Under the Continuum Hypothesis all nonreflexive Banach space ultrapowers are primary.

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Abstract

In this note a large class of primary Banach spaces is characterized. Namely, it will be demonstrated that - under the Continuum Hypothesis - if E is any infinite dimensional nonsuperreflexive Banach space, then its ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ is always primary. Consequently, any infinite dimensional nonsuperreflexive Banach space can be isometrically embedded into its primary ultrapowers.

Keywords and phrases. Banach space ultrapowers, primary Banach spaces, Stone-Cech remainder, Representation Theorem, superreflexivity.

2010 *Mathematical Subject Classification*. 46B08, 46B20, 46B25.

1. Introduction.

Recall that a Banach space E is called *primary* if, for every bounded projection operator P on E, either PE or $(I-P)E$ is isomorphic to E (where I is the identity operator). It was proved that many classical function spaces $C(K)$, i.e., the space of continuous scalar-valued functions on any infinite metrizable compact space K and $L_p(0,1)$ spaces (for $1 \leq p \leq \infty$) are all primary. Also it was demonstrated that Pełczyński's universal basis space U_1 and the space J of James are primary ([5, 7, 8, 12]).

In this note the methodology of Banach space ultrapowers will be employed in order to single out a large class of Banach spaces which are primary. Namely, it will be shown that if the C*ontinuum Hypothesis* (CH) holds and E is any infinite dimensional nonsuperreflexive Banach space, then the ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ is always primary. Therefore, any infinite dimensional nonsuperreflexive Banach space E can be isometrically embedded into the primary Banach space of the form $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$.

2. Banach space ultrapowers and the Representation Theorem

Recall that a Banach space E is said to be finite dimensional if and only if its unit ball is compact, i.e., if and only if for every bounded family $(x_i)_{i\in I}$ and for every ultrafilter $\mathcal{U}$ on the set I, the so-called $\mathcal{U}-$limit $\lim_{i,\mathcal{U}} x_i$ exists. The sequence $(x_i)_{i\in I}$ converges to x with respect to the ultrafilter $\mathcal{U}$ if and only if the set $\{i \in I : x_i \in V\}$ is contained in $\mathcal{U}$ for any open neighborhood V of x. This $\mathcal{U}-$limit is simply denoted by $\lim\limits_{i,\mathcal{U}} x_i$. If a Banach space E is infinite dimensional and $\mathcal{U}$ is an ultrafilter on the set I, then it is possible to enlarge E to a Banach space $\widehat{E}$ by adjoining for every bounded family $(x_i)_{i\in I}$ in E an element $\widehat{x} \in \widehat{E}$ such that the following equality holds: $\|\widehat{x}\| = \lim\limits_{i,\mathcal{U}} \|x_i\|$. This construction is called *Banach space ultrapower*. From the model-theoretical point of view Banach space ultrapowers can be considered as the result of erazing the elements with infinite norm from an ordinary (i.e., algebraic) ultrapower and dividing it by infinitesimals. Suppose that $(E_i)_{i\in I}$ is a collection of Banach spaces. Then define

$$\ell_\infty (E_i) = \{(x_i) : x_i \in E_i \text{ and } \|(x_i)\|_\infty < \infty\},$$

i.e., the Banach space of all bounded families $(x_i) \in \prod\limits_{i\in I} E_i$ endowed with the norm given by $\|(x_i)\|_\infty = \sup\limits_{i\in I} \|x_i\|_{E_i}$. If $\mathcal{U}$ is an ultrafilter on the index set I, then $\lim\limits_{i,\mathcal{U}} \|x_i\|_{E_i}$ always exists. Thus it is possible to define a seminorm on $\ell_\infty(E_i)$ by assuming that $\mathcal{N}((x_i)) = \lim\limits_{i,\mathcal{U}} \|x_i\|_{E_i}$. Then the kernel of $\mathcal{N}$ is given by

$$\mathcal{N}_\mathcal{U} = \left\{x = (x_i) \in \ell_\infty (E_i) : \lim_{i,\mathcal{U}} \|x_i\| = 0\right\}$$

and it is straightforward to observe that $\mathcal{N}_\mathcal{U}$ constitutes a closed ideal in the Banach space $\ell_\infty (E_i)$. Next, define the quotient space of the form

$$\ell_\infty (E_i) / \mathcal{N}_\mathcal{U}.$$

This space is said to be the *ultraproduct* of the family of Banach spaces $(E_i)_{i\in I}$. If $E_i = E$ for every $i \in I$, then the space $\ell_\infty(E_i)/\mathcal{U}$ is termed the *ultrapower* of E and is denoted by $E^I/\mathcal{U}$ or by $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$. If (x_i) is the sequence in the Banach space $\prod\limits_{i\in I} (E_i)$, then $(x_i)_\mathcal{U}$ denotes the *equivalence class* of (x_i) in $\ell_\infty(E_i)/\mathcal{U}$. On the other hand, if $E^I/\mathcal{U}$ is an ultrapower of the Banach space E, then the mapping $x \to (x_i)_\mathcal{U}$, where $x_i = x$ for every $i \in I$, constitutes a classical embedding of E into $E^I/\mathcal{U}$. It can be asserted that if $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ is a Banach space ultrapower, then $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ contains a subspace isometrically isomorphic to E ([1, 2, 3, 4, 6, 9, 10]).

A Banach space E is said to be superreflexive if and only if each ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ is reflexive ([1, 9]). There exist many alternative characterizations of superreflexivity (Theorem 2.1 and 2.2 in [17], cf. [11]). In our paper the condition of superreflexivity of the Banach space E will be characterized by the containment of the copy of ℓ_∞ in $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$.

In ([16]) the *Representation Theorem* for nonreflexive Banach space ultrapowers was obtained. In order to render to this result recall that the *Stone-Cech* *compactification* of the discrete space $\omega = \{1, 2, ...\}$, denoted by $\beta\omega$, can be identified with the space of all ultrafilters on ω equipped with the topology generated by sets of the form $\{F : U \in F\}$ for $U \subseteq \omega$. Then the points of $\beta\omega$ may be viewed as the ultrafilters on ω with the points of ω identified with the principal ultrafilters. Denote by ω^* (or by $\beta\omega\backslash\omega$) the so-called *Stone-Cech remainder* of $\beta\omega$. In this case the points of ω^* can be identified with free ultrafilters on ω ([5, 15]). From *Set-Theoretical Topology* it is known that the so-called Parovicenco space X can be regarded as a topological space satisfying the following requirements: 1) X is compact and Hausdorff, 2) X has no isolated points, 3) X has weight $\mathfrak{c}$, 4) every nonempty G_δ subset of X has nonempty interior, 5) every two disjoint open F_σ subsets of X have disjoint closures. Parovičenko proved that assuming CH evey Parovicenko space X is isomorphic to ω^* ([15]). The above mentioned Representation Theorem ascertains that - under CH - all nonreflexive Banach space ultrapowers can be represented in the form of the space of continuous, bounded and real-valued functionsons the Parovicenco space ω^*. Suppose that the symbol $\cong$ denotes the relation of isometric isomorphism between Banach spaces. Then it is possible to formulate the following theorem (Corollary 3 in [16]):

Theorem 1 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space and let $\ell_\infty(E)/N_{\mathcal{U}}$ be its ultrapower. Then the Banach space $\ell_\infty(E)/N_{\mathcal{U}}$ is isometrically isomorphic to the space of continuous, bounded and real-valued functions of the Stone-Cech remainder ω^*, i.e.,*

$$\ell_\infty(E)/\mathcal{N}_{\mathcal{U}} \cong C(\omega^*).$$

In order to outline the proof of this theorem suppose that the set ω has the discrete topology and E is any infinite dimensional nonsuperreflexive Banach space. Then $\ell_\infty(E) = C(\omega)$. Defining the restriction mapping $R : C(\beta\omega) \to C(\omega)$ by $R(\widehat{f}) = \widehat{f} \upharpoonright \omega$ for each $\widehat{f} \in C(\beta\omega)$ it can be concluded that R is a linear isometry of $C(\beta\omega)$ onto $C(\omega)$. Consequently, it can be stated that if we assume CH, then the $\ell_\infty-$sum of countably many copies of any infinite dimensional nonsuperreflexive Banach space E is isometrically isomorphic to the space $C(\beta\omega)$, i.e., $\ell_\infty(E) \cong C(\beta\omega)$ (Proposition 1 in [16]). Next, suppose that every function $f \in C(\omega)$ has a unique norm-preserving extension $\widehat{f} \in C(\beta\omega)$. Therefore, it is possible to define the closed ideal I in the space $C(\beta\omega)$ consisting of functions which vanish on ω^*, i.e., $I = \left\{\widehat{f} \in C(\beta\omega) : \widehat{f}(t) = 0 \text{ for all } t \in \omega^*\right\}$. Further, assume that the space $c_0(\omega)$ consists of functions in $C(\omega)$ which vanish at infinity, i.e., $c_0(\omega) = \{f \in C(\omega) : \text{ for each } \varepsilon > 0,\ \{t \in \omega : |f(t)| > \varepsilon\} \text{ is finite}\}$. Observe that for $\lim_{i,\mathcal{U}} x_i = x$ the set $\{i \in I : x_i \notin V\}$ is finite. Then it becomes obvious that $c_0(\omega) = \mathcal{N}_{\mathcal{U}}$ and the restriction mapping $R : I \to \mathcal{N}_{\mathcal{U}}$ defines a linear isometry from I onto $\mathcal{N}_{\mathcal{U}}$. Consequently, $\mathcal{N}_{\mathcal{U}} \cong I$ (Proposition 2 in [16]). Summing up these two facts it follows that $\ell_\infty(E)/\mathcal{N}_{\mathcal{U}} \cong C(\omega^*)$.

In this place it shuold be noted that in our Representation Theorem it is supposed that all considered infinite dimensional Banach spaces are nonsuperreflexive and - consequently - their ultrapowers are nonreflexive. It is unknown if the condition of nonsuperreflexivity can be weakened (or modified) in order to represent Banach space ultrapowers in the form $C(\omega^*)$.

In this paper a large class of primary Banach spaces will be singled out. Also it will be shown that - assuming CH - every infinite dimensional nonsuperreflexive Banach space can be isometrically embedded into its primary ultrapower.

3. All nonreflexive Banach space ultrapowers are primary

Recall that if X is any Banach space, then a sequence $\{X_n\}_{n=1}^{\infty}$ of its closed subspaces is said to be a *Schauder decomposition* of X if every element $x \in X$ can be uniquely represented in the form $x = \sum_{n=1}^{\infty}$, where $x_n \in X_n$ for every n. Drewnowski and Roberts showed that if CH holds, then there exists a Schauder decomposition of the space $C(\omega^*)$ (Corollary 2.5 in [8], cf. [7]). Using their result and the fact that the spaces $C(\omega^*)$ and $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ (where E is any infinite dimensional nonsuperreflexive Banach space) are congruent the following theorem can be formulated:

Theorem 2. *Let E be any infinite dimensional nonsuperreflexive Banach space and let $\ell_\infty(E)/N_\mathcal{U}$ be its ultrapower. Then for every (finite or infinite) Schauder decomposition*

$$\ell_\infty(E)/\mathcal{N}_\mathcal{U} = X_1 + X_2 + ...$$

of Banach space ultrapower $\ell_\infty(E)/N_\mathcal{U}$ at least one of the summands X_n has a subspace which is isomorphic to $\ell_\infty(E)/N_\mathcal{U}$ and is complemented in $\ell_\infty(E)/N_\mathcal{U}$.

Proof. cf. [8]. □

In order to completely grasp the primariness of Banach space ultrapowers let us recall some fact from *Set-Theoretical Topology.* Namely, Negrepontis proved under CH the following result (Corollary 3.2 in [14]):

Proposition 3 (CH). *If A is an open F_σ subset of the Stone-Cech remainder ω^*, then $\overline{A}$ constitutes a retract of ω^*.*

Now, suppose that the space $\ell_\infty\left(\ell_\infty\left(E\right)/\mathcal{N}_\mathcal{U}\right)$ denotes the $\ell_\infty-$sum of countable many copies of Banach space ultrapower $\ell_\infty\left(E\right)/\mathcal{N}_\mathcal{U}$, i.e.,

$$\ell_\infty\left(\ell_\infty\left(E\right)/\mathcal{N}_\mathcal{U}\right): = \left(\ell_\infty\left(E\right)/\mathcal{N}_\mathcal{U} \oplus \ell_\infty\left(E\right)/\mathcal{N}_\mathcal{U} \oplus ...\right).$$

In our next theorems we are going to show that if E is any infinite dimensional nonsuperreflexive Banach space, then the space $\ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U})$ is isomorphic to a complemented subspace of Banach space ultrapower $\ell_\infty\left(E\right)/\mathcal{N}_\mathcal{U}$. Consequently, it will be straightforward to conclude that for any infinite dimensional nonsuperreflexive Banach space E the spaces $\ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U})$ and $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$

are isomorphic, i.e., $\ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \approx \ell_\infty(E)/\mathcal{N}_\mathcal{U}$. Also it should be noted that if E is any infinite dimensional nonsuperreflexive Banach space, then every infinite-dimensional complemented subspace of $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ contains an isomorph of the space $\ell_\infty(E)$ (cf. [7, 8]).

Theorem 4 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space. Then the Banach space $\ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U})$ is isometric to a complemented subspace of the Banach space ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$.*

Proof. Suppose that $\{A_n\}_{n=1}^\infty$ is any infinite sequence of disjoint and nonempty clopen subsets of the Parovicenco space ω^*. Suppose that A denotes the union of this sequence. Then the Banach space $\ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U})$ is isomorphically isometric to the Banach space $C_b(A)$ of bounded continuous functions on A. This fact follows easily from the congruence between the spaces $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ and $C(\omega^*)$. Also it can be observed that $\overline{A} = \beta A$ and - consequently - $C_b\left(\overline{A}\right) \cong C(\overline{A})$. As the result it is obtained that $\ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \cong C(\overline{A})$. From Proposition 3 it is deducible that there exists a retraction r of ω^* onto $\overline{A}$. If we define the corresponding composition operator $R : C(\overline{A}) \to \ell_\infty(E)/\mathcal{N}_\mathcal{U}$, then the mapping $x \to x \circ r$ can be identified with an isometry and the operator P given by $P(x) = R(x_{|A})$ is a projection from $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ onto $R[C(\overline{A})]$. $\square$

The proof of Theorem 5 relies mainly on Pełczyński's decomposition method (cf. [2, 5, 8, 12]).

Theorem 5. *Let E be any infinite dimensional nonsuperreflexive Banach space. Then the Banach space $\ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U})$ is isomorphic to the Banach space ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$, i.e.,*

$$\ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \approx \ell_\infty(E)/\mathcal{N}_\mathcal{U}.$$

Proof. From Theorem 4 and Pełczyński's decomposition technique it follows that if $\ell_\infty(E)/\mathcal{N}_\mathcal{U} \approx \ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \oplus Z$ for some Banach space Z, then

$$\begin{aligned}\ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) &\approx \ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \oplus \ell_\infty(E)/\mathcal{N}_\mathcal{U} \\ &\approx \ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \oplus \ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \oplus Z \\ &\approx \ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \oplus Z \\ &\approx \ell_\infty(E)/\mathcal{N}_\mathcal{U}. \ \square\end{aligned}$$

From this result it is possible to deduce the additional characterization of nonsuperreflexivity ([13]). Namely, the following proposition can be formulated:

Proposition 6. *If the Banach space ℓ_∞ of all bounded sequences is contained in the ultrapower $\ell_\infty(E)/N_\mathcal{U}$ of an infinite dimensional Banach space E, then the Banach space E is not superreflexive.*

Now we are ready to prove our main Theorem asserting that under CH all nonreflexive Banach space ultrapowers are primary (cf. [7, 8]).

Theorem 7 (CH). *Let E be any infinite dimensional nonsuperreflexive Banach space and let $\ell_\infty(E)/\mathcal{N}_\mathcal{U} = X_1 \oplus X_2 \oplus ...$ be a (finite or infinite)*

Schauder decomposition of its ultrapower. Then there exists an index m such that $X_m \approx \ell_\infty(E)/\mathcal{N}_\mathcal{U}$. Particularly, the Banach space ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ is primary.

Proof. From Theorem 1 it follows that it is possible to indicate an index m such that the space X_m contains a subspace V which is isomorphic to $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ and is complemented in $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$. For instance, suppose that $m = 1$ and denote $X = X_1$, $Y = X_2 \oplus X_3 \oplus ...$. Hence we obtain:

$$\ell_\infty(E)/\mathcal{N}_\mathcal{U} = X \oplus Y,\ X = U \oplus V \text{ and } V \approx \ell_\infty(E)/\mathcal{N}_\mathcal{U}$$

for some subspace U of X. From Theorem 5 we have that $\ell_\infty(E)/\mathcal{N}_\mathcal{U} \approx \ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U})$. Applying Pelczynski's decomposition method we arrive at the following formula:

$$\begin{aligned} X &\approx U \oplus \ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \approx U \oplus \ell_\infty(E)/\mathcal{N}_\mathcal{U} \oplus \ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \\ &\approx X \oplus \ell_\infty(X \oplus Y) \approx X \oplus \ell_\infty(X) \oplus \ell_\infty(Y) \approx \ell_\infty(X) \oplus \ell_\infty(Y) \\ &\approx \ell_\infty(X \oplus Y) \approx \ell_\infty(\ell_\infty(E)/\mathcal{N}_\mathcal{U}) \approx \ell_\infty(E)/\mathcal{N}_\mathcal{U}. \ \square \end{aligned}$$

4. Concluding remarks

In this paper it was demonstrated that if the *Continuum Hypothesis* holds, then all nonreflexive Banach space ultrapowers are primary. As the consequence of this fact it can be observed that every infinite dimensional nonsuperreflexive Banach space E is isometrically embeddable into its primary ultrapower $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$. It was also indicated that the presence of the copy of ℓ_∞ in $\ell_\infty(E)/\mathcal{N}_\mathcal{U}$ (where E is infinite dimensional Banach space) implies that E is not superreflexive.

References

[1] Aksoy, A. G. and Khamsi, M. A., Nonstandard methods in fixed point theory, Springer - Verlag, New York, 1990.

[2] Albiac, F. and Kalton, N. J., Topics in Banach space theory, Springer - Verlag, New York 2006.

[3] Ben Yaacov, I., Berenstein, A., Henson, C. W., Usvyatsov, A.: Model theory for metric structures, [in:] Chatzidakis, Z., Pillay, A., and Wilkie, A., Model theory with applications to algebra and analysis. Vol. II, Cambridge University Press, Cambridge 2008, $315 - 427$.

[4] Bretagnolle, J., Dacunha-Castelle, D., Krivine, J.-L.: Lois stables et espaces L^p, *Ann. Inst. H. Poincaré Sect. B* **2** (1965/1966), $231 - 259$.

[5] Carothers, N. L., A short course on Banach space theory, Cambridge University Press, Cambridge 2005.

[6] Dacunha-Castelle, D., Krivine, J.-L.*:* Applications des ultraproduits à l'étude des espaces et des algèbres de Banach, *Studia Math.* **41** (1972), 315 – 334.

[7] Drewnowski, L.: Primariness of some spaces of continuous functions, *Rev. Mat. Complut.* **2** (1989), 119 – 127.

[8] Drewnowski, L. and J. W. Roberts, J. W.: On the primariness of the Banach space ℓ_∞/c_0, *Proc. Amer. Math. Soc.* **112** (1991), 949 – 957.

[9] Heinrich, S.: Ultraproducts in Banach space theory, *J. Reine Angewandte Math.* **313** (1980), 72 – 104.

[10] Henson, C. W., Iovino, J.: Ultraproducts in analysis, [in:] C. W. Henson, C. W., Iovino, J., Kechris, A. S., and Odell, E. W., Analysis and logic, Cambridge University Press, Cambridge 2003, 1 – 113.

[11] James, R. C.: Superreflexive Banach spaces, *Canad. J. Math.* **24** (1972), 896 – 904.

[12] J. Lindenstrauss, J. and Tzafriri, L., Classical Banach spaces *I*, Springer - Verlag, Berlin/Heidelberg/New york 1996.

[13] Martínez-Abejón, A.: *Private information communicated by e-mail* 26 April 2011.

[14] Negrepontis, S.: The Stone space of the saturated Boolean algebras, *Trans. Amer. Math. Soc.* **13** (1969), 515 – 527.

[15] Walker, R. C., The Stone-Cech compactification, Springer - Verlag, New York, 1974.

[16] Wilczek, P.: Some representation theorem for nonreflexive Banach space ultrapowers under the Continuum Hypothesis, *preprint.*

[17] Wiśnicki, A.: Towards the fixed point property for superreflexive spaces, *Bull. Austral. Math. Soc.* **64** (2001), 435 – 444.

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