Invariant Measures with Bounded Variation Densities for Piecewise Area Preserving Maps

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Abstract

We investigate the properties of absolutely continuous invariant probability measures (ACIPs) for piecewise area preserving maps (PAPs) on $\mathbb{R}^d$. This class of maps unifies piecewise isometries (PWIs) and piecewise hyperbolic maps where Lebesgue measure is locally preserved. In particular for PWIs, we use a functional approach to explore the relationship between topological transitivity and uniqueness of ACIPs, especially those measures with bounded variation densities. Our results "partially" answer one of the fundamental questions posed in [12] - determine all invariant non-atomic probability Borel measures in piecewise rotations. When reducing to interval exchange transformations (IETs), we demonstrate that for non-uniquely ergodic IETs with two or more ACIPs, these ACIPs have very irregular densities (namely of unbounded variation and discontinuous everywhere) and intermingle with each other.

Contents

1 Introduction

Conservative systems are often used as models of the physical world, where conservative is usually understood as energy preserving (i.e., where energy is invariant under the time evolution). In this article we consider conservative systems that are governed by discrete time dynamical systems. In particular, we focus on multidimensional piecewise area preserving maps (PAPs), which are a general extension of interval exchange transformations (IETs) into $\mathbb{R}^d$. Regarding IETs, Keane conjectures that minimality implies unique ergodicity in [15] and this conjecture holds for IETs with two or three intervals. However, counterexamples have been constructed; see [16, 18]. Thereafter, Masur [20] and Veech [24] independently demonstrated that almost every minimal (transitive) IET (with respect to Lebesgue measure) is uniquely ergodic; simultaneously, Keane & Rauzy [17] revealed that unique ergodicity holds for a (Baire) residual subset of the space of IETs. To fully understand the densities of invariant measures for these non-uniquely ergodic counterexamples, it is natural to explore equivalent conditions to the minimality (transitivity) in IETs in terms of absolutely continuous invariant probability measures (ACIPs).

When extending the above question to multidimensional PAPs, there are at least two technical obstacles: complicated topology in high dimensions and distance may not preserved locally. For the class of PAPs preserving distance locally, a special case of interest is the class of piecewise isometries (PWIs). Establishing their ACIPs contributes to determine all their invariant non-atomic probability Borel measures. This is one of the fundamental questions posed in [12] for two-dimensional piecewise rotations that is still open.

For the class of PAPs that do not preserve distance locally, a particular case is piecewise hyperbolic maps. For these maps, transitivity along with the uniqueness of physical measure has been widely studied (see works of Boyarsky & Góra, Viana [3, 25]). These studies use a functional analytic approach by choosing a "reasonable" function space and applying a *transfer operator* on this space. They study statistical properties of the system by looking at the operator fixed point and determining if there is a spectral gap. In one-dimensional piecewise expanding maps, the space of bounded variation functions is demonstrated to be such a reasonable space [3, 25]. In higher dimensions, the space of multidimensional bounded variation functions can still be chosen under certain assumptions [5, 23] and contains a classical anisotropic Sobolev space of Triebel-Lizorkin type which is shown by Baladi & Gouëzel [2] to be such a "reasonable" space.

In this article, our interest is to explore the structure of ACIPs and the relationship between the uniqueness of such measures and topological properties (e.g. existence of dense orbits, topological transitivity and minimality) for multidimensional PAPs (particularly for PWIs) by applying the functional approach. Definitions of PAPs as well as PWIs are given below.

Let X be a compact subset of $\mathbb{R}^d$ and $(X, \mathfrak{B}, m)$ be a probability space. For convenience, m always denotes $d-$dimension normalized Lebesgue measure on X, and $\mathfrak{B}$ is the Borel σ-field. We say $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$ is a *topological partition* of X if: (**i**) $\omega_i \cap \omega_j = \emptyset$, for $i \neq j$; (**ii**) $\bigcup_{i=0}^{r-1} \omega_i = X$; and (**iii**) for each ω_i, $\mathrm{int}(\omega_i) \neq \emptyset$ and $m(\partial \omega_i) = 0$. Here each ω_i is called an atom; $\mathrm{int}\, A$ and ∂A are the interior and boundary of A respectively.

Definition 1 *A nonsingular map $f : (X, \mathfrak{B}, m) \to (X, \mathfrak{B}, m)$ with a topological partition $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$ is called a* piecewise area preserving map *(PAP) if $f|_{\mathrm{int}(\omega_i)} \in C^1$ for each ω_i and $|\det Df(x)| \equiv 1$ for $x \in \bigcup_{i=0}^{r-1} \mathrm{int}(\omega_i)$. Here nonsingularity means that f is measurable (with respect to $\mathfrak{B}$) and $m(A) = 0$ implies $m(f^{-1}(A)) = m(f(A)) = 0$ for any $A \in \mathfrak{B}$; and Df*

refers to the Jacobian Matrix. We say a PAP f is piecewise invertible area preserving if $f|_{\omega_i}$ is invertible for each ω_i, and say f is invertible PAP if f is globally invertible. In particular, if each ω_i is a connected polyhedral region and $f|_{\text{int}\,\omega_i}$ is isometry (i.e., preserving Euclidean distance), we say f is a piecewise isometry *(PWI).*

PWIs include reflections while PAPs also include piecewise (non-)uniformly hyperbolic maps with determinant ± 1, e.g. *Baker's map, Arnold's cat map*, area preserving *Hénon map* and *Standard map* [19]. Piecewise versions of these maps can be realized by considering them on the torus.

For a PAP $f : X \to X$, we classify its ACIPs based on their density properties as [1]

$$\begin{aligned} \mathcal{M}_I(f): &= \{\mu \text{ is an ACIP with respect to } f\}, \\ \mathcal{M}_{IB}(f): &= \{\mu \in \mathcal{M}_I(f) : \frac{d\mu}{dm} = \eta|_X \text{ for some } \eta \in BV(U), \forall U \text{ open } \supset X\}, \\ \mathcal{M}_{IC}(f): &= \{\mu \in \mathcal{M}_I(f) : \frac{d\mu}{dm} \text{ is m-a.e. continuous}\}, \end{aligned}$$

where $BV(U)$ is the space of bounded variation functions (see Definition 2). It is reasonable to discuss $\mathcal{M}_{IB}, \mathcal{M}_{IC}$ as

- these spaces are "large enough" Banach subspaces of $L^1(m)$ (i.e. containing some discontinuous functions since L^1 is a non-separable space) [22];
- functions in these spaces have "good" geometric intuitiveness, e.g. if $\chi_E \in BV(U)$, then the measurable subset $E \subset U$ has finite perimeter [9];
- these spaces coincide with those chosen in piecewise hyperbolic maps in [2, 5, 23];
- these spaces are invariant under the *transfer operator* (defined in Section 2.1) for PWIs shown in Lemma 4. This invariance is necessary for a space to be "good" as explained in [2].

It is clear that $\mathcal{M}_{IB} \subset \mathcal{M}_{IC} \subset \mathcal{M}_I$ in $\mathbb{R}$, while for a general invertible PAP f in higher dimension, $m \in \mathcal{M}_{IB}(f), \mathcal{M}_{IC}(f) \subset \mathcal{M}_I(f)$. For non-invertible PAPs, $\mathcal{M}_I(f)$ is possibly empty; and conditions for $\mathcal{M}_I(f) \neq \emptyset$ is discussed in Section 3.2.

The novelty of this paper is that we introduce multidimensional bounded variation functions to analyze ACIPs in PAPs, especially in PWIs. Particularly, we explore the relationship between the existence of dense orbits and uniqueness of $\mathcal{M}_{IC}(f)$ or $\mathcal{M}_{IB}(f)$ for invertible multidimensional PAPs in Theorem 1. Reducing to non-uniquely ergodic IETs, this indicates the irregularity of densities of ACIPs in examples constructed by Keynes & Newton and Keane. Moreover, in Theorem 2 for invertible PWIs, we give an approach to construct invariant measures in $\mathcal{M}_{IB}(f)$ by taking accumulation point in the Birkhoff average of the *transfer operator* acting on bounded variation functions. These results together with the extension in piecewise invertible area preserving maps in Section 3.2, "partially" answer one of Goetz's questions in [12].

The paper is organized as following: preliminaries and the main results are stated in Section 2 with proofs given in Section 4. Applications along with discussions are in Section 3.

[1] $\frac{d\mu}{dm} \in L^1(m)$ is $m - a.e.$ continuous means that its equivalence class contains an $m - a.e.$ continuous representative.

2 Preliminaries and Main results

In this section, we state the main results obtained by employing the *transfer operator* and *multidimensional bounded variation*, whose explicit definitions are given in the following for the completeness. The main results is then connected to one of the open questions in [12]. Note that m always denotes normalized Lebesgue measure on a compact subset X of $\mathbb{R}^d$.

2.1 Transfer operator

Let $(X, \mathfrak{B}, m)$ be a probability space and $f : X \to X$ be nonsingular. The *transfer operator* $\mathcal{L}_f : L^1(m) \to L^1(m)$ associated with f is defined up to $m - a.e.$ equivalence as [3]:

$$\int_A \mathcal{L}_f \varphi dm = \int_{f^{-1}(A)} \varphi dm,\ \ \varphi \in L^1(m),\ \ A \in \mathfrak{B}.$$

This *transfer operator* possesses the following dual property [25]

$$\int (\mathcal{L}_f \varphi)\psi dm = \int \varphi \cdot (\psi \circ f) dm, \quad \varphi \in L^1(m), \psi \in L^\infty(m).$$

For an invertible PAP $f : X \to X$ with a topological partition $\mathcal{P} := \{\omega_0, \cdots, \omega_{r-1}\}$, the *transfer operator* can be simplified to $\mathcal{L}_f(\varphi) = \varphi \circ f^{-1},\ \forall \varphi \in L^1(m)$. In particular, by choosing a representative φ from the $m - a.e.$ equivalent class, we can adjust $\mathcal{L}_f$ to ensure

$$\mathcal{L}_f \varphi(x) = \varphi \circ f^{-1}(x),\ \ x \in \bigcup_{i=0}^{r-1} \operatorname{int} \omega_i.$$

2.2 Multidimensional bounded variation functions

There are various equivalent definitions of multidimensional bounded variation function; see Appendix A.1 (we refer to [9, 26] for overview). In one dimension, the Definition 2 below reduces to the usual notation of bounded variation as described in [3], see also Appendix A.1. We also state here Helly's Theorem, which is used in the proof of Theorem 2.

Definition 2 *[9] Let U be an open set of $\mathbb{R}^d$. A function $\eta \in L^1(U)$ is a* bounded variation *function ($\eta \in BV(U)$) if*

$$\operatorname{var}(\eta) := \sup\left\{ \int_U \eta \ \operatorname{div} \overrightarrow{\phi} \ dm : \overrightarrow{\phi} \in C_c^1(U, \mathbb{R}^d), |\overrightarrow{\phi}| \leq 1 \right\} < \infty \tag{1}$$

where $\overrightarrow{\phi} = (\phi_i)_{i=1}^d$, $\operatorname{div} \overrightarrow{\phi} = \sum_{i=1}^d \frac{\partial \phi_i}{\partial x_i}$, and $\overrightarrow{\phi} \in C_c^1(U, \mathbb{R}^d)$ means that $\overrightarrow{\phi} \in C^1$ and its compact support set is contained in U. We define a norm on BV(U) by $||\eta||_{BV(U)} := ||\eta||_1 + \operatorname{var}(\eta)$.

Helly's Theorem [9] *Let $U \subset \mathbb{R}^d$ be open and bounded, with ∂U Lipschitz. Assume that $\{\eta_n\}_{n=1}^\infty$ is a sequence in BV(U) satisfying $\sup_k ||\eta_k||_{BV(U)} < \infty$, then there exist a subsequence $\{\eta_{n_j}\}_{j=1}^\infty$ and a function $\eta \in BV(U)$ such that $\eta_{k_j} \to \eta$ in $L^1(U)$ as $j \to \infty$.*

2.3 Main results

Let X be a compact subset of $\mathbb{R}^d$, for an invertible map $f : X \to X$, we say f *admits a dense orbit* if there exists $x_0 \in X$ with $O_f(x_0) := \{f^i(x_0)|i \in \mathbb{Z}\}$ dense in X. A representative $x^* \in O_f(x_0)$ of a dense orbit is called a *nomadic point*. If every point $x_0 \in X$ is nomadic, then f is called *minimal*.

Theorem 1 *Let $(X, \mathfrak{B}, m)$ be a probability space and $f : X \to X$ be an invertible PAP with a topological partition $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$, then the following hold.*

(i) If f admits a dense orbit, then $\mathcal{M}_{IB}(f) \cup \mathcal{M}_{IC}(f) = \{m\}$.

(ii) If $f|_{\omega_i}$ is a homeomorphism for each ω_i, then f admits a dense orbit if and only if $\mathcal{M}_{IC}(f) = \{m\}$.

Corollary 1 *For an invertible PAP f, if $f|_{\omega_i}$ is a homeomorphism for each ω_i, then $\mathcal{M}_{IC}(f) = \{m\}$ implies $\mathcal{M}_{IB}(f) = \{m\}$.*

Theorem 2 *For an invertible PWI $f : X \to X$, given any $\eta \in BV(U)$ with $\eta|_X > 0$, where $U \supset X$ is open bounded and ∂U is Lipschitz, then there is a subsequence of the Birkhoff average sequence of the transfer operator $\mathcal{L}_f$*

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathcal{L}_f^i \eta \to \eta^* \in BV(U), \text{ as } k \to \infty$$

and $d\mu := \eta^*|_X dm \in \mathcal{M}_{IB}(f)$; moreover, $\mathrm{var}(\eta^*) \leq \mathrm{var}(\eta)$ and $\sup(\eta^*) \leq \sup(\eta)$.

Note that all the invariant measures in $\mathcal{M}_{IB}(f)$ can be constructed using Theorem 2; the reason being the following. Suppose $\eta \in BV(U)$ and $d\mu := \eta|_X dm \in \mathcal{M}_{IB}(f)$, then it is clearly to see that $\mathcal{L}_f \eta = \eta$, which follows $\lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \mathcal{L}_f^j \eta = \eta$. This construction in principle determines all invariant measures in $\mathcal{M}_{IB}$ of piecewise rotations (see Appendix A.2 for the definition).

Remark 1 *Theorem 1 and Theorem 2 also hold for $m-$a.e. invertible (i.e., $m(\{y|\#f^{-1}(y) \neq 1\}) = 0$).*

The above theorems together with Lemma 5 and Proposition 2 in the following sections "partially" answer the open question posed in [12] - "determine all invariant non-atomic probability Borel measure for two dimensional piecewise rotation" and we summarise this "partial" answer as the following corollary.

Corollary 2 *Suppose $f : X \to X$ is an invertible piecewise rotation, then*

(i) $\mathcal{M}_I(f) = \{\varphi dm : \varphi = \mathbb{E}(\varphi|\mathcal{I}), \varphi \in L^1(m)\}$, where $\mathcal{I} = \{B \in \mathcal{B} : f^{-1}(B) = B \mod m\}$;

(ii) $\mathcal{M}_{IB}(f) = \{\eta^*|_X dm : \eta^*$ is an accumulation point of $\{\frac{1}{n} \sum_{i=0}^{n-1} \mathcal{L}_f^i \eta\}_{n\in\mathbb{Z}}$, where $\eta \in BV(U)$ and $\eta|_X > 0\}$; and if f admits a dense orbit then $\mathcal{M}_{IB}(f) = \{m\}$;

(iii) $\mathcal{M}_{IC}(f) = \{m\}$ if and only if f admits a dense orbit.

Suppose f is non-invertible. Let $X^+ = \bigcap_{i=0}^{\infty} f^i(X)$ and define $f^+ : \overline{X^+} \to \overline{X^+}$ as in (2) in Section 3.2, then f^+ is m − a.e. invertible. Furthermore, if $m(\overline{X^+}) > 0$, then above statements (i), (ii) and (iii) hold for f^+ and

$$\mathcal{M}_I(f) = \{\mu(\cdot) := \nu(\cdot \cap \overline{X^+}), \forall \nu \in \mathcal{M}_I(f^+)\};$$

if $m(\overline{X^+}) = 0$, then $\mathcal{M}_I(f) = \mathcal{M}_{IB}(f) = \mathcal{M}_{IC}(f) = \emptyset$.

Concerning the open question [12] in piecewise rotations, this Corollary gives a universal approach to construct all invariant measures with bounded variation densities and gives a necessary and sufficient condition for uniqueness of ACIPs for any piecewise rotations. However, ACIPs only gives a subset of non-atomic probability Borel measures. Thereby, to fully answer the question in piecewise rotations (or in PAPs), we have to explore singular (with respect to Lebesgue measure) non-atomic probability invariant measures, e.g. Hausdorff measure when it is probabilistic. This is discussed at the end of Section 3.2.

3 Applications and Discussions

In this section, we firstly apply Theorem 1 to IETs (see Appendix A.3 for the definition), particularly to examples of non-uniquely ergodic IETs and show the irregularity in the densities of their ACIPs. Secondly, we apply Theorem 1 and Theorem 2 to multidimensional piecewise invertible area preserving maps and give a short discussion at the end particularly on the open question proposed in [12] for piecewise rotations. Recall that m always denotes normalized Lebesgue measure on a compact subset X of $\mathbb{R}^d$.

3.1 Interval exchange transformations

In IETs, $\mathcal{M}_{IC}$ can be refined to

$$\mathcal{M}'_{IC}(f) := \{\mu \in \mathcal{M}_I(f) : \frac{d\mu}{dm} := \varphi \text{ has at most countably many discontinuity points}\},$$

then $\mathcal{M}_{IB} \subset \mathcal{M}'_{IC} \subset \mathcal{M}_{IC}$. Note that topological transitivity[2] implies minimality for IETs (see e.g. Corollary 14.5.11 in [14]) and so by applying Theorem 1, it is straightforward to prove the following corollary which characterizes the minimality in terms of the uniqueness of ACIPs with particular properties.

Corollary 3 *For any IET $f : [0, 1) \to [0, 1)$,*

$$f \text{ is minimal} \Leftrightarrow \mathcal{M}_{IC}(f) = \{m\} \Leftrightarrow \mathcal{M}'_{IC}(f) = \{m\}.$$

This corollary can be used to investigate Keane Conjecture - minimality implies unique ergodicity for IETs [15] and its counterexamples. Firstly, we review two well-known counterexamples.

In [18] Keynes and Newton construct $T_\gamma(x) = x + \gamma \pmod 1$ where $\gamma \in (0, 1)$ was an irrational number. By choosing certain β and γ (see [6, 18]) and defining $\widehat{T}_{\gamma\beta} : [0, 1 + \beta) \to [0, 1 + \beta)$ as

$$\widehat{T}_{\gamma\beta}(x) = \begin{cases} x + 1, & \text{if } 0 \le x < \beta \\ x + \gamma \pmod 1, & \text{if } \beta \le x < 1 + \beta, \end{cases}$$

[2]Topological transitivity means that for any open sets U and V, there exists $n \in \mathbb{Z}$ such that $f^n(U)) \cap V \neq \emptyset$.

then $T_{\gamma\beta}(x) := \frac{1}{1+\beta}\widehat{T}_{\gamma\beta}(x(1+\beta))$ has an eigenvalue -1, implying that $T^2_{\gamma\beta}$ is not uniquely ergodic and its ergodic measures belong to $\mathcal{M}_I$ (see [18] for details).

In [16], Keane constructs an interval exchange transformation with four intervals satisfying a strong irrationality condition that implies minimality. Under certain conditions (see [16]), there exist two different ergodic measures μ_1 and μ_2. Moreover, such ergodic measures are either both in $\mathcal{M}_I$ or one is Lebesgue measure and the other is singular. The Hausdorff dimension of singular measures in this example has been recently estimated in [7]; here we discuss ACIPs. Combining these results gives a better understanding of ergodic measures for non-uniquely ergodic IETs.

For the examples above, explicit formulae of their densities (if they belong to $\mathcal{M}_I$) are not clear. The difficulty in constructing counterexamples can be seen from Corollary 3 as these ergodic measures are in $\mathcal{M}_I(f)\backslash\mathcal{M}_{IC}(f)$, which means that their densities are Lebesgue integrable but the points of discontinuity have a positive Lebesgue measure, and so the densities have unbounded variation. The following proposition provides an explicit description of topological properties for the density of invariant measures of non-uniquely ergodic IETs.

Proposition 1 *For any topologically transitive IET, if $m \neq \mu \in \mathcal{M}_I$, the following holds:*

(i) the density of μ is a simple function (i.e. a combination of finitely many characteristic functions);

(ii) for any representative from the equivalence class $\varphi := \frac{d\mu}{dm}$, φ is discontinuous everywhere and $\operatorname{supp}\mu = [0,1)$ (i.e. if $x \notin \operatorname{supp}\mu$, then there exists an open ball B such that $\mu(B) = 0$).

Remark 2 *For the two ergodic measures $\mu_1, \mu_2 \in \mathcal{M}_I$ in the examples of Keynes & Newton[18] and Keane [16], we can derive from Proposition 1 that $\mu_1 \perp \mu_2$, $\mu_1, \mu_2 \ll m$, and $\operatorname{supp}\mu_1 = \operatorname{supp}\mu_2 = [0,1)$. Hence, in some sense the measure μ_1 and μ_2 intermingle with each other.*

3.2 Piecewise invertible area preserving maps

In this subsection, we aim to understand the structure of ACIPs for non-invertible PAPs, particularly for piecewise invertible area preserving maps $f : X \to X$, where X is a compact subset of $\mathbb{R}^d$. Let $X^+ := \bigcap_{i=0}^{\infty} f^i(X)$, which is invariant under f, i.e., $f(X^+) = X^+$ [11]. In particular for PWIs, X^+ is shown to be almost closed (i.e., $m(X^+) = m(\overline{X^+})$) in [1]. Here we show such almost closedness of X^+ holds for a large proportion of piecewise invertible area preserving maps.

Lemma 1 *Let $f : X \to X$ be piecewise invertible area preserving with a topological partition $\mathcal{P} = \{\omega_0, \cdots, \omega_{r-1}\}$ and $f_i := f|_{\operatorname{int}\omega_i}$ is Lipschitz for each ω_i, then $m(X^+) = m(\overline{X^+})$ and $f|_{X^+}$ is $m-a.e.$ invertible.*

Under the conditions of Lemma 1, $\overline{X^+}$ is not necessarily invariant under f, however, there still exists a map f^+ (not necessarily unique) that is $m-a.e.$ equal to f for which $\overline{X^+}$ is invariant. This f^+ can be constructed as follows. Since each f_i is Lipschitz, there exists a continuous extension $\widehat{f_i} : \overline{\operatorname{int}\omega_i} \to \overline{f_i(\operatorname{int}\omega_i)}$. Hence for any $x \in (\bigcup_{i=0}^{r-1} \partial\omega_i) \cap \overline{X^+}$, we can define $g(x) := \widehat{f_{i^*}}(x)$, where $i^* := \min\{i : x \in \partial\omega_i\}$. Thereby, $f^+ : \overline{X^+} \to \overline{X^+}$ can be defined as

$$f^+(x) = \begin{cases} f(x), & x \in \operatorname{int}(\omega_i) \cap X^+ \\ g(x), & \text{otherwise.} \end{cases} \tag{2}$$

Moreover, if f_i is bi-Lipschitz, f^+ can be shown to be non-singular and $m-a.e.$ invertible. The non-singularity and $m-a.e.$ invertibility allow the analysis of the ACIPs of f^+ analogous to the proofs of Theorem 1 and Theorem 2 by replacing X and $\mathrm{int}(\cdot)$ with $\overline{X^+}$ and $\mathrm{int}(\cdot)\cap\overline{X^+}$ respectively, though $\overline{X^+}$ may be not sufficiently regular in the sense that $m(\partial\overline{X^+})>0$. The ACIPs of f^+ can be further used to construct all the ACIPs of f. This gives the following proposition.

Proposition 2 *Let $f : X \to X$ be a piecewise invertible area preserving map with topological partition $\mathcal{P} := \{\omega_0, \cdots, \omega_{r-1}\}$ and assume that $f_i := f|_{\mathrm{int}\,\omega_i}$ is bi-Lipschitz continuous. Moreover, let $f^+ : \overline{X^+} \to \overline{X^+}$ be defined as in (2), then f^+ is non-singular and $m-a.e.$ invertible. Moreover the following hold:*

(i) if $m(\overline{X^+})>0$, then $\mathcal{M}_I(f) = \left\{\mu(\cdot) := \nu(\cdot\cap\overline{X^+}), \forall\nu\in\mathcal{M}_I(f^+)\right\}$;

(ii) if $m(\overline{X^+})=0$, then $\mathcal{M}_I(f)=\emptyset$.

When $m(\overline{X^+})=0$, it is natural to consider invariant measures that are absolutely continuous with respect to Hausdorff measure $\mathcal{H}^s$, where $s=\dim_H\overline{X^+}$. If f^+ satisfies the following conditions:

(1) $0<\mathcal{H}^s(X^+)=\mathcal{H}^s(\overline{X^+})<\infty$;

(2) $\mathcal{H}^s$ is an invariant measure for f^+;

(3) f^+ is non-singular with respect to $\mathcal{H}^s$, i.e., $\mathcal{H}^s\left((f^+)^{-1}(A)\right)=\mathcal{H}^s(f^+(A))=0$ whenever $\mathcal{H}^s(A)=0$;

then by an analogous argument to Proposition 2, f^+ is $\mathcal{H}^s-a.e.$ invertible. Correspondingly, Theorem 1 and Theorem 2 hold for f^+ when Lebesgue measure is replaced by $\mathcal{H}^s$.

The above three conditions are possible to achieve for interval translation maps (which is a special class of piecewise invertible area preserving maps on $\mathbb{R}$, see Appendix A.3). In fact, condition (2) is demonstrated in [4, Theorem 2] while condition (3) can be inferred by combining condition (2) and the definition of Hausdorff measure. Moreover, the $\mathcal{H}^s-a.e.$ closedness can be shown by analogous arguments to the proof of Lemma 1. Eventually, by [10, Theorem 9.3], condition (1) can be achieved for particular interval translation maps where $\overline{X^+}$ are self similar sets satisfying an open set condition and positive Hausdorff dimension [4].

Particular to the open question in determining all invariant non-atomic probability Borel measures of non-invertible piecewise rotations when $m(\overline{X^+})=0$, we consider absolutely continuous (with respect to $\mathcal{H}^s$) invariant probability measures. Let f^+ be defined as (2), then we have the following proposition.

Proposition 3 *Let f be a two-dimensional piecewise rotation, with $m(\overline{X^+})=0$ and $s:=\dim_H\overline{X^+}>1$, then X^+ is $\mathcal{H}^s-a.e.$ closed and f^+ is non-singular. Moreover, let*

$$\mathcal{H}_I(f) := \{\nu \text{ probability invariant measure of } f : \nu \ll \mathcal{H}^s\},$$

then the following hold:

(i) if $0<\mathcal{H}^s(\overline{X^+})<\infty$, then f^+ is $\mathcal{H}^s-a.e.$ invertible, $\mathcal{H}^s$ is invariant under f^+ and $\mathcal{H}_I(f)=\left\{\mu(\cdot):=\nu(\cdot\cap\overline{X^+}),\forall\nu\in\mathcal{H}_I(f^+)\right\}$;

(ii) if $\mathcal{H}^s(\overline{X^+}) = 0$*, then* $\mathcal{H}_I(f) = \emptyset$.

The proof of Proposition 3 is analogous to the proofs of Proposition 2 and Lemma 1 where we need the condition $s := \dim_H \overline{X^+} > 1$. This condition does not always hold. For instance, taking the Cartesian product of interval translation maps in [4] gives some examples of piecewise rotations with Hausdorff dimension ranging $0 \leq s \leq 1$.It might be interesting to explore conditions for $s > 1$. For the case $0 \leq s \leq 1$, we suspect that the approaches would be different.

To fully answer the open question in piecewise rotations, we suggest to consider the structure of $\mathcal{H}_I(f^+)$ in the case of $\mathcal{H}^s(\overline{X^+}) = \infty$. In this case, the corresponding approaches in Theorem 1 and 2 on f^+ are invalid due to lack of $\mathcal{H}^s - a.e.$ invertibility of f^+. Note that by [10, Theorem 6.2], there exists a compact subset $E \subset \overline{X^+}$ with $0 < \mathcal{H}^s(E) < \infty$. We suggest to establish a non-atomic probability invariant measures of f^+ induced by E as a reference measure and leave this for further studies.

4 Proofs

To demonstrate the proofs of our theorems and propositions, we first state the following two basic results regarding $\mathcal{M}_I(f)$, i.e., the set of all ACIPs of a PAP $f : X \to X$ with standard proofs. Recall m always denotes normalized Lebesgue measure on a compact subset X of $\mathbb{R}^d$.

Proposition 4 *For a nonsingular map* $f : X \to X$ *and* $\varphi \in L^1(m)$*, we have* $\mathcal{L}_f\varphi = \varphi$ *if and only if* $d\mu := \varphi dm \in \mathcal{M}_I(f)$.

Proof: Suppose $\mathcal{L}_f\varphi = \varphi$. Let $d\mu := \varphi dm$, then for each $A \subset X$,

$$\mu(A) = \int_A \varphi dm = \int_A \mathcal{L}_f\varphi dm = \int_{f^{-1}(A)} \varphi dm = \mu(f^{-1}(A)),$$

which implies $\mu \in \mathcal{M}_I(f)$.

On the other hand, if $d\mu := \varphi dm$ is an invariant measure of f, i.e., $\mu(f^{-1}(A)) = \mu(A)$ for any Borel set $A \subset X$, then $\int_A \mathcal{L}_f\varphi dm = \int_{f^{-1}(A)} \varphi dm = \mu(f^{-1}(A)) = \mu(A) = \int_A \varphi dm$, which implies that $\mathcal{L}_f\varphi = \varphi$. □

Proposition 5 *For an invertible PAP* $f : X \to X$*,* m *is ergodic if and only if* $\mathcal{M}_I(f) = \{m\}$.

Proof: Suppose $\mu \in \mathcal{M}_I(f)$, then by Proposition 4, $\frac{d\mu}{dm} \circ f = \frac{d\mu}{dm}$. Combining with the facts that μ is a probability measure and m is ergodic, gives $\frac{d\mu}{dm} \equiv 1$, i.e., $\mathcal{M}_I = \{m\}$.

On the other hand, if m is not ergodic, then there exists an invariant subset $B \subset X$ with $0 < m(B) < 1$. Let $m^*(F) := \frac{m(F \cap B)}{m(B)}$ for each $F \in \mathfrak{B}$, then

$$m^*(f^{-1}(F)) = \frac{m(f^{-1}(F) \cap B)}{m(B)} = \frac{m(f^{-1}(F \cap B))}{m(B)} = m^*(F),$$

showing that $m^* \in \mathcal{M}_I(f)$. Note that $m^* \neq m$, which is contradictory to $\mathcal{M}_I(f) = \{m\}$. □

4.1 Proof of Theorem 1

To prove the statement (i), we firstly show $\mathcal{M}_{IC} = \{m\}$ and then by combining some particular properties on bounded variation and Lebesgue point, we show $\mathcal{M}_{IB} = \{m\}$. For the statement (ii), we start with a lemma showing the equivalence between topological transitivity and the existence of a nomadic point.

Without loss of generality, we assume that $x^* \in \bigcup_{i=0}^{r-1} \operatorname{int}\omega_i$ is a nomadic point.

Proof of the statement (i) in Theorem 1: In the case of $\mu \in \mathcal{M}_{IC}$, take a representative $\varphi = \frac{d\mu}{dm}$ from the $m - a.e.$ equivalent class such that φ is $m - a.e.$ continuous and a point $x' \in \bigcup_{i=0}^{r-1} \operatorname{int}\omega_i$ for which φ is continuous, hence there exists a subsequence $\{f^{k_t}(x^*)\}$ such that as $|t| \to \infty$, $f^{k_t}(x^*) \to x'$. Simultaneously, $\varphi \circ f^{-1}(x) = \varphi(x)$ holds for each $x \in \bigcup_{i=0}^{r-1} \operatorname{int}\omega_i$. By the continuity of the point x', $\varphi(x') = \varphi(x^*)$, which implies $\varphi \equiv 1$, i.e. $\mathcal{M}_{IC} = \{m\}$.

In the following, we consider the case of $\mu \in \mathcal{M}_{IB}(f)$. Take any $x_0 \in \bigcup_{i=0}^{r-1} \operatorname{int}\omega_i$, then there exists an open ball $B \subset \operatorname{int}\omega_i$, whose center is x_0. Since $\mu \in \mathcal{M}_{IB}$, (i.e., $\varphi = \frac{d\mu}{dm}$ with $\varphi = \eta|_X$ for some $\eta \in BV(U)$, $\forall U \supset X$), there exists a unit vector $\hat{a} \in \mathbb{R}^d$ splitting B into:

$$B_{(\hat{a},x_0)} := B \cap \{x \in \mathbb{R}^d : \langle x - x_0, \hat{a}\rangle > 0\} \text{ and } B_{(-\hat{a},x_0)} := B \cap \{x \in \mathbb{R}^d : \langle x - x_0, -\hat{a}\rangle > 0\}$$

(where $\langle \cdot, \cdot \rangle$ is the inner product), and the limits

$$\lim_{\substack{x\to x_0\\ x\in B_{(\hat{a},x_0)}}} \varphi(x) \quad \text{and} \quad \lim_{\substack{x\to x_0\\ x\in B_{(-\hat{a},x_0)}}} \varphi(x)$$

exist for $m - a.e.$ $x_0 \in \bigcup_{i=0}^{r-1} \operatorname{int}\omega_i$ (x_0 is then called a *regular point*) [26, page 178]. Hence by analogous arguments to the proof of the uniqueness of measures in $\mathcal{M}_{IC}$ (recall x^* is the nomadic point),

$$\lim_{\substack{x\to x_0\\ x\in B_{(\hat{a},x_0)}}} \varphi(x) = \lim_{\substack{x\to x_0\\ x\in B_{(-\hat{a},x_0)}}} \varphi(x) = \varphi(x^*),$$

By [26, page 168], it follows that $\lim_{x\to x_0} \varphi(x) = \varphi(x^*)$. In addition, since $\varphi \in L^1(m)$, then $m - a.e.$ $x \in X$ is a Lebesgue point of φ i.e.,

$$\lim_{r\to 0^+} \frac{1}{m(B(x,r))} \int_{B(x,r)} |\varphi(y) - \varphi(x)| dm(y) = 0.$$

Therefore, if the regular point x_0 is also a Lebesgue point,

$$\begin{aligned} 0 \le |\varphi(x_0) - \varphi(x^*)| &= \lim_{r\to 0^+} \frac{1}{m(B(x_0,r))} \int_{B(x_0,r)} |\varphi(x_0) - \varphi(x^*)| dm(y) \\ &\le \lim_{r\to 0^+} \frac{1}{m(B(x_0,r))} \int_{B(x_0,r)} |\varphi(y) - \varphi(x_0)| dm(y) \\ &+ \lim_{r\to 0^+} \frac{1}{m(B(x_0,r))} \int_{B(x_0,r)} |\varphi(y) - \varphi(x^*)| dm(y) = 0, \end{aligned}$$

which implies that $\varphi(x_0) = \varphi(x^*)$, meaning that $\varphi \equiv 1$. □

Before prooving statement (ii), we first formulate an equivalent condition for topological transitivity.

Lemma 2 *Let $f : X \to X$ on $\mathbb{R}^d$ be an invertible PAP with a topological partition $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$. Suppose $f|_{\omega_i}$ is homeomorphism for each ω_i, then the following are equivalent.*

(*i*) *f admits a dense orbit;*

(*ii*) *f is strong topologically transitive(i.e. For any open sets U, V, there exists $n \in \mathbb{Z}$ such that int$(f^n(U)) \cap V \neq \emptyset$);*

(*iii*) *f is topologically transitive (i.e. for any open sets U, V, there exists $n \in \mathbb{Z}$ such that $f^n(U)) \cap V \neq \emptyset$).*

Note that since PAPs are typically not continuous at every point, the proof for the above equivalence will not be standard (see [27] for the standard proof), therefore, we provide the details of the proof here.

Proof of Lemma 2: "(ii) implies (iii)" is directly and we only need to prove (i) implies (ii) and (iii) implies (i). For convenience, we denote $\mathcal{P}^{(n)} := \bigvee_{i=0}^{n} f^{-i}(\mathcal{P})$ and $\omega^{(n)}$ one of the atoms in the topological partition $\mathcal{P}^{(n)}$ for any $n \in \mathbb{Z}$.

"$(i) \Rightarrow (ii)$" We prove by contradiction. Suppose there exist open sets $U, V \neq \emptyset$ such that for any $n \in \mathbb{Z}$, $\mathrm{int}(f^n(U)) \cap V = \emptyset$ and a nomadic point x^* of f, then there exist $n_1, n_2 \in \mathbb{Z}$ such that $f^{n_1}(x^*) \in U$ and $f^{n_2}(x^*) \in V$. Let $t := n_2 - n_1$, then there exists an atom $\omega^{(t)} \in \mathcal{P}^{(t)}$ such that $f^{n_1}(x^*) \in \omega^{(t)}$.

If $f^{n_1}(x^*) \in \mathrm{int}\, \omega^{(t)}$, then $f^t(U \cap \mathrm{int}\, \omega^{(t)}) = \mathrm{int}\, f^t(U \cap \mathrm{int}\, \omega^{(t)})$ since $f^t|_{\mathrm{int}\, \omega^{(t)}}$ is homeomorphic. Therefore,

$$f^{n_2}(x^*) = f^t(f^{n_1})(x^*) \in f^t(U \cap \mathrm{int}\, \omega^{(t)}) \subset \mathrm{int}\, f^t(U),$$

which implies $f^{n_2}(x^*) \in \mathrm{int}(f^t(U)) \cap V$. This is a contradiction.

Suppose $f^{n_1}(x^*) \in \partial \omega^{(t)}$, let $l := n_1 + n_2$, then there exists an atom $\omega^{(l)}$ such that $x^* \in \omega^{(l)}$. Note that $f^{n_1}|_{\omega^{(l)}}, f^{n_2}|_{\omega^{(l)}}$ are continuous, therefore, there exists $x' \in \omega^{(l)}$ sufficiently close to x^* such that $f^{n_1}(x') \in \mathrm{int}\, \omega^{(t)} \cap U$ and $f^{n_2}(x') \in V$. Repeat the same process as the above case, using x' in place of x^* and this completes the proof.

"$(iii) \Rightarrow (i)$" Let $\{U_i\}_{i=1}^{\infty}$ be a countable base for X and $n_1 \in \mathbb{Z}$ such that $f^{n_1}(U_1) \cap U_2 \neq \emptyset$. We firstly show $\mathrm{int}(f^{n_1}(U_1)) \cap U_2 \neq \emptyset$. Let $y \in f^{n_1}(U_1) \cap U_2$. Hence there exists $x \in \omega^{(n_1)} \in \mathcal{P}^{(n_1)}$ such that $y = f^{n_1}(x)$. If $x \in \mathrm{int}\, \omega^{(n_1)}$, then there exists an open ball $x \in B_x \subset \omega^{(n_1)}$ such that $f^{n_1}(B_x) \subset \mathrm{int}\, f^{n_1}(U_1)$. Therefore, $y \in \mathrm{int}(f^{n_1}(U_1)) \cap U_2$. Otherwise, if $x \in \partial \omega^{(n_1)}$, by the analogous approach to the case "$(i) \Rightarrow (ii)$", there exists x' such that $x' \in \mathrm{int}\, \omega^{(n_1)} \cap U_1$ and $y' := f^{n_1}(x') \in \mathrm{int}(f^{n_1}(U_1)) \cap U_2$. Hence $\mathrm{int}(f^{n_1}(U_1)) \cap U_2 \neq \emptyset$.

Therefore, there exists a closed ball B_2 such that $B_2 \subset \mathrm{int}(f^{n_1}(U_1)) \cap U_2 \cap \mathrm{int}\, \omega^{(n_1)}$. Moreover, $f^{n_1}|_{B_2}$ is a homeomorphism meaning that $V_1 := f^{-n_1}(B_2)$ is a nonempty closed set. Analogously, for open sets $\mathrm{int}\, B_2$ and U_3 there exist $n_2 \in \mathbb{Z}$ and a closed ball $B_3 \subset \mathrm{int}(f^{n_2}(B_2)) \cap U_3$, with $f^{n_2}|_{B_3}$ a homeomorphism. Let $V_2 := f^{-n_2}(B_3) \subset B_2$, then V_2 is a nonempty closed set and $f^{-n_1}(V_2) \subset V_1$.

Continue this process and eventually there exist $\{n_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ such that $f^{-n_i}(V_{i+1}) \subset V_i$ for each i. By taking $x^* \in \bigcap_{i=1}^{\infty} f^N(V_{i+1})$, where $N = -\sum_{j=1}^{i} n_j$, it is straightforward to see that x^* is a nomadic point, which completes the proof. □

Remark 3 *There does exist a map [21] that is not continuously extend to the boundary from its interior and is topologically transitive but does not admit dense orbit.*

Proof of statement (ii) in Theorem 1: We show the converse by contraction. Suppose that f is not topologically transitive in X, by Lemma 2 there exist two open intervals $U, V \subset X$ such that $\mathrm{int}(f^n(U)) \cap V = \emptyset$ for all $n \in \mathbb{Z}$. Therefore, $\mathrm{int}(f^{i+n}(U)) \cap \mathrm{int}\, f^i(V) = \emptyset \mod m$, for any $n, i \in \mathbb{Z}$. Let

$$U^* := \bigcup_{i=-\infty}^{\infty} \operatorname{int} f^i(U) \text{ and } V^* := \bigcup_{i=-\infty}^{\infty} \operatorname{int} f^i(V),$$

then $U^* \cap V^* = \emptyset \mod m$ and both U^* and V^* are invariant under f. Hence both $m|_{U^*}$ and $m|_{V^*}$ are invariant measures and $m|_{U^*} \neq m|_{V^*}$. Note the facts that $\partial U^* \subset \bigcup_{i=-\infty}^{\infty} \partial f^i(U)$ and $\partial V^* \subset \bigcup_{i=-\infty}^{\infty} \partial f^i(V)$ implying that $m(\partial U^*) = m(\partial V^*) = 0$, that is χ_{U^*} and χ_{V^*} have Lebesgue zero measure points of discontinuity. Therefore, $m|_{U^*}, m|_{V^*} \in \mathcal{M}_{IC}$, which contradicts to the uniqueness of measure in $\mathcal{M}_{IC}$. □

Remark 4 *When reducing to IETs, each open set is a union of countably many open intervals, therefore, χ_{U^*} and χ_{V^*} have at most countably many discontinuities, which will be used in the proof of Corollary 3.*

4.2 Proof of Theorem 2

As any bounded variation function η is defined on an open set, for any open set $U \supset X$, we define $\bar{f} : U \to U$ by:

$$\bar{f}(x) = \begin{cases} f(x), & x \in X \\ x, & x \in U \backslash X, \end{cases} \tag{3}$$

then $\bar{f}$ is still a PAP. We first show that $\mathcal{L}_{\bar{f}}$ preserves the variation of bounded variation functions by applying the following lemma and eventually, use Helly's Theorem to illustrate the structure for $\mathcal{M}_{IB}$ of PWI.

Lemma 3 *[Coordinate Transformation][8] Let $\psi : W \to \Omega$ be a C^2- diffeomorphism where W, Ω are open subset in $\mathbb{R}^d$. Given $\overrightarrow{\phi} \in C^1(\Omega, \mathbb{R}^d)$, let $\overrightarrow{\phi}^{\psi}(y) := D\psi^{-1}(y)\overrightarrow{\phi}(\psi(y))$ then*

$$\operatorname{div}(|\det D\psi|\overrightarrow{\phi}^{\psi}) = (\operatorname{div} \overrightarrow{\phi}) \circ \psi \cdot |\det D\psi|. \tag{4}$$

Lemma 4 *Let $f : X \to X$ be an invertible piecewise isometry. For any $\eta \in BV(U), X \subset U \subset \mathbb{R}^d$, then $\operatorname{var} \mathcal{L}_{\bar{f}}\eta = \operatorname{var} \eta$.*

Proof Recall Definition 2 of bounded variation

$$\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) = \sup\left\{ \int_U (\eta \circ \overline{f}^{-1} \cdot \operatorname{div} \overrightarrow{\phi}) dm \ : \ \overrightarrow{\phi} \in C_c^1(U, \mathbb{R}), |\overrightarrow{\phi}| \leq 1 \right\}.$$

Let $f_i := f|_{\omega_i}$, where $\omega_i \in \mathcal{P}$. For any $\varepsilon > 0$, there exists $\overrightarrow{\phi} \in C_c^1(U, \mathbb{R}), |\overrightarrow{\phi}| \leq 1$ such that

$$\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) - \varepsilon \leq \int_U (\eta \circ \overline{f}^{-1} \cdot \operatorname{div} \overrightarrow{\phi}) dm = \sum_{i=0}^{r-1} \int_{\operatorname{int} \omega_i} (\eta \circ f_i^{-1} \cdot \operatorname{div} \overrightarrow{\phi}) dm + \int_{U \backslash X} \eta \cdot \operatorname{div} \overrightarrow{\phi} dm.$$

By applying the coordinate transformation ($x = f_i^{-1}(y)$), for each ω_i

$$\int_{\operatorname{int} \omega_i} \eta(f_i^{-1}(y)) \cdot \operatorname{div} \overrightarrow{\phi}(y) dm(y) = \int_{f_i^{-1}(\operatorname{int} \omega_i)} \eta(x) \cdot \operatorname{div}(\overrightarrow{\phi})(f_i(x)) dm(f_i(x))$$

$$= \int_{f_i^{-1}(\operatorname{int} \omega_i)} \eta \cdot (\operatorname{div} \overrightarrow{\phi}) \circ f_i dm = \int_{f_i^{-1}(\operatorname{int} \omega_i)} \eta \cdot \operatorname{div} \overrightarrow{\phi}^{f_i} dm,$$

where $\overrightarrow{\phi}^{f_i} := (Df_i)^{-1} \cdot \overrightarrow{\phi} \circ f_i$. The second equality is due to $|\det Df_i(x)| \equiv 1$ while the third is due to (4) in Lemma 3. Hence,

$$\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) - \epsilon \leq \sum_{i=1}^{r-1} \int_{f_i^{-1}(\operatorname{int}\omega_i)} \eta(x) \cdot \operatorname{div} \overrightarrow{\phi}^{f_i} + \int_{U \setminus X} \eta \cdot \operatorname{div} \overrightarrow{\phi}\, dm. \tag{5}$$

Moreover, since f is an invertible PWI, it can be written in form of $f_i(x) = Df_i \cdot x + b_i$. Note that the rotation matrix Df_i preserves the Euclidean metric, then

$$\sup_{x \in \operatorname{int}\omega_i} |\overrightarrow{\phi}^{f_i}(x)| = \sup_{x \in \operatorname{int}\omega_i} |(Df_i)^{-1}\overrightarrow{\phi}(f_i(x))| = \sup_{x \in \operatorname{int}\omega_i} |\overrightarrow{\phi}(f_i(x))| = \sup_{y \in f_i(\omega_i)} |\overrightarrow{\phi}(y)|, \tag{6}$$

For given ϵ, there exists $\delta > 0$ such that by letting

$$\Delta_i := \{x \in \omega_i : d(x,z) \geq \delta, \forall z \in \partial\omega_i\},$$

we can then define

$$\widehat{\phi}(x) = \begin{cases} \overrightarrow{\phi}^{f_i}, & \text{if } x \in \Delta_i \\ \text{smooth connecting with the boundary}, & \text{otherwise}, \end{cases}$$

such that

$$\left| \int_U \eta \cdot \operatorname{div} \widehat{\phi} dm - \left(\sum_{i=1}^{r-1} \int_{f_i^{-1}(\operatorname{int}\omega_i)} \eta(x) \cdot \operatorname{div} \overrightarrow{\phi}^{f_i} + \int_{U \setminus X} \eta \cdot \operatorname{div} \overrightarrow{\phi}\, dm \right) \right| \leq \epsilon/2.$$

Moreover, based on (6) and each ω_i being a connected polyhedral region, it is straightforward to see that for any given $\overrightarrow{\phi} \in C_c^1(U, \mathbb{R}^d)$ satisfying $|\overrightarrow{\phi}| \leq 1$, then $\widehat{\phi}(x) \in C_c^1(U, \mathbb{R}^d)$ is well defined and $|\widehat{\phi}| \leq 1$. Therefore,

$$\begin{aligned} \operatorname{var}(\mathcal{L}_{\overline{f}}\eta) - \epsilon &\leq \sum_{i=1}^{r-1} \int_{f_i^{-1}(\operatorname{int}\omega_i)} \eta(x) \cdot \operatorname{div} \overrightarrow{\phi}^{f_i} + \int_{U \setminus X} \eta \cdot \operatorname{div} \overrightarrow{\phi}\, dm \\ &\leq \int_U \eta \cdot \operatorname{div} \widehat{\phi} dm + \epsilon/2 \leq \operatorname{var}(\eta) + \epsilon/2, \end{aligned}$$

following $\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) \leq \operatorname{var}(\eta)$ due to arbitrariness of ε. By the invertibility of $\overline{f}$, $\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) \geq \operatorname{var}(\eta)$. Hence $\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) = \operatorname{var}(f)$. □

Proof of Theorem 2: Since $\mathcal{L}_{\bar{f}}(\eta) = \eta \circ \bar{f}^{-1}$, it follows that $\operatorname{var}(\mathcal{L}_{\bar{f}}^n\eta) = \operatorname{var}(\eta)$ from Lemma 4 and $\operatorname{esssup}(\mathcal{L}_{\bar{f}}^n\eta) = \operatorname{esssup}(\eta)$, for $n \geq 1$. Let

$$\eta_n = \frac{1}{n}\sum_{j=0}^{n-1} \mathcal{L}_{\bar{f}}^j \eta = \frac{1}{n}\sum_{j=0}^{n-1} \eta \circ \bar{f}^{-j},$$

then $\operatorname{var}(\eta_n) \leq \operatorname{var}(\eta)$ and $\operatorname{esssup}(\eta_n) \leq \operatorname{esssup}(\eta)$. By Helly's Theorem [9], there exists a subsequence $(\eta_{n_k})_{k=0}^{\infty}$ converging in $L^1(m)$ to some function $\eta^* \in BV$. By the triangle inequality,

$$||\mathcal{L}_{\bar{f}}\eta^* - \eta^*||_1 \leq ||\mathcal{L}_{\bar{f}}\eta^* - \mathcal{L}_{\bar{f}}\eta_{n_k}||_1 + ||\mathcal{L}_{\bar{f}}\eta_{n_k} - \eta_{n_k}||_1 + ||\eta_{n_k} - \eta^*||_1. \tag{7}$$

The third term tends to 0 because $||\eta_{n_k} - \eta^*||_1 \to 0$, as $k \to \infty$ while the first term tends to 0 since $||\mathcal{L}_{\bar{f}}\eta^* - \mathcal{L}_{\bar{f}}\eta_k||_1 \le ||\eta^* - \eta_k||_1$. For the second term,

$$||\mathcal{L}_{\bar{f}}\eta_{n_k} - \eta_{n_k}||_1 = \left|\left|\frac{1}{n_k}\sum_{j=1}^{n_k}\eta\circ\bar{f}^{-j} - \frac{1}{n_k}\sum_{j=0}^{n_k-1}\eta\circ\bar{f}^{-j}\right|\right|_1 = \frac{1}{n_k}||\eta\circ\bar{f}^{-n_k} - \eta||_1 \le \frac{2}{n_k}||\eta||_1 \to 0.$$

Hence $\mathcal{L}_{\bar{f}}\eta^* = \eta^*$, implying that $d\mu = \eta^* dm$ is an invariant measure of $\bar{f}$ with $\mathrm{var}(\eta^*) \le \mathrm{var}(\eta)$ and $\mathrm{esssup}(\eta^*) \le \mathrm{esssup}(\eta)$. Note that $\eta|_X > 0$, therefore, $||\eta^*|_X||_1 = ||\eta|_X||_1$, by normalization $d\mu = \eta^*|_X dm \in \mathcal{M}_{IB}(f)$. □

4.3 Proof of Proposition 1

To prove Proposition 1, we first show the following lemma.

Lemma 5 *Let $(X, \mathfrak{B}, m)$ be a probability space and $f : X \to X$ be an invertible PAP, then for any $\varphi \in L^1(m)$, the conditional expectation $\mathbb{E}(\varphi|\mathcal{I})$ is a fixed point of $\mathcal{L}_f$, where $\mathcal{I} := \{B \in \mathfrak{B}, f^{-1}(B) = B \mod m\}$ is a sub $\sigma-$field. Moreover, a measure $d\mu := \varphi dm \in \mathcal{M}_I(f)$ if and only if $\varphi = \mathbb{E}(\varphi|\mathcal{I})$.*

Proof: Without loss of generality, for any fixed $L \in \mathbb{N}$, we define

$$\varphi_L := \min\{\varphi, L\} \in L^1(m).$$

By the Birkhoff Ergodic Theorem, it follows that for $m - a.e.\ x \in X$.

$$\mathcal{L}_f(\mathbb{E}(\varphi_L|\mathcal{I}))(x) = \mathbb{E}(\varphi_L|\mathcal{I})\circ f^{-1}(x) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=-1}^{n-2}\varphi_L\circ f^i(x)$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}(\sum_{i=0}^{n-1}\varphi_L\circ f^i)(x) + \frac{1}{n}\left(\varphi_L\circ f^{-1}(x) - \varphi_L\circ f^{n-1}(x)\right)\right].$$

Since

$$\lim_{n\to\infty}\frac{1}{n}\left(\varphi_L\circ f^{-1}(x) - \varphi_L\circ f^{n-1}(x)\right) \le \lim_{n\to\infty}\frac{2L}{n} = 0,\ m - a.e.\ x \in X,$$

which implies that $\mathcal{L}_f\mathbb{E}(\varphi_L|\mathcal{I}) = \mathbb{E}(\varphi_L|\mathcal{I})$. By the Monotone convergence theorem,

$$\mathcal{L}_f\mathbb{E}(\varphi|\mathcal{I}) = \mathbb{E}(\varphi|\mathcal{I}). \tag{8}$$

Next, we show the second part of the lemma. If $d\mu = \varphi dm \in \mathcal{M}_I(f)$, then by Proposition 4, $\varphi\circ f = \varphi$. Combining with the Birkhoff Ergodic Theorem gives

$$\varphi = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi\circ f^i = \mathbb{E}(\varphi|\mathcal{I}).$$

The converse is clear to see by the invariance of $\mathbb{E}(\varphi|\mathcal{I})$ under $\mathcal{L}_f$. □

We say a $\sigma-$field $\mathcal{I}$ is *finitely generated*, if there exists a partition $\mathcal{A} := \{A_i\}_{i=0}^{r-1} \subseteq \mathcal{I}$, and for each $B \in \mathcal{I}$, there exist finitely many $A_{i_1}, \cdots, A_{i_l} \in \mathcal{A}$ such that $B = \bigcup_{k=1}^{l} A_{i_k} \mod m$.

Proof of Proposition 1: (i) For any topologically transitive IET f, there are only finitely many ergodic measures $\{\nu_i\}_{i=1}^n \in \mathcal{M}_I(f)$ [15]. By using the ergodic decomposition theorem, it follows that the $\sigma-$field $\mathcal{I}$ is finitely generated by a partition, say $\mathcal{A} :=$

$\{A_1, \cdots, A_{r-1}\}$ with $0 < m(A_i) < 1$ for each i. Moreover, each ergodic $\nu = m|_{A_i} \in \mathcal{M}_I$, which implies that $\frac{d\nu_i}{dm} = \chi_{A_i}$, hence by Lemma 5 and [3], it is straightforward to see that if $d\mu := \varphi dm \in \mathcal{M}_I$, then

$$\varphi = \mathbb{E}(\varphi|\mathcal{I}) = \sum_{i=0}^{r-1} \frac{1}{m(A_i)} \int_{A_i} \varphi dm \cdot \chi_{A_i}, \quad \forall\, \varphi \in L^1(m), \tag{9}$$

which completes the proof of the first statement.

(*ii*) For any $\mu \neq m \in \mathcal{M}_I$, take any representative $\varphi := \frac{d\mu}{dm}$ from the equivalence class, and define

$$S := \{x \mid \varphi \text{ is discontinuous at } x\};$$

$$T := \{x \mid \exists\, n \in \mathbb{Z}, \text{ s.t. } f^n \text{ is discontinuous at } x\}.$$

By Theorem 1, we know that $m(S) > 0$. Moreover, since f is an IET, $m(T) = 0$. Therefore, $m(S\backslash T) > 0$. By choosing a $x_0 \in S\backslash T$, such that $\varphi \circ f^n(x_0) = \varphi(x_0)$ for any $n \in \mathbb{Z}$, we have $\{f^n(x_0)\}_{n\in\mathbb{Z}} \subset S$. Note that topologically transitive implies minimality for IETs [14]. Hence S dense in X. By specializing to $\varphi = \chi_{A_i}$, A_i is then dense in X. Combining with (9), it is straightforward to see that $\operatorname{supp}\mu = [0, 1)$ and the density φ is discontinuous everywhere. □

4.4 Proofs of Lemma 1 and Proposition 2

Lemma 6 *Suppose $\hat{f}$ is Lipschitz continuous, then $m(\hat{f}(A)) = 0$ if $m(A) = 0$.*

We give the proof of this Lemma even though it is standard from [10].

Proof: If $m(A) = 0$, by the definition of Lebesgue measure, for any $\epsilon > 0$, there exists a countable covering $\{U_i\}$ such that $A \subset \cup_i U_i$ and $\sum_i \operatorname{diam}(U_i) < \epsilon$. Hence

$$\sum_i \operatorname{diam}(U_i \cap A) \leq \sum_i \operatorname{diam}(U_i) \leq \epsilon. \tag{10}$$

By the Lipschitz property of $\hat{f}$, there exists a universal constant C such that,

$$\sum_i \operatorname{diam} \hat{f}(U_i \cap A) \leq C \sum_i \operatorname{diam}(U_i \cap A). \tag{11}$$

Note that $\{\hat{f}(U_i \cap A)\}_i$ is also a covering of $\hat{f}(A)$. Hence, by (10) and (11)

$$m(\hat{f}(A)) \leq \sum_i \operatorname{diam}(\hat{f}(U_i \cap A)) \leq C\epsilon$$

for any $\epsilon > 0$, implying that $m(\hat{f}(A)) = 0$. □

Proof of Lemma 1: We first show almost closedness of $\overline{X^+}$. Since f_i is Lipschitz countinuous, it can be continuously extended from $\operatorname{int}\omega_i$ onto $\overline{\operatorname{int}\omega_i}$. Denoting its continuous extension by $\widehat{f_i} : \overline{\operatorname{int}\omega_i} \to \widehat{f_i}(\overline{\operatorname{int}\omega_i})$, it is a straightforward consequence that each $\widehat{f_i}$ is also Lipschitz continuous. Moreover, if for $m - a.e.\ x \in A$, $x \in B$, we say $A \subset B$ mod m. Then using the non-singularity of f and Lemma 6, we know that

$$\begin{array}{rcl} \overline{X^+} := \text{closure}\left(\bigcap_{j=0}^{\infty} f^j(\bigcup_{i=0}^{r-1} \omega_i)\right) & \subseteq & \bigcap_{j=0}^{\infty} \text{closure}\left(\bigcup_{i=0}^{r-1} \widehat{f}_i^j(\operatorname{int}\omega_i)\right) \text{ mod } m \\ \subseteq \bigcap_{j=0}^{\infty} \bigcup_{i=0}^{r-1} \widehat{f}_i^j(\overline{\operatorname{int}\omega_i}) \text{ mod } m & = & \bigcap_{j=0}^{\infty} \bigcup_{i=0}^{r-1} f_i(\omega_i) \text{ mod } m = X^+ \text{ mod } m, \end{array} \tag{12}$$

which means $m(X^+) = m(\overline{X^+})$.

Denote $\omega_i^+ := \omega_i \cap X^+$, then $X^+ = \cup_{i=0}^{r-1}\omega_i^+$. Consequently,

$$\sum_{i=0}^{r-1} m(f(\omega_i^+)) = \sum_{i=0}^{r-1} m(\omega_i^+) = m(\bigcup_{i=0}^{r-1}\omega_i^+) = m(X^+) = m(f(X^+)) = m(\cup_{i=0}^{r-1} f(\omega_i^+)),$$

which follows that $m(f(\omega_i^+) \cap f(\omega_j^+)) = 0$, which implies that $f|_{X^+}$ is $m - a.e.$ invertible. □

Proof of Proposition 2: The $m-a.e.$ invertibility of f^+ is straightforward from the $m-a.e.$ invertibility of $f|_{X^+}$. By Lemma 6 and the definition of f^+, for any Borel subset $A \subset \overline{X^+}$, $m(f^+(A)) = 0$. Meanwhile, as $f|_{\omega_i}$ is bi-Lipschitz, by symmetry it follows $m((f^+)^{-1}(A)) = 0$. Hence f^+ is non-singular.

When $m(\overline{X^+}) > 0$, $f|_{\overline{X^+}}$ can be seen as a first return map of f on $\overline{X^+}$. Therefore, by applying the proposition in [3, prop 3.6.2], it follows that

$$\left\{\mu(\cdot) := \nu(\cdot \cap \overline{X^+}), \forall \nu \in \mathcal{M}_I(f^+)\right\} \subseteq \mathcal{M}_I(f).$$

On the other side, for any $\mu \in \mathcal{M}_I(f)$, since $\mu(X) = 1$ and $f^{-1} \circ f(X) = X$, hence $\mu(f(X)) = 1$. This implies that $\mu(X^+) = 1$ and completes the proof of statement (i).

For statement (ii), we prove by contraction. Suppose that there exists $\mu \in \mathcal{M}_I(f)$, it follows that $\mu(\overline{X^+}) = 0$ since $m(\overline{X^+}) = 0$. This is a contradiction with $\mu(X^+) = 1$. □

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A Appendix

A.1 Equivalence of definitions of multidimensional bounded variation function

We introduce the usual notion of bounded variation in one dimension followed by anther two equivalent definitions of multidimensional bounded variation functions.

Let $\eta \in L^1$ be a function defined on the entire real axis and $[a, b]$ a segment outside of which $\eta(x) = 0$. The *total variation* of η is defined to be

$$V(\eta) := \sup \sum_{i=0}^{i-1} |\eta(x_{i+1}) - \eta(x_i)|,$$

where sup is taken over all possible finite partitions of the segment $[a, b]$ by means of the points $x_0 = a < x_1 < \cdots < x_r = b$. The *essential total variation* of η is defined in [26] as $\bar{V}(\eta) := \inf_z V(\eta + u)$, where the "inf" is taken over all functions u that equal zero almost everywhere on $[a, b]$.

Next, we proceed to multidimensional definitions of bounded variation. Let $\eta \in L^1$ be a function defined on $\mathbb{R}^d$. We regard $\eta(x)$ as a function of the variable x_i for the other variables fixed and denote by $\bar{V}_i(x_i')$ the essential total variation of the function η with respect to x_i for a fixed point $x_i' = \{x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_d\}$.

Definition 3 *[26] A function η defined on $\mathbb{R}^d$ is said to be a function with* essentially bounded variation *if the integrals*

$$\int \bar{V}_i(x_i')dx_i' \qquad (i = 1, 2, \cdots, n)$$

exist. η is called a bounded variation *function if it has essentially bounded variation.*

Note the fact that the above definition of bounded variation reduces to the usual notion as described in [3]. The equivalence between Definition 3 and Definition 2 in Section 2.2 is shown via the below definition of bounded variation.

Definition 4 *[26, p158] Suppose U is an open subset in $\mathbb{R}^d$ and $\eta \in L^1(U)$, then η is a* bounded variation *function if and only if there exists a constant K such that*

$$\left| \int_U \frac{\partial \phi}{\partial x_i} \eta dx \right| \leq K \sup_{x \in U} |\phi(x)|, \quad (i = 1, 2, \cdots, d) \tag{13}$$

for all $\phi \in C_0^1(U, \mathbb{R})$, where $\phi \in C_0^1(U, \mathbb{R})$ means that $\phi \in C^1$ and $|\int_U \phi dm| < \infty$.

A necessary and sufficient condition for a function η to be bounded variation (as defined in Definition 4) is that η has *essentially bounded variation* [26], meaning that Definition 4 of bounded variation agrees with Definition 3. Moreover, we show that Definition 4 is also equivalent to Definition 2 via the following proposition.

Proposition 6 *Suppose $\eta \in L^1(U)$, then $\mathrm{var}(\eta) < \infty$ (recalling $\mathrm{var}(\cdot)$ is defined in (1)) if and only if (13) holds for any $\phi \in C_0^1(U, \mathbb{R})$.*

Proof "$\Rightarrow$" Suppose that $\mathrm{var}(\eta) < \infty$, then it is straightforward to see that

$$\left| \int_U \eta(x) \,\mathrm{div}\, \vec{\phi}(x) dx \right| \leq \mathrm{var}(\eta) ||\vec{\phi}||_\infty, \quad \forall \vec{\phi} \in C_c^1(U, \mathbb{R}^d). \tag{14}$$

For any $\phi \in C_c^1(U, \mathbb{R})$, let $\vec{\phi_i} = (\overbrace{0, \cdots,}^{i-1} \phi, \cdots, 0)$, then (14) implies (13). Hence, we can define a continuous linear functional on the linear subspace $C_c^1(U, \mathbb{R}) \subset C_0^1(U, \mathbb{R})$ by

$$\int_U \eta \,\mathrm{div}(\cdot) dx : \vec{\phi} \mapsto \int_U \eta \,\mathrm{div}\, \vec{\phi}\, dx$$

By the *Hahn-Banach theorem*, it follows that this continuous functional extends continuously to $C_0^1(U, \mathbb{R})$.

"$\Leftarrow$" It is straightforward to see that $\mathrm{var}(\eta) \leq d \cdot K < \infty$ for the dimension constant d.
□

Note that the standard Sobolev space $W^{1,1}(U) \subsetneq BV(U)$ and $BV(U)$ with the norm $||\cdot||_{BV} = ||\cdot||_1 + \mathrm{var}(\cdot)$ is a non-separable Banach subspace that is dense in L^1.

A.2 Piecewise rotations

Definition 5 (Piecewise rotation) *[13] Let Ω be a compact convex connected polygon in $\mathbb{C}$. A map $T : \Omega \to \Omega$ is called a* piecewise rotation *with atoms $\mathcal{P} := \{\omega_0, \cdots, \omega_{r-1}\}$ if*

$$T|_{\omega_j} x = \rho_j x + z_j \quad \text{if} \quad x \in \omega_j$$

for some complex numbers: z_j and ρ_j such that $|\rho_j| = 1$ for all $j = 0, 1, \cdots, r-1$. The atoms are assumed to be mutually disjoint convex connected polygons.

It is trivial to see that piecewise rotations are PWIs in $\mathbb{R}^2$ with a topological partition and are homeomorphism when restricting on each atom.

A.3 Interval translation maps and interval exchange transformations

Definition 6 (Interval translation map) *[4] Let* $0 = \beta_0 < \beta_1 < \cdots < \beta_r = 1, I = [0, 1)$ *and for* $i = 0, \cdots, r, \beta_i = [\beta_{i-1}, \beta_i)$*. An* interval translation mapping *is an interval map* $T : I \to I$ *given by*

$$T(x) = x + \gamma_i, \quad \text{if } x \in B_i,$$

where γ_i *are fixed numbers such that* T *maps* I *into itself. Define* $\Omega_0 := I$*, and* $\Omega_n := T(\Omega_{n-1})$*. Then define* $\Omega := \cap_n \Omega_n$*. Particularily, if* $\Omega = \Omega_0 = I$*, then* T *is called an* interval exchange transformation.

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