}_{TM} : \mathcal{I}^\perp \ni \begin{pmatrix} \rho \\ X^a \\ 0 \end{pmatrix} \mapsto X^a \in TM.$$

In a splitting defined by a metric g_{ab} the tractor connection $\overline{\nabla}$ is given by the following formula,

$$\overline{\nabla}_a \begin{pmatrix} \rho \\ X^b \\ \sigma \end{pmatrix} = \begin{pmatrix} \nabla_a \rho - \mathsf{P}_{ab} X^b \\ \nabla_a X^b + \rho \delta_a^b + \sigma \mathsf{P}_a{}^b \\ \nabla_a \sigma - g_{ab} X^b \end{pmatrix}, \tag{23}$$

where ∇ is the Levi-Civita connection and P the Schouten tensor of the metric g. The curvature of this connection is given as

$$\overline{R}_{ab} \begin{pmatrix} \rho \\ X^c \\ \sigma \end{pmatrix} = \begin{pmatrix} -A_{dab} X^d \\ \sigma A^c{}_{ab} + C_{ab}{}^c{}_d X^d \\ 0 \end{pmatrix}. \tag{24}$$

It is a well known fact, that covariantly constant sections of $\overline{\nabla}$ are in one-to-one correspondence with local Einstein metrics on an open and dense subset of M. In fact, if $X^B = \begin{pmatrix} \rho \\ X^b \\ \sigma \end{pmatrix}$ is covariantly constant for $\overline{\nabla}$ if and only if $\sigma^{-2} g$ is an Einstein metric on the open and dense complement of the zero set of $\sigma = X^A I_A$. Indeed, X^B is parallel if and only if $X^b = \nabla^b \sigma$ and $\nabla_a \nabla_b \sigma + \sigma P_{ab} + \rho g_{ab} = 0$, where the latter is equivalent to $\sigma^{-2} g$ being an Einstein metric outside the zero set of σ. Furthermore, applying the tractor curvature $\overline{R}_{ab}$ to a parallel tractor X^B we get the integrability conditions $\overline{R}_{ab} X^C = 0$, i.e. for $X_c = \nabla_c \sigma$ we get

$$\begin{aligned} A^c{}_{ab} X_c &= 0 \\ \sigma A_{cab} + C_{abc}{}^d X_d &= 0. \end{aligned}$$

Setting $\sigma = \mathrm{e}^{-\Upsilon}$ we get $X_a = -\sigma\Upsilon_a$ with $\Upsilon_a = \nabla_a\Upsilon$ and obtain

$$A_{cab} - C_{abc}{}^{d}\Upsilon_d = 0,$$

as one of the integrability conditions that was derived in [5].

5. Tractorial characterisation of aligned pure radiation metrics

Now we will describe the situation in which the normal conformal tractor holonomy admits an invariant plane. The case when the plane is non-degenerate implies that the conformal class contains a product of Einstein metrics with related Einstein constants (see [1] for Riemannian conformal classes and related results in the unpublished parts of [9] for arbitrary signature). Here we will deal with the case that the null plane is totally null in arbitrary signature. The following theorem is a generalisation to arbitrary signature of the corresponding result for Lorentzian conformal classes which we proved in [7].

Theorem 2. *Let $(M, [g])$ be a pseudo-Riemannian conformal manifold of dimension $n > 2$. Then the normal conformal Cartan connection admits a parallel totally null plane if and only if, on an open and dense subset of M, there is an aligned pure radiation metric g_{ab} in the conformal class $[g]$.*

Proof. Let $\mathcal{T}$ be the normal conformal tractor bundle. First, we prove a Lemma.

Lemma 1. *Given a bundle of null lines in TM spanned by K^a and a metric g_{ab} in the conformal class, then the following statements are equivalent:*

(i) *g_{ab} is pure radiation metric with parallel rays spanned by K^a,*

(ii) *W.r.t. the splitting of $\mathcal{T} = \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}}$ given by g_{ab}, the plane bundle $\mathcal{H}$ spanned by*

$$K^A = \begin{pmatrix} 0 \\ K^a \\ 0 \end{pmatrix} \text{ and } J^A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{25}$$

is parallel for the tractor connection.

Proof. Since the tractor connection preserves the metric, $\mathcal{H}$ is parallel if and only if

$$\mathcal{H}^{\perp} = \left\{ \begin{pmatrix} 0 \\ X^b \\ \sigma \end{pmatrix} \mid K_a X^a = 0, \sigma \in \mathbb{R} \right\}$$

is parallel. By pairing with K^A and J^A we get that $\mathcal{H}^{\perp}$ being parallel is equivalent to

$$P_{ab}X^b = 0 \text{ and } \left(\nabla_a X^b - \sigma P_a{}^b\right) K_b = 0,$$

which is equivalent to $R_{ab}X^b = 0$ and $\nabla_a K^b = f_a K^b$. □

The 'if'-statement of the theorem follows immediately from the lemma: If g_{ab} is an aligned pure radiation metric in the conformal class, we split the tractor bundle with respect to g_{ab} and get a parallel null plane $\mathcal{H}$ for the tractor connection as in (25).

Now we show the 'only if'-statement in the theorem. Assume that $\mathcal{H}$ is a null plane bundle that is parallel for the tractor connection. We have to find a null vector K^a and a metric in the conformal class such that $\mathcal{H}$ is spanned by K^A and J^A as in as in (25). First, using that $\mathcal{H}$ is parallel and some basic linear algebra, we will define the null line. Let $\mathcal{I} \subset \mathcal{T}$ be the null line bundle defined by the conformal structure. Then we define

$$\mathcal{L} := \mathcal{I}^{\perp} \cap \mathcal{H} = \{X^A \in \mathcal{H} \mid X^A I_A = 0\}$$

a subbundle of $\mathcal{H}$. We have:

Lemma 2. *$\mathcal{L}$ is a null line bundle and, over an open and dense subset of M, its invariant projection $\mathrm{pr}_{TM}(\mathcal{L})$ is not zero.*

Proof. As $\mathcal{H}$ is bundle of 2-planes, for dimensional reasons we have $\mathcal{L} \neq \{0\}$. Hence, the rank of $\mathcal{L}$ is either one or two. Then we fix *some* metric in the conformal class and w.r.t to the induced splitting $\mathcal{T} = \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}}$ of the tractor bundle we get

$$\mathcal{L} = \left\{ L^A = \begin{pmatrix} \rho \\ L^a \\ 0 \end{pmatrix} = \rho I^A + L^a \in \mathcal{H} \right\}.$$

If $\mathcal{L}$ had rank two over an open set U, we would get $\mathcal{H}|_U \subset (\underline{\mathbb{R}} \oplus TM)|_U = \mathcal{I}^\perp|_U$. Hence, with $\mathcal{H}$ being parallel, the tractor derivative of every $L^A \in \mathcal{H}|_U$ is again in $\mathcal{H}|_U$ and hence in $\mathcal{I}$. Differentiating in any direction using formula (23), over U we get

$$0 = \overline{\nabla}_b L^A I_A = -g_{ba} L^a.$$

But this means that $L^a = 0$ which excludes then rank of $\mathcal{L}|_U$ being two. Hence there is no open set over which $\mathcal{L}$ has rank two. Therefore it must have rank one over all of M and hence, $\mathcal{L}$ is a line bundle.

Furthermore, assume that $\mathrm{pr}_{TM}\mathcal{L}\{0\}$ over an open set U of M. Then, over U, every section of $\mathcal{L}$ would be of the form ρI^A with $\rho \in C^\infty(U)$. Differentiating yields

$$\overline{\nabla}_b(\rho I^A) = \nabla_b \rho\, I^A + \rho \nabla_b I^A = \nabla_b \rho I^A + \rho \delta_b{}^a.$$

Since $\mathcal{L} \subset \mathcal{H}$ we get that $\overline{\nabla}_b(\rho I^A) = \nabla_b \rho I^A + \rho \delta_b{}^a$ is again in $\mathcal{H}$ because $\mathcal{H}$ was parallel. Since U was open, we can differentiate in any direction of TM, which yields acontradiction to $\mathcal{H}$ being a plane. □

The lemma shows that, over an open and dense subset of M, we can define a line bundle in TM as

$$L := \mathrm{pr}_{TM}\mathcal{L}$$

in an invariant way. In the following we will restrict our computations to this open and dense subset without explicitly mentioning it again. Since $\mathcal{H}$ and hence $\mathcal{L}$ were totally null, L is a bundle of null lines. Also, its orthogonal complement, $L^\perp \subset TM$, is given as a projection from $\mathcal{T}$:

Lemma 3. *The orthogonal complement of L satisfies $L^\perp = \mathrm{pr}_{TM}\left(\mathcal{L}^\perp \cap \mathcal{I}^\perp\right)$.*

Proof. Let $\mathcal{L}$ be spanned locally by $I^A + K^a$ with a null vector field K^a. If $\rho I^A + X^a \in \mathcal{L}^\perp \cap \mathcal{I}^\perp$, then $X^b K_b = 0$, i.e. $\mathrm{pr}_{TM}\left(\mathcal{L}^\perp \cap \mathcal{I}^\perp\right) \subset L^\perp$. On the other hand, if $X^a \in L^\perp$, then $I^A + X^a \in \mathcal{L}^\perp \cap \mathcal{I}^\perp$. □

Next, we prove

Lemma 4. *$\mathcal{L}^\perp = \mathcal{H}^\perp \oplus \mathcal{I}$*

Proof. First note that $\mathcal{L}^\perp$ contains I^A. On the other hand, $\mathcal{L} \subset \mathcal{H}$ gives $\mathcal{H}^\perp \subset \mathcal{L}^\perp$. Recalling that $\mathcal{L} = \mathcal{H} \cap \mathcal{I}^\perp$ is a line in the plane $\mathcal{H}$ shows that there is an element $X^A \in \mathcal{H}$ such that $X^A I_A = 1$. This implies $\mathcal{H}^\perp \cap \mathcal{I} = \{0\}$ and counting dimensions completes the proof. □

Then Lemmas 3 and 4 imply that

$$L^\perp = \mathrm{pr}_{TM}\left(\mathcal{H}^\perp \cap \mathcal{I}^\perp\right), \tag{26}$$

which provides us with

Lemma 5. *The hyperplane bundle $L^\perp \subset TM$ is integrable.*

Proof. Let X^a and Y^b two local sections of $L^\perp$ such that $X_aY^a = 0$. Then there is a smooth function ρ such that $\rho I^a + Y^b \in \mathcal{H}^\perp \cap \mathcal{I}^\perp \subset \mathcal{H}^\perp$. Since $\mathcal{H}^\perp$ is parallel, we get that

$$X^a\overline{\nabla}_a \begin{pmatrix} \rho \\ Y^b \\ 0 \end{pmatrix} = \begin{pmatrix} X^a(\nabla_a\rho - \mathsf{P}_{ab}Y^b) \\ X^a\nabla_aY^b + \rho X^b \\ 0 \end{pmatrix} \in \Gamma(\mathcal{H}^\perp \cap \mathcal{I}^\perp),$$

By (26) this shows that $X^a\nabla_aY^B \in \Gamma(L^\perp)$ for $X^a, Y^b \in \Gamma(L^\perp)$ orthogonal to each other. By fixing a basis of $\mathcal{L}^\perp$ that consists of mutually orthogonal vectors, the vanishing of the torsion of ∇ implies that $L^\perp$ is integrable. □

Hence, so far, given the parallel plane distribution $\mathcal{H}$ in $\mathcal{T}$, we have invariantly constructed a null line bundle L in TM such that the $L^\perp$ is integrable.

The integrability of $L^\perp$ allows us to define a second fundamental form for $L^\perp$ in the following way. From now on we fix a null vector field K^a spanning L and define a bilinear form on $L^\perp$ by

$$\Pi_{ab} := \nabla_aK^cg_{cb}|_{L^\perp\times L^\perp}.$$

Since $L^\perp$ is integrable, Π^K is symmetric. We define the trace of Π^K as

$$H := g^{ab}\Pi_{ij}E_i^aE_i^b \in C^\infty(M)$$

where $E_1^a, \dots, E_{n-2}^a$ are linearly independent in $L^\perp$ and i, j range over $1, \dots, n-2$. Since $K^a\Pi_{ab} = 0$, this is independent of the chosen E_i^a's. Now we claim that there is a metric $\hat{g}_{ab} = \mathrm{e}^{2\Upsilon}g_{ab}$ in the conformal class such that the corresponding function $\hat{H}$ is zero. To this end we notice that the transformation formula for $\hat{\Pi}$ is given by

$$X^aY^b\hat{\Pi}_{ab} = X^aY^b\nabla_a\hat{g}_{cb}K^c = \mathrm{e}^{2\Upsilon}\left(\Pi_{ab} + K^c\Upsilon_cg_{ab}\right)X^aY^b,$$

for $X^a, Y^b \in L^\perp$. Hence,

$$\hat{H} = \mathrm{e}^{2\Upsilon}\left(H + (n-2)K^a\Upsilon_a\right).$$

Now the differential equation

$$K^a\Upsilon_a = \frac{H}{n-2}$$

has always a solution Υ, which ensures that we can chose $\hat{g}_{ab}$ such that $\hat{H} \equiv 0$. Finally, to conclude the proof, we fix this metric, omit the hat, and split the tractor bundle with respect to the new metric. Now let $\rho I^A + K^a$ be an arbitray section of $\mathcal{L}$. Since $\mathcal{H}$ is parallel, differentiating in direction $Y^b \in L^\perp$ yields that

$$Y^b\overline{\nabla}_b \begin{pmatrix} \rho \\ K^a \\ 0 \end{pmatrix} = \begin{pmatrix} Y^b\nabla_b\rho - \mathsf{P}_{ab}Y^bK^a \\ Y^b\nabla_bK^a + \rho Y^a \\ 0 \end{pmatrix}$$

is still a section of $\mathcal{H}$, but also of $\mathcal{L}$ since the last component vanishes. This means that $Y^b\nabla_bK^a + \rho Y^a$ is in L which implies that

$$\Pi_{ab}Y^aY^b + \rho g_{ab}Y^aY^b = 0$$

for all $Y^a \in L^\perp$. Taking the trace, $H = 0$ gives $0 = \rho(n-2)$, which results in $\mathcal{L} = \mathbb{R}\cdot \begin{pmatrix} 0 \\ K \\ 0 \end{pmatrix}$.

This, on the other hand, means the parallel null plane $\mathcal{H}$ is given by

$$\mathcal{H} = \left\{ \begin{pmatrix} 0 \\ X \\ \sigma \end{pmatrix} \mid X \in L, \sigma \in \mathbb{R} \right\}.$$

But this was equivalent to properties (6) and (4). □

Remark 2. This tractorial characterisation in Theorem 2 yields integrability conditions in terms of the tractor curvature, which, on the other hand, imply exactly the obstructions (13) and (14) for the existence of an aligned pure radiation metric in a conformal class given in Theorem 1. Indeed, when splitting the tractor bundle with respect to the aligned pure radiation metric g_{ab} in the conformal class, the parallel null plane bundle $\mathcal{H}$ contains $K^B = \begin{pmatrix} 0 \\ K^b \\ 0 \end{pmatrix}$ and $\mathcal{H}^\perp$ contains $X^B = \begin{pmatrix} 0 \\ X^b \\ 0 \end{pmatrix}$ with X^b orthogonal to K^a. Hence, with respect to a splitting in another metric $\hat{g}_{ab} = \mathrm{e}^{2\Upsilon} g_{ab}$, using formula (22), K^B and X^B are given as

$$K^B = \begin{pmatrix} \Upsilon_a K^a \\ K^b \\ 0 \end{pmatrix}, \quad X^B = \begin{pmatrix} \Upsilon_a X^a \\ X^b \\ 0 \end{pmatrix},$$

and still contained in $\mathcal{H}$ and $\mathcal{H}^\perp$, respectively. Hence, since $\mathcal{H}$ is parallel and thus invariant under the tractor curvature, applying the tractor curvature to K^B yields a tractor that is again contained in $\mathcal{H}$. Pairing $\overline{R}_{abCD} K^C$ with X^D and using (24), this implies that

$$0 \;=\; \overline{R}_{abCD} K^C X^D \;=\; C_{abcd} K^c X^d,$$

for all X^d orthogonal to K^c, which is equation (13). On the other hand, we have seen that $\mathcal{H}$ also contains the section $J^A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, when splitting the tractor bundle with the aligned pure radiation metric g_{ab}. In another metric $\hat{g}_{ab} = \mathrm{e}^{2\Upsilon} g_{ab}$, again using (22), J^A is given as

$$J^A \;=\; \begin{pmatrix} -\frac{1}{2}\Upsilon^b \Upsilon_b \\ \Upsilon_a \\ 1 \end{pmatrix}.$$

Since $\mathcal{H}^\perp$ is parallel and thus curvature invariant, pairing $\overline{R}_{abCD} X^C$ with J^D yields

$$0 \;=\; \overline{R}_{abCD} X^C J^D \;=\; -A_{dab} X^d + C_{ab}{}^d{}_c X^c \Upsilon_d,$$

in which we have used formula (24) for the tractor curvature. But this is equation (14) of Theorem 1.

Finally, from Theorem 2, we get a characterisation of conformal pp–waves that is based on their characterisation as pure radiation metrics in Proposition 3.

Corollary 2. *Let $(M, [g])$ be a pseudo-Riemannian conformal manifold of dimension $n > 2$. Then, on an open and dense subset of M, $[g]$ contains a pp–wave metric if and only if the normal conformal Cartan connection admits a parallel totally null plane subbundle $\mathcal{H}$ and the tractor curvature $\overline{R}$ satisfies*

$$\overline{R}_{abCD} X^C Y^D = 0, \tag{27}$$

for all X^C and Y^D orthogonal to $\mathcal{H}$.

Proof. If g is a pp–wave in the conformal class, then by Proposition 3 it is an aligned pure radiation metric, and hence, by Theorem 2 defines a parallel plane distribution in the tractor bundle. Using the pp–wave metric to split the tractor bundle, we have seen in the proof of the theorem that

$$\mathcal{H}^\perp = \left\{ X^B = \begin{pmatrix} 0 \\ X^b \\ \sigma \end{pmatrix} \mid K_a X^a = 0, \sigma \in \mathbb{R} \right\}.$$

Hence, for $X^C = \begin{pmatrix} 0 \\ X^c \\ \sigma \end{pmatrix}$ and $Y^D = \begin{pmatrix} 0 \\ Y^d \\ \tau \end{pmatrix}$ from $\mathcal{H}^\perp$ with X^c and Y^d orthogonal to K^a, the formula (24) for the tractor curvature yields

$$\overline{R}_{abC}{}^{D} X^C Y_D \;=\; -\tau A_{dab} X^d + \sigma A^c{}_{ab} Y_c + C_{ab}{}^c{}_d X^d Y_c \;=\; 0, \tag{28}$$

because of Propositions 1 and 3.

On the other hand, by Theorem 2 the existence of the parallel plane distribution yields an aligned pure radiation metric g in the conformal class. With this metric as a gauge the curvature condition (27) then spells out as equation (28). Now, since g is an aligned pure radiation metric, by Proposition 1 we have that the first two terms in (28) vanish and we are left with

$$C_{ab}{}^c{}_d X^d Y_c = 0$$

for all X^d and Y^c orthogonal to K^a. Then, from Proposition 4 we know that g is indeed a pp–wave. □

References

[1] S. Armstrong. Definite signature conformal holonomy: A complete classification. *J. Geom. Phys.*, 57(10):2024–2048, 2007.

[2] T. N. Bailey, M. G. Eastwood, and A. R. Gover. Thomas's structure bundle for conformal, projective and related structures. *Rocky Mountain J. Math.*, 24(4):1191–1217, 1994.

[3] H. Baum. The conformal analog of Calabi-Yau manifolds. In *Handbook of Pseudo-Riemannian Geometry*, IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society, 2007. In press.

[4] A. Čap and A. R. Gover. Tractor calculi for parabolic geometries. *Trans. Amer. Math. Soc.*, 354(4):1511–1548 (electronic), 2002.

[5] A. R. Gover and P. Nurowski. Obstructions to conformally Einstein metrics in n dimensions. *J. Geom. Phys.*, 56(3):450–484, 2006.

[6] S. Kobayashi. *Transformation groups in differential geometry*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1972 edition.

[7] T. Leistner. Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds. *Differential Geom. Appl.*, 24(5):458–478, 2006.

[8] T. Leistner and P. Nurowski. Ambient metrics for n-dimensional *pp*-waves. *Comm. Math. Phys.*, 296(3):881–898, 2010.

[9] F. Leitner. Normal conformal Killing forms, 2004. arXiv.org:math/0406316.

[10] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt. *Exact solutions of Einstein's field equations*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, second edition, 2003.

(Leistner) School of Mathematical Sciences, University of Adelaide, SA 5005, Australia
E-mail address: thomas.leistner@adelaide.edu.au

(Nurowski) Wydział Fizyki, Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoża 69, 00-681 Warszawa, Poland
E-mail address: nurowski@fuw.edu.pl