A CONSTRUCTION OF 2-COFILTERED BILIMITS OF TOPOI

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INTRODUCTION

We show the existence of bilimits of 2-cofiltered diagrams of topoi, generalizing the construction of cofiltered bilimits developed in [2]. For any given such diagram, we show that it can be represented by a 2-cofiltered diagram of small sites with finite limits, and we construct a small site for the inverse limit topos. This is done by taking the 2-filtered bicolimit of the underlying categories and inverse image functors. We use the construction of this bicolimit developed in [4], where it is proved that if the categories in the diagram have finite limits and the transition functors are exact, then the bicolimit category has finite limits and the pseudocone functors are exact. An application of our result here is the fact that every Galois topos has points [3].

1. Background, terminology and notation

In this section we recall some 2-category and topos theory that we shall explicitly need, and in this way fix notation and terminology. We also include some in-edit proofs when it seems necessary. We distinguish between *small* and *large* sets. Categories are supposed to have small hom-sets. A category with large hom-sets is called *illegitimate*.

Bicolimits

By a *2-category* we mean a $\mathcal{C}at$ enriched category, and *2-functors* are $\mathcal{C}at$ functors, where $\mathcal{C}at$ is the category of small categories. Given a 2-category, as usual, we denote horizontal composition by juxtaposition, and vertical composition by a $''\circ''$. We consider juxtaposition more binding than $''\circ''$ (thus $xy \circ z$ means $(xy) \circ z$). If $\mathcal{A}$, $\mathcal{B}$ are 2-categories ($\mathcal{A}$ small), we will denote by $[[\mathcal{A}, \mathcal{B}]]$ the 2-category which has as objects the 2-functors, as arrows the *pseudonatural transformations*, and as 2-cells the *modifications* (see [5] I,2.4.). Given F, G, $H : \mathcal{A} \longrightarrow \mathcal{B}$, there is a functor:

$$[[\mathcal{A}, \mathcal{B}]](G, H) \times [[\mathcal{A}, \mathcal{B}]](F, G) \longrightarrow [[\mathcal{A}, \mathcal{B}]](F, H) \tag{1.1}$$

To have a handy reference we will explicitly describe these data in the particular cases we use.

A *pseudocone* of a diagram given by a 2-functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ to an object $X \in \mathcal{B}$ is a pseudonatural transformation $F \xrightarrow{h} X$ from F to the 2-functor which is constant at X. It consists of a family of arrows $(h_A : FA \to X)_{A \in \mathcal{A}}$, and a family of invertible 2-cells $(h_u : h_A \to h_B \circ Fu)_{(A \xrightarrow{u} B) \in \mathcal{A}}$. A morphism $g \overset{\varphi}{\Longrightarrow} h$ of pseudocones (with same vertex) is a modification, as such, it consists of a family of 2-cells $(g_A \overset{\varphi_A}{\Longrightarrow} h_A)_{A \in \mathcal{A}}$. These data is subject to the following:

1.2 (*Pseudocone and morphism of pseudocone equations)*.

pc0. $h_{id_A} = id_{h_A}$, *for each object* A

pc1. $h_v Fu \circ h_u = h_{vu}$, *for each pair of arrows* $A \xrightarrow{u} B \xrightarrow{v} C$

pc2. $h_B F\gamma \circ h_v = h_u$, *for each 2-cell* $A \underset{v}{\overset{u}{\underset{\gamma\Downarrow}{\rightrightarrows}}} B$

pcM. $h_u \circ \varphi_A = \varphi_B Fu \circ g_u$, *for each arrow* $A \xrightarrow{u} B$

We state and prove now a lemma which, although expected, needs nevertheless a proof, and for which we do not have a reference in the literature. As the reader will realize, the statement concerns general pseudonatural transformations, but we treat here the particular case of pseudocones.

1.3. **Lemma.** *Let* $\mathcal{A} \xrightarrow{F} \mathcal{B}$ *be a 2-functor and* $F \xrightarrow{g} X$ *a pseudocone. Let* $FA \xrightarrow{h_A} X$ *be a family of morphisms together with invertible* 2*-cells* $g_A \overset{\varphi_A}{\Longrightarrow} h_A$*. Then, conjugating by* φ *determines a pseudocone structure for* h*, unique such that* φ *becomes an isomorphism of pseudocones.*

Proof. If φ is to become a pseudocone morphism, the equation pcM. $\varphi_B Fu \circ g_u = h_u \circ \varphi_A$ must hold. Thus, $h_u = \varphi_B Fu \circ g_u \circ \varphi_A^{-1}$ determines and defines h. The pseudocone equations 1.2 for h follow from the respective equations for g:

pc0. $h_{id_A} \;=\; \varphi_A \circ g_{id_A} \circ \varphi_A^{-1} \;=\; \varphi_A \circ id_{g_A} \circ \varphi_A^{-1} \;=\; id_{h_A}$

pc1. $A \xrightarrow{u} B \xrightarrow{v} C$:

$$\begin{aligned} h_v Fu \circ h_u &= (\varphi_C Fv \circ g_v \circ \varphi_B^{-1}) Fu \circ \varphi_B Fu \circ g_u \circ \varphi_A^{-1} = \\ &= \varphi_C F(vu) \circ g_v Fu \circ \varphi_B^{-1} Fu \circ \varphi_B Fu \circ g_u \circ \varphi_A^{-1} = \\ &= \varphi_C F(vu) \circ g_v Fu \circ g_u \circ \varphi_A^{-1} = \\ &= \varphi_C F(vu) \circ g_{vu} \circ \varphi_A^{-1} = h_{vu} \end{aligned}$$

pc2. For $A \underset{v}{\overset{u}{\underset{\Uparrow\gamma}{\rightrightarrows}}} B$ we must see $h_B F\gamma \circ h_v = h_u$. This is the same as $h_B F\gamma \circ \varphi_B Fv \circ g_v \circ \varphi_A^{-1} = \varphi_B Fu \circ g_u \circ \varphi_A^{-1}$. Canceling φ_A^{-1} and composing with $(\varphi_B Fu)^{-1}$ yields (1) $(\varphi_B Fu)^{-1} \circ h_B F\gamma \circ \varphi_B Fv \circ g_v = g_u$. From the compatibility between vertical and horizontal composition it follows $(\varphi_B Fu)^{-1} \circ h_B F\gamma \circ \varphi_B Fv \;= (\varphi_B^{-1} \circ h_B \circ \varphi_B)(Fu \circ F\gamma \circ Fv) = g_B F\gamma$. Thus, after replacing, (1) becomes $g_B F\gamma \circ g_v = g_u$. □

Given a small 2-diagram $\mathcal{A} \xrightarrow{F} \mathcal{B}$, the category of pseudocones and its morphisms is, by definition, $pc\mathcal{B}(F, X) = [[\mathcal{A}, \mathcal{B}]](F, X)$. Given a pseudocone $F \xrightarrow{f} Z$ and a 2-cell $Z \underset{t}{\overset{s}{\underset{\xi\Downarrow}{\rightrightarrows}}} X$, it is clear and straightforward how to define a morphism of pseudocones $F \underset{tf}{\overset{sf}{\underset{\xi f\Downarrow}{\rightrightarrows}}} X$ which is the composite $F \xrightarrow{f} Z \underset{t}{\overset{s}{\underset{\xi\Downarrow}{\rightrightarrows}}} X$. This is a particular case of 1.1, thus composing with f determines a functor (denoted ρ_f) $\mathcal{B}(Z, X) \xrightarrow{\rho_f} pc\mathcal{B}(F, X)$.

1.4. **Definition.** *A pseudocone* $F \xrightarrow{\lambda} L$ *is a* bicolimit *of* F *if for every object* $X \in \mathcal{B}$*, the functor* $\mathcal{B}(L,\, X) \xrightarrow{\rho_\lambda} pc\mathcal{B}(F,\, X)$ *is an equivalence of categories. This amounts to the following:*

bl) Given any pseudocone $F \xrightarrow{h} X$*, there exists an arrow* $L \xrightarrow{\ell} X$ *and an invertible morphism of pseudocones* $h \overset{\theta}{\Longrightarrow} \ell\lambda$*. Furthermore, given any other* $L \xrightarrow{t} X$ *and* $h \overset{\varphi}{\Longrightarrow} t\lambda$*, there exists a* unique *2-cell* $\ell \overset{\xi}{\Longrightarrow} t$ *such that* $\varphi = (\xi\lambda) \circ \theta$ *(if* φ *is invertible, then so it is* ξ*).*

1.5. **Definition.** *When the functor* $\mathcal{B}(L,\, X) \xrightarrow{\rho_\lambda} pc\mathcal{B}(F,\, X)$ *is an isomorphism of categories, the bicolimit is said to be a* pseudocolimit*.*

It is known that the 2-category $\mathcal{C}at$ of small categories has all small pseudocolimits, then a "fortiori" all small bicolimits (see for example [7]). Given a 2-functor $\mathcal{A} \xrightarrow{F} \mathcal{C}at$ we denote by $\underrightarrow{\mathcal{L}im}\, F$ the vertex of a bicolimit cone.

In [4] a special construction of the pseudocolimit of a 2-filtered diagram of categories (not necessarily small) is made, and using this construction it is proved a result (theorem 1.6 below) which is the key to our construction of small 2-filtered bilimits of topoi. Notice that even if the categories of the system are large, condition *bl)* in definition 1.4 makes sense and it defines the bicolimit of large categories.

We denote by $\mathcal{CAT}_{fl}$ the *illegitimate* (in the sense that its hom-sets are large) 2-category of finitely complete categories and exact (that is, finite limit preserving) functors.

1.6. **Theorem** ([4] Theorem 2.5)**.** $\mathcal{CAT}_{fl} \subset \mathcal{CAT}$ *is closed under 2-filtered pseudocolimits. Namely, given any 2-filtered diagram* $\mathcal{A} \xrightarrow{F} \mathcal{CAT}_{fl}$*, the pseudocolimit pseudocone* $FA \xrightarrow{\lambda_A} \underrightarrow{\mathcal{L}im}\, F$ *taken in* $\mathcal{CAT}$ *is a pseudocolimit cone in* $\mathcal{CAT}_{fl}$*. If the index 2-category* $\mathcal{A}$ *as well as all the categories* FA *are small, then* $\underrightarrow{\mathcal{L}im}\, F$ *is a small category.* □

Topoi

By a *site* we mean a category furnished with a (Grothendieck) topology, and a small set of objects capable of covering any object (called *topological generators* in [1]). *To simplify we will consider only sites with finite limits.* A *morphism* of sites with finite limits $\mathcal{D} \xrightarrow{f} \mathcal{C}$ is a *continous* (that is, cover preserving) and exact functor in the other direction $\mathcal{C} \xrightarrow{f^*} \mathcal{D}$. A 2-cell $\mathcal{D} \underset{g}{\overset{f}{\underset{\gamma\Downarrow}{\rightrightarrows}}} \mathcal{C}$ is a natural transformation $\mathcal{C} \underset{f^*}{\overset{g^*}{\underset{\gamma\Downarrow}{\rightrightarrows}}} \mathcal{D}$ [1]. Under the presence of topological generators it can be easily seen there is only a small set of natural transformations between any two continous functors. We denote by $\mathcal{S}it$ the resulting 2-category of sites with finite limits. We denote by $\mathcal{S}it^*$ the 2-category whose objects are the sites, but taking as arrows and 2-cells the functors f^* and natural transformations respectively. Thus $\mathcal{S}it$ is obtained by formally inverting the arrows and the 2-cells of $\mathcal{S}it^*$. We have by definition $\mathcal{S}it(\mathcal{D}, \mathcal{C}) = \mathcal{S}it^*(\mathcal{C}, \mathcal{D})^{op}$.

[1]Notice that 2-cells are also taken in the opposite direction. This is Grothendieck original convention, later changed by some authors.

A topos (also "Grothendieck topos") is a category equivalent to the category of sheaves on a site. Topoi are considered as sites furnishing them with the canonical topology. This determines a full subcategory $\mathcal{T}op^* \subset \mathcal{S}it^*$, $\mathcal{T}op^*(\mathcal{F},\, \mathcal{E}) \;=\; \mathcal{S}it^*(\mathcal{F},\, \mathcal{E})$.

A morphism of topoi (also "geometric morphism") $\mathcal{E} \xrightarrow{f} \mathcal{F}$ is a pair of adjoint functors $f^* \dashv f_*$ (called inverse and direct image respectively) $\mathcal{E} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathcal{F}$ together with an adjunction isomorphism $[f^*C, D] \xrightarrow{\cong} [C, f_*D]$. Furthermore, f^* is required to preserve finite limits. Let $\mathcal{T}op$ be the 2-category of topos with geometric morphisms. 2-arrows are pairs of natural transformations $(f^* \Rightarrow g^*,\ g_* \Rightarrow f_*)$ compatible with the adjunction (one of the natural transformations completely determines the other). The inverse image f^* of a morphism is an arrow in $\mathcal{T}op^* \subset \mathcal{S}it^*$. This determines a forgetful 2-functor (identity on the objects) $\mathcal{T}op \longrightarrow \mathcal{S}it$ which establish an equivalence of categories $\mathcal{T}op(\mathcal{E},\, \mathcal{F}) \;\cong\; \mathcal{S}it(\mathcal{E},\, \mathcal{F})$. Notice that $\mathcal{T}op(\mathcal{E}, \mathcal{F}) \cong \mathcal{T}op^*(\mathcal{F}, \mathcal{E})^{op}$, not an equality.

We recall a basic result in the theory of morphisms of Grothendieck topoi [1] expose IV, 4.9.4. (see for example [6] Chapter VII, section 7).

1.7. **Lemma.** *Let $\mathcal{C}$ be a site with finite limits, and $\mathcal{C} \xrightarrow{\epsilon^*} \widetilde{\mathcal{C}}$ the canonical morphism of sites to the topos of sheaves $\widetilde{\mathcal{C}}$. Then for any topos $\mathcal{F}$, composing with ϵ^* determines a functor $\mathcal{T}op^*(\widetilde{\mathcal{C}},\, \mathcal{F}) \xrightarrow{\cong} \mathcal{S}it^*(\mathcal{C},\, \mathcal{F})$ which is an equivalence of categories. Thus, $\mathcal{T}op(\mathcal{F},\, \widetilde{\mathcal{C}}) \xrightarrow{\cong} \mathcal{S}it(\mathcal{F},\, \mathcal{C})$.*

By the comparison lemma [1] Ex. III 4.1 we can state it in the following form, to be used in the proof of lemma 2.3.

1.8. **Lemma.** *Let $\mathcal{E}$ be any topos and $\mathcal{C}$ any small set of generators closed under finite limits (considered as a site with the canonical topology). Then, for any topos $\mathcal{F}$, the inclusion $\mathcal{C} \subset \mathcal{E}$ induce a restriction functor $\mathcal{T}op^*(\mathcal{E}, \mathcal{F}) \xrightarrow{\rho} \mathcal{S}it^*(\mathcal{C}, \mathcal{F})$ which is an equivalence of categories.*

2. 2-cofiltered bilimits of topoi

Our work with sites is auxiliary to prove our results for topoi, and for this all we need are sites with finite limits. The 2-category $\mathcal{S}it$ has all small 2-cofiltered pseudolimits, which are obtained by furnishing the 2-filtered pseudocolimit in $\mathcal{CAT}_{fl}$ (1.6) of the underlying categories with the coarsest topology making the cone injections site morphisms. Explicitly:

2.1. **Theorem.** *Let $\mathcal{A}$ be a small 2-filtered 2-category, and $\mathcal{A}^{op} \xrightarrow{F} \mathcal{S}it$ ($\mathcal{A} \xrightarrow{F} \mathcal{S}it^*$) a 2-functor. Then, the category $\underrightarrow{\mathcal{L}im}\, F$ is furnished with a topology such that the pseudocone functors $FA \xrightarrow{\lambda_A^*} \underrightarrow{\mathcal{L}im}\, F$ become continuous and induce an isomorphism of categories $\mathcal{S}it^*[\underrightarrow{\mathcal{L}im}\, F,\, \mathcal{X}] \xrightarrow{\rho_\lambda} \mathcal{PCS}it^*[F,\, \mathcal{X}]$. The corresponding site is then a pseudocolimit of F in the 2-category $\mathcal{S}it^*$. If each FA is a small category, then so it is $\underrightarrow{\mathcal{L}im}\, F$.*

Proof. Let $FA \xrightarrow{\lambda_A} \underrightarrow{\mathcal{L}im}\, F$ be the colimit pseudocone in $\mathcal{CAT}_{fl}$. We give $\underrightarrow{\mathcal{L}im}\, F$ the topology generated by the families $\lambda_A c_\alpha \longrightarrow \lambda_A c$, where $c_\alpha \longrightarrow c$ is a covering in some FA, $A \in \mathcal{A}$. With this topology, the functors λ_A become continuous, thus they correspond to site morphisms. This determines the upper horizontal arrow in the following diagram (where the vertical arrows are full subcategories and the lower horizontal arrow is an isomorphism):

$$\begin{array}{ccc} \mathcal{S}it[\underrightarrow{\mathcal{L}im}\, F,\, \mathcal{X}] & \longrightarrow & pc\mathcal{S}it[F,\, \mathcal{X}] \\ \downarrow & & \downarrow \\ \mathcal{C}at_{fl}[\underrightarrow{\mathcal{L}im}\, F,\, \mathcal{X}] & \xrightarrow{\cong} & pc\mathcal{C}at_{fl}[F,\, \mathcal{X}] \end{array}$$

To show that the upper horizontal arrow is an isomorphism we have to check that given a pseudocone $h \in pc\mathcal{S}it[F,\, \mathcal{X}]$, the unique functor $f \in \mathcal{C}at_{fl}[\underrightarrow{\mathcal{L}im}\, F,\, \mathcal{X}]$, corresponding to h under the lower arrow, is continuous. But this is clear since from the equation $f\lambda = h$ it follows that it preserves the generating covers, and thus all covers as well. Finally, by the construction of $\underrightarrow{\mathcal{L}im}\, F$ in [4] we know that every object in $\underrightarrow{\mathcal{L}im}\, F$ is of the form $\lambda_A c$ for some $A \in \mathcal{A}$, $c \in FA$. It follows then that the collection of objects of the form $\lambda_A c$, with c varying on the set of topological generators of each FA, is a set of topological generators for $\underrightarrow{\mathcal{L}im}\, F$. □

In the next proposition we show that any 2-diagram of topoi restricts to a 2-diagram of small sites with finite limits by means of a 2-natural (thus a fortiori pseudonatural) transformation.

2.2. **Proposition.** *Given a 2-functor $\mathcal{A}^{op} \xrightarrow{\mathcal{E}} \mathcal{T}op$ there exists a 2-functor $\mathcal{A}^{op} \xrightarrow{\mathcal{C}} \mathcal{S}it$ such that:*

i) For any $A \in \mathcal{A}$, $\mathcal{C}_A$ is a small *full generating subcategory of $\mathcal{E}_A$ closed under finite limits, considered as a site with the canonical topology.*

ii) The arrows and the 2-cells in the $\mathcal{C}$ diagram are the restrictions of those in the $\mathcal{E}$ diagram: For any 2 cell $A \underset{v}{\overset{u}{\underset{\gamma\Downarrow}{\rightrightarrows}}} B$ in $\mathcal{A}$, the following diagram commutes (where we omit notation for the action of the 2 functors on arrows and 2-cells):

$$\begin{array}{ccc} \mathcal{E}_A & \underset{v^*}{\overset{u^*}{\underset{\gamma\Downarrow}{\rightrightarrows}}} & \mathcal{E}_B \\ {\scriptstyle i_A}\uparrow & & \uparrow{\scriptstyle i_B} \\ \mathcal{C}_A & \underset{v^*}{\overset{u^*}{\underset{\gamma\Downarrow}{\rightrightarrows}}} & \mathcal{C}_B \end{array}$$

Proof. It is well known that any small set $\mathcal{C}$ of generators in a topos can be enlarged so as to determine a (non canonical) small full subcategory $\overline{\mathcal{C}} \supset \mathcal{C}$ closed under finite limits: Choose a limit cone for each finite diagram, and repeat this in a denumarable process. On the other hand, for the validity of condition ii) it is enough that for each transition functor $\mathcal{E}_A \xrightarrow{u^*} \mathcal{E}_B$

and object $c \in \mathcal{C}_A$, we have $u^*(c) \in \mathcal{C}_B$ (with this, natural transformations restrict automatically).

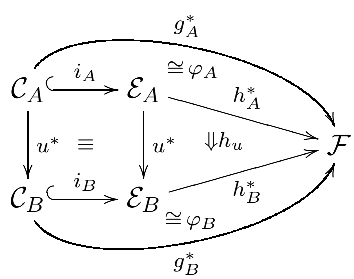
Let's start with any set of generators $\mathcal{R}_A \subset \mathcal{E}_A$ for all $A \in \mathcal{A}$. We will naively add objects to these sets to remedy the failure of each condition alternatively. In this way we achieve simultaneously the two conditions:

Define $\mathcal{C}^0_A = \overline{\mathcal{R}}_A \supset \mathcal{R}_A$. Define $\mathcal{R}^{n+1}_A = \bigcup\limits_{X \xrightarrow{u} A} u^*(\mathcal{C}^n_X)$. $\mathcal{R}^{n+1}_A$ is small because $\mathcal{A}$ is small. $\mathcal{C}^n_X \subset \mathcal{R}^{n+1}_A$ due to id_A. Suppose now $c \in \mathcal{R}^{n+1}_A$, $c = u^*(d)$ with $d \in \mathcal{C}^n_X$, and let $A \xrightarrow{v} B$ in $\mathcal{A}$. We have $v^*(c) = v^*u^*(d) = (vu)^*(d)$, thus $v^*(c) \in \mathcal{R}^{n+1}_B$. Define $\mathcal{C}^{n+1}_A = \overline{\mathcal{R}^{n+1}_A} \supset \mathcal{R}^{n+1}_A$. Then, it is straightforward to check that $\mathcal{C}_A = \bigcup\limits_{n \in \mathbb{N}} \mathcal{C}^n_A$ satisfy the two conditions. □

A generalization of lemma 1.8 to pseudocones holds.

2.3. **Lemma.** *Given any 2-diagram of topoi $\mathcal{A}^{op} \xrightarrow{\mathcal{E}} \mathcal{T}op$, a restriction $\mathcal{A}^{op} \xrightarrow{\mathcal{C}} \mathcal{S}it$ as before, and any topos $\mathcal{F}$, the inclusions $\mathcal{C}_A \subset \mathcal{E}_A$ induce a restriction functor $pc\mathcal{T}op^*(\mathcal{E}, \mathcal{F}) \xrightarrow{\rho} pc\mathcal{S}it^*(\mathcal{C}, \mathcal{F})$ which is an equivalence of categories.*

Proof. The restriction functor ρ is just a particular case of 1.1, so it is well defined. We will check that it is essentially surjective and fully-faithful. The following diagram illustrates the situation:



essentially surjective: Let $g \in pc\mathcal{S}it^*(\mathcal{C}, \mathcal{F})$. For each $A \in \mathcal{A}$, take by lemma 1.8 $\mathcal{E}_A \xrightarrow{h^*_A} \mathcal{F}$, φ_A, $h^*_A i_A \overset{\varphi_A}{\cong} g^*_A$. By lemma 1.3, h^*i inherits a pseudocone structure such that φ becomes a pseudocone isomorphism. For each arrow $A \xrightarrow{u} B$ we have $(h^*i)_A \overset{(h^*i)_u}{\Rightarrow} (h^*i)_B u^*$. Since ρ_A is fully-faithful, there exists a unique $h^*_A \overset{h_u}{\Rightarrow} h^*_B u^*$ extending $(h^*i)_u$. In this way we obtain data $h^* = (h^*_A, h_u)$ that restricts to a pseudocone. Again from the fully-faithfulness of each ρ_A it is straightforward to check that it satisfies the pseudocone equations 1.2.

fully-faithful: Let $h^*, l^* \in pc\mathcal{T}op^*(\mathcal{E}, \mathcal{F})$ be two pseudocones, and let $\widetilde{\eta}$ be a morphism between the pseudocones h^*i and l^*i. We have natural transformations $h^*_A i_A \xRightarrow{\widetilde{\eta_A}} l^*_A i_A$. Since the inclusions i_A are dense, we can extend $\widetilde{\eta_A}$ uniquely to $h^*_A \xRightarrow{\eta_A} l^*_A$ such that $\widetilde{\eta} = \eta\, i$. As before, from the fully-faithfulness of each ρ_A it is straightforward to check that $\eta = (\eta_A)$ satisfies the morphism of pseudocone equation 1.2. □

2.4. **Theorem.** *Let $\mathcal{A}^{op}$ be a small 2-filtered 2-category, and $\mathcal{A}^{op} \xrightarrow{\mathcal{E}} \mathcal{T}op$ be a 2-functor. Let $\mathcal{A}^{op} \xrightarrow{\mathcal{C}} \mathcal{S}it$ be a restriction to small sites as in 2.2. Then, the topos of sheaves $\widetilde{\underrightarrow{\mathcal{L}im}\, \mathcal{C}}$ on the site $\underrightarrow{\mathcal{L}im}\, \mathcal{C}$ of 2.1 is a bilimit of $\mathcal{E}$ in $\mathcal{T}op$, or, equivalently, a bicolimit in $\mathcal{T}op^*$.*

Proof. Let λ^* be the pseudocolimit pseudocone $\mathcal{C}_A \xrightarrow{\lambda^*_A} \underrightarrow{\mathcal{L}im}\, \mathcal{C}$ in the 2-category $\mathcal{S}it^*$ (2.1). Consider the composite pseudocone $\mathcal{C}_A \xrightarrow{\lambda^*_A} \underrightarrow{\mathcal{L}im}\, \mathcal{C} \xrightarrow{\varepsilon} \widetilde{\underrightarrow{\mathcal{L}im}\, \mathcal{C}}$ and let l^* be a pseudocone from $\mathcal{E}$ to $\widetilde{\underrightarrow{\mathcal{L}im}\, \mathcal{C}}$ such that $l^* i \simeq \epsilon^* \lambda^*$ given by lemma 2.3. We have the following diagrams commuting up to an isomorphism:

$$\begin{array}{ccccc} \mathcal{F} \longleftarrow & \widetilde{\underrightarrow{\mathcal{L}im}\, \mathcal{C}} & \xleftarrow{\varepsilon^*} & \underrightarrow{\mathcal{L}im}\, \mathcal{C} \\ & \uparrow {\scriptstyle l^*} & \cong & \uparrow {\scriptstyle \lambda^*} \\ & \mathcal{E} & \xleftarrow{\;\; i \;\;} & \mathcal{C} \end{array} \qquad \begin{array}{ccc} \mathcal{T}op^*(\widetilde{\underrightarrow{\mathcal{L}im}\, \mathcal{C}},\, \mathcal{F}) & \xrightarrow{\rho_\varepsilon} & \mathcal{S}it^*(\underrightarrow{\mathcal{L}im}\, \mathcal{C},\, \mathcal{F}) \\ \downarrow {\scriptstyle \rho_l} & \cong & \downarrow {\scriptstyle \rho_\lambda} \\ pc\mathcal{T}op^*(\mathcal{E},\, \mathcal{F}) & \xrightarrow{\;\;\rho\;\;} & pc\mathcal{S}it^*(\mathcal{C},\, \mathcal{F}) \end{array}$$

In the diagram on the right the arrows ρ_ε, ρ_λ and ρ are equivalences of categories (1.7, 2.1 and 2.3 respectively), so it follows that ρ_l is an equivalence. This finishes the proof. □

This theorem shows the existence of small 2-cofiltered bilimits in the 2-category of topoi and geometric morphisms. But, it shows more, namely, that given any small 2-filtered diagram of topoi, without loss of generality, we can construct a small site with finite limits for the bilimit topos out of a 2-cofiltered sub-diagram of small sites with finite limits. However, this depends on the *axiom of choice* (needed for Proposition 2.2). We notice for the interested reader that if we allow large sites (as in Theorem 2.1), we can take the topoi themselves as sites, and the proof of theorem 2.4 with $\mathcal{C} = \mathcal{E}$ does not use Proposition 2.2. Thus, without the use of choice we have:

2.5. **Theorem.** *Let $\mathcal{A}^{op}$ be a small 2-filtered 2-category, and $\mathcal{A}^{op} \xrightarrow{\mathcal{E}} \mathcal{T}op$ be a 2-functor. Then, the topos of sheaves $\widetilde{\underrightarrow{\mathcal{L}im}\, \mathcal{E}}$ on the site $\underrightarrow{\mathcal{L}im}\, \mathcal{E}$ of 2.1 is a bilimit of $\mathcal{E}$ in $\mathcal{T}op$, or, equivalently, a bicolimit in $\mathcal{T}op^*$.*

References

[1] Artin M, Grothendieck A, Verdier J., *SGA 4 , (1963-64)*, Lecture Notes in Mathematics 269 Springer, (1972).

[2] Artin M, Grothendieck A, Verdier J., *SGA 4 , (1963-64)*, Springer Lecture Notes in Mathematics 270 (1972).

[3] Dubuc, E. J., *2-Filteredness and the point of every Galois topos*, Proceedings of CT2007, Applied Categorical Structures, Volume 18, Issue 2, Springer Verlag (2010).

[4] Dubuc, E. J., Street, R., *A construction of 2-filtered bicolimits of categories*, Cahiers de Topologie et Geometrie Differentielle, (2005).

[5] Gray J. W., *Formal Category Theory: Adjointness for 2-Categories*, Springer Lecture Notes in Mathematics 391 (1974).

[6] Mac Lane S., Moerdijk I., *Sheaves in Geometry and Logic*, Springer Verlag, (1992).

[7] Street R.,*Limits indexed by category-valued 2-functors* J. Pure Appl. Alg. 8 (1976).