RANGE DESCRIPTION FOR A SPHERICAL MEAN TRANSFORM ON THE HYPERBOLIC SPACES

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Abstract. We consider the spherical mean transform $\mathcal{R}(f)$ of a function f defined on the hyperbolic space $\mathbb{H}^n$. Let $\mathbb{S} \subset \mathbb{H}^n$ be the unit sphere centered at the origin, and $\mathcal{R}_{\mathbb{S}}$ be the restriction of $\mathcal{R}$ on the sets of spheres centered on $\mathbb{S}$. We prove the range description of the operator $\mathcal{R}_{\mathbb{S}}$. The description consists of the smoothness & support and orthogonality conditions. This description resembles the known result for the Euclidean spaces.

1. Introduction

Let us first recall the unit ball model of the hyperbolic space $\mathbb{H}^n$. It is $\mathbb{B}_E \equiv \{x \in \mathbb{R}^n : |x| < 1\} \subset \mathbb{R}^n$ equipped with the metric [1]:

$$ds^2(x) = \frac{4}{(1-|x|^2)^2} dx^2.$$

The distance $d = d_{\mathbb{H}^n}(x, y)$ between two points x and y is:

$$d_{\mathbb{H}^n}(x, y) = \log \left(\frac{\sqrt{1 - 2\langle x, y\rangle + |x|^2|y|^2} + |x-y|}{\sqrt{1 - 2\langle x, y\rangle + |x|^2|y|^2} - |x-y|} \right).$$

A sphere $\mathbb{S}_r(x) \subset \mathbb{H}^n$ is defined as

$$\mathbb{S}_r(x) = \{y \in \mathbb{H}^n : d_{\mathbb{H}^n}(x, y) = r\}.$$

It is quite easy to observe that $\mathbb{S}_r(x)$ is, in fact, an Euclidean sphere in $\mathbb{B}_E$. However, its (Euclidean) center and radius are not x and r. We now define the spherical mean transform of a function $f \in C_0^\infty(\mathbb{H}^n)$:

$$\mathcal{R}(f)(x, r) = \frac{1}{|\mathbb{S}_r(x)|} \int\limits_{\mathbb{S}_r(x)} f(y) d\sigma(y).$$

Here, $d\sigma(y)$ is the measure induced by the above metrics and $|\mathbb{S}_r(x)|$ is the measure of $\mathbb{S}_r(x)$.

The spherical mean transform on the Euclidean spaces have been intensively investigated due to its applications in PDEs, biomedical/geophysical imaging, and approximation theory, e.g., [Joh81, CH62, LP94, AKQ07, FR06, FR07, FR, AQ96, AFK09, AN, GGG03, EK93, CQ80]. The transform on the hyperbolic spaces also attracted great attentions, e.g. [Vol03, Hel84, BZ80, GGG03].

[1] We follow here the definition in [BZ80]. This is different by a factor of 4 from other standard references, e.g. [Hel84].

Let us consider the Laplace-Beltrami operator on $\mathbb{H}^n$:

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial_i}\left(\sqrt{g} g^{ij} \frac{\partial}{\partial_j}\right) = \frac{1}{4}(1-|x|^2)^n \nabla\left((1-|x|^2)^{2-n}\nabla\right).$$

In terms of polar coordinates,

$$\Delta = \frac{\partial^2}{\partial r^2} + (n-1)\coth(r)\frac{\partial}{\partial r} + \frac{1}{\sinh^2(r)}\Delta_\theta, \tag{1}$$

where $r = d_{\mathbb{H}^n}(x,0)$ and $\theta = \frac{x}{|x|}$. Due to [Ole44], $G(x,r) = \mathcal{R}(f)(x,r)$ satisfies the following Darboux-type equation

$$\begin{cases} [\partial_r^2 + (n-1)\coth(r) - \Delta]G(x,r) = 0, (x,t) \in \mathbb{H}^n \times \mathbb{R}_+. \\ G(x,0) = f(x),\ G_r(x,0) = 0, \quad x \in \mathbb{H}^n. \end{cases} \tag{2}$$

Here, $\mathbb{R}_+$ is the set of nonnegative real numbers. Conversely, if $G(x,t) \in C^\infty(\mathbb{H}^n \times \mathbb{R}_+)$ satisfies the above equation, one can prove that $G(x,t) = \mathcal{R}(f)(x,t)$. Let $\mathcal{R}_{\mathbb{S}}(f)$ be the restriction of $\mathcal{R}$ to the set of spheres centered on the unit sphere $\mathbb{S} \subset \mathbb{H}^n$. The analog of this restriction on the Euclidean spaces plays an important role in thermoacoustic tomography, an emerging biomedical imaging modality (see, e.g., [FPR04, FHR07, KK08]).

In this article, we study the range description of $\mathcal{R}_{\mathbb{S}}$. That is, we investigate the necessary and sufficient conditions of a function g defined on $\mathbb{S} \times \mathbb{R}_+$ such that

$$g(x,r) = \mathcal{R}_{\mathbb{S}}(f)(x,r), \quad (x,t) \in \mathbb{S} \times \mathbb{R}_+,$$

for some function $f \in C_0^\infty(\overline{\mathbb{B}})$. Here, $\mathbb{B}$ is the unit ball in $\mathbb{H}^n$ whose boundary is $\mathbb{S}$.

Similar question for the Euclidean spaces has been resolved. In the two dimensional space $\mathbb{R}^2$, the description was proven by Ambartsoumian and Kuchment [AK06]. It includes the **smoothness & support**, **orthogonality**, and **moment** conditions. Finch and Rakesh [FR07] proved the range description for a related transform in the odd dimensional spaces. It only includes two conditions, the smoothness & support and orthogonality conditions. Agranovsky, Kuchment, and Quinto [AKQ07] then proved the range description for arbitrary dimensions. It includes three conditions: smoothness & support, orthogonality, and moment conditions. They also proved that for odd dimensions, the moment condition is not needed. Then, Agranovsky, Finch, and Kuchment [AFK09] proved that for all dimensions the moment condition follows from the other two. This, indirectly, shows that the smoothness & support and orthogonality conditions are sufficient for the range description in all dimensions. Recently, Agranovsky and the author took a different approach to prove that the two aforementioned conditions are sufficient for the range description [AN]. Our proof relies on the extendibility of solutions of the internal Darboux-Euler-Poisson equation.

Let us return to the question in the hyperbolic space. We now describe the necessary conditions. The first condition, which we call **support & smoothness** condition is quite easy to observe: $g \in C_0^\infty(\mathbb{S} \times [0,2])$. The second one, **orthogonality** condition, comes from the PDE characterization (2) of the spherical mean transform. To fully understand that condition, let us make a detour to some basic harmonic analysis on hyperbolic spaces.

We recall that a horosphere in $\mathbb{H}^n$ is a sphere tangent to $\mathbb{S}_E = \partial\mathbb{B}_E \subset \mathbb{R}^n$ at a point $\eta \in \mathbb{S}_E$. Horospheres in a hyperbolic space play the same role as hyperplanes in an Euclidean space. The tangent points $\eta's$ determine the "normal directions" of the horospheres. Let $x \in \mathbb{B}_E$ and $\eta \in \mathbb{S}_E$, we define:

$$\langle x, \eta \rangle = \log \frac{1-|x|^2}{|x-\eta|^2}.$$

This is the distance between the origin 0 and the horosphere tangent to $\mathbb{S}_E$ at η that contains x. Then for a fix $\mu \in \mathbb{C}$, the function $e^{\mu\langle x,\eta\rangle}$ is a "plane wave" moving along the direction η. For a given number $\lambda \in \mathbb{R}$, letting $\mu = \mu(\lambda) = \frac{i\lambda+(n-1)}{2}$, one has (e.g., [Hel84]):

$$\Delta_x e^{\mu\langle x,\eta\rangle} = -\frac{\lambda^2+(n-1)^2}{4} e^{\lambda\langle x,\eta\rangle}. \tag{3}$$

Let Y_i^m be a spherical harmonic of degree m. That is, $Y_i^m = Y_i^m(\eta)$ is a function on $\mathbb{S}_E$ satisfying the equation

$$\Delta_{\mathbb{S}_E} Y_i^m(\eta) = -m(m+n-2) Y_i^m(\eta).$$

We define the function

$$\Phi_{\lambda,i}^m(x) = \frac{1}{\omega_{n-1}} \int\limits_{\mathbb{S}} e^{\mu\langle x,\eta\rangle} Y_i^m(\eta) d\sigma(\eta). \tag{4}$$

It is known that (e.g., [EH95]), there is a function $h_{\lambda,i}^m : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\Phi_{\lambda,i}^m(x) = h_{\lambda,i}^m(r) Y_i^m(\eta).$$

Here, $r = d_{\mathbb{H}^n}(x,0)$ and $\eta = \frac{x}{|s|}$. Due to (3):

$$\Delta \Phi_{\lambda,i}^m(x) = -(\lambda^2+1)\Phi_{\lambda,m}^i(x).$$

From the polar coordinate decomposition (1), we obtain:

$$\left[\partial_r^2 + (n-1)\coth(r)\partial_r - \frac{m(m+n-2)}{\sinh^2(r)}\right] h_{\lambda,i}^m(r) = -\frac{(n-1)+\lambda^2}{4} h_{\lambda,i}^m(r).$$

Due to (4), we also observe that $h_{\lambda,i}^m(0) = 1$ and $\frac{dh_{\lambda,i}^m}{dr}(0) = 0$. When $m = 0$, for each λ there is only one such function $h_{\lambda,i}^m$, which we will denote by $h(\lambda, .)$. The above equation implies

$$\begin{cases} \left[\partial_r^2 + (n-1)\coth(r)\partial_r\right] h(\lambda,r) = -\frac{(n-1)+\lambda^2}{4} h(\lambda,r), \\ h(\lambda,0) = 1, \quad h_r(\lambda,0) = 0. \end{cases} \tag{5}$$

Due to the uniqueness of the solution of this equation, one deduces $h(\lambda,r) = h(-\lambda,r)$.

It is known that the Laplace-Beltrami operator in $\mathbb{B}$ with zero Dirichlet condition on $\mathbb{S}$ has a discrete set of eigenvalue-eigenvector's $\left\{\left(-\frac{(n-1)^2+\lambda_k^2}{4}, \varphi_k\right)\right\}_{k=1}^{\infty}$. For each k, let $u_k(x,r) = h(\lambda_k,r)\varphi(x)$. Then, u_k satisfies the equation

$$\begin{cases} [\partial_r^2 + (n-1)\coth(r) - \Delta]u(x,r) = 0, \ (x,r) \in \mathbb{B} \times \mathbb{R}_+, \\ u_r(x,0) = 0, \ \forall x \in \mathbb{B}, \\ u(y,r) = 0, \ \forall (y,r) \in \mathbb{S} \times \mathbb{R}_+. \end{cases} \tag{6}$$

We are now ready to describe the **orthogonality** condition. Multiplying the equation (2) by $\sinh^{n-1}(r)u_k(x,r)$ and then taking integration over the domain

$\mathbb{B} \times \mathbb{R}_+$, we obtain

$$\int_{\mathbb{R}_+} \int_{\mathbb{B}} [\partial_r^2 + (n-1)\coth(r)\partial_r - \Delta] G(x,r)] u_k(x,r) \sinh^{n-1}(r) dx dr = 0.$$

Taking integration by parts and using equation (6), we arrive to

$$\int_{\mathbb{R}_+} \int_{\mathbb{S}} g(x,r) \partial_\nu u_k(x,r) \sinh^{n-1}(r) d\sigma(x) dr = 0.$$

Here, ∂_ν is the outward normal derivative. Since $u_k(x,r) = h(\lambda_k, r)\varphi_k(x)$, we obtain the following orthogonality condition:

$$\int_{\mathbb{R}_+} \int_{\mathbb{S}} g(x,r) \partial_\nu \varphi_k(x) h(\lambda_k, r) \sinh^{n-1}(r) d\sigma(x) dr = 0.$$

In this article, we prove that the aforementioned conditions are sufficient for range description of $\mathcal{R}_{\mathbb{S}}$:

Theorem 1.1. *Let g be a function defined on $\mathbb{S} \times \mathbb{R}_+$. Then, there is a function $f \in C_0^\infty(\overline{\mathbb{B}})$ such that $g = \mathcal{R}_{\mathbb{S}}(f)$ if and only if:*

1. *Smoothness & support condition: $g \in C_0^\infty(\mathbb{S} \times [0,2])$.*
2. *Orthogonality condition:*

$$\int_{\mathbb{R}_+} \int_{\mathbb{S}} g(x,r) \partial_\nu \varphi_k(x) h(\lambda_k, r) \sinh^{n-1}(r) d\sigma(x) dr = 0. \tag{7}$$

2. Proof of the main result

The necessity of the condition in Theorem 1.1 was discussed in Section 1. We now prove the sufficiency. Due to the uniqueness of solution of equation (2), $g = \mathcal{R}_{\mathbb{S}}(f)$ if and only if there is a solution G of the following equation:

$$\begin{cases} [\partial_r^2 + (n-1)\coth(r) - \Delta] G(x,r) = 0, \ (x,t) \in \mathbb{H}^n \times \mathbb{R}_+. \\ G(x,0) = f(x), G_r(x,0) = 0, \quad x \in \mathbb{H}^n, \\ G|_{\mathbb{S} \times [0,\infty)} = g. \end{cases}$$

We now prove the existence of such a global solution by following the strategy in [AKQ07] and [AN]. The proof consists of two main steps. The first one is to consider the boundary value time-reversed equation

$$\begin{cases} [\partial_r^2 + (n-1)\coth(r)\partial_r - \Delta] U(x,r) = 0, \text{ in } \mathbb{B} \times (0,2], \\ U(x,r)|_{\mathbb{S} \times [0,2]} = g(x,r), \quad (x,r) \in \mathbb{S} \times (0,2]. \\ U(x,2) = 0, \ U_r(x,2) = 0, \ \forall x \in \mathbb{B}. \end{cases} \tag{8}$$

We will prove that the solution U converges to a function $f^* \in C^\infty(\overline{\mathbb{B}})$ as $r \to 0^+$. The second step is to prove that f^* vanishes up to infinite order on the boundary $\mathbb{S}$ of $\mathbb{B}$. Letting f be the zero extension of f^* to $\mathbb{H}^n$, we will then show that $G = \mathcal{R}(f)$ is the required solution.

2.1. **Extendibility to $r=0$.** We now carry out the first step of the proof by proving:

Theorem 2.1. *Assume the smoothness & support and the orthogonality conditions. Let U be the solution of (8), then there is a function $f^* \in C^\infty(\overline{\mathbb{B}})$ such that $U(.,r) \to f^*$. The convergence is in the topology of $C^\infty(\overline{\mathbb{B}})$.*

Equation (8) is hyperbolic for all $r > 0$. Standard theory for hyperbolic equation shows that $U \in C^\infty(\overline{\mathbb{B}} \times (0,2])$. However, when $r = 0$ the equation is singular. This prevents us from using any standard theory to derive the regularity of the solution at $r = 0$. Our strategy is to get rid of that singularity by transforming (8) to a wave equation.

Let us introduce the following analog of the Fourier-Bessel transform

$$\mathcal{F}(k)(\lambda) = \widehat{k}(\lambda) = \int\limits_0^\infty k(r) h(\lambda, r) \sinh^{n-1}(r) dr,$$

where $h(\lambda, r)$ is defined in equation (5). Although in the above definition we only need the values of k on $[0,\infty)$, we will always consider k as an even function $\mathbb{R} \to \mathcal{B}$, where $\mathcal{B}$ is some Banach space. Let us extend $g(x,r)$ to an even function in r. The orthogonality condition (7) is equivalent to:

$$\int\limits_{\mathbb{S}} \widehat{g}(x, \lambda_k) \partial_\nu \varphi_k(x) d\sigma(x) = 0. \tag{9}$$

Here, $\widehat{g}(x,\lambda)$ is the $\mathcal{F}$ transform of g with respect to the variable r. The following result is analogous to [AKQ07, Lemma 1]:

Lemma 2.2. *Let $\Phi(\lambda)$ be a $\mathcal{B}$-valued function on $\mathbb{R}$. Then $\Phi = \mathcal{F}(k)$ where $k \in C^\infty(\mathbb{R})$ is an even function supported in $[-a,a]$ if and only if the following conditions are satisfied:*

1. *$\Phi(\lambda)$ is even λ.*
2. *$\Phi(\lambda)$ extends to an entire function in $\mathbb{C}$ with Paley-Wiener estimates*

$$\|\Phi(\lambda)\| \leq C_N (1 + |\lambda|)^{-N} e^{a|\Im(\lambda)|}.$$

When $\mathcal{B} = \mathbb{C}$, the above result follows directly from the Paley-Wiener theorem for the Fourier transform in hyperbolic space (see [Hel84]). In the general case, the proof can be proceeded as that of [AKQ07, Lemma 1]. We now apply it to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\widehat{U} = \mathcal{F}(U)$ be the $\mathcal{F}$ transform of U with respect to the variable r. Straight forward calculations give:

$$\mathcal{F}\left[(\partial_{rr} + (n-1)\coth(r)\partial_r) U\right] = -\lambda^2 \mathcal{F}(U) = -\lambda^2 \widehat{U}.$$

Let us at the moment assume that $U \in C^\infty(\overline{\mathbb{B}} \times \mathbb{R}_+)$. From equation (8), we obtain

$$\begin{cases} \lambda^2 \widehat{U}(x,\lambda) + \Delta \widehat{U}(x,\lambda) = 0, \; x \in \mathbb{B}, \\ \widehat{U}(x,\lambda) = \widehat{g}(x,\lambda), \; x \in \mathbb{S}. \end{cases} \tag{10}$$

Applying the inverse Fourier transform to the variable λ, we arrive to

$$\begin{cases} V_{tt}(x,t) - \Delta V(x,t) = 0, \; x \in \mathbb{B}, \\ V(x,t) = h(x,t), \quad x \in \mathbb{S}. \end{cases} \tag{11}$$

Here, $h(x,t)$ is the inverse one dimensional Fourier transform of the function $\widehat{g}(x,\lambda)$ with respect to λ. In order to prove that U extends smoothly to $r=0$, we first prove that the above equation has a solution $V \in C^\infty(\overline{\mathbb{B}} \times \mathbb{R})$ which is even in t and $supp(V) \subset \overline{\mathbb{B}} \times [-2,2]$.

Let us consider the following initial value problem:

$$\begin{cases} V_{tt}(x,t) - \Delta V(x,t) = 0, \ x \in \mathbb{B}, \ t \geq -2, \\ V(x,t) = h(x,t), \ x \in \mathbb{S}, \\ V(x,-2) = 0, \ V_t(x,-2) = 0, \ x \in \mathbb{B}. \end{cases} \tag{12}$$

For any $s > 0$, due to the smoothness & support condition: $g \in C^\infty(\mathbb{R}, H^s(\mathbb{S}))$ and $supp(g) \subset [-2,2]$. From Lemma 2.2, $\widehat{g} : \mathbb{R} \to H^s(\mathbb{S})$ is even in λ and extends to an entire function with the Paley-Wiener estimate:

$$\|\widehat{g}(.,\lambda)\|_{H^s(\mathbb{S})} \leq C(1+|\lambda|)^{-N} e^{2|\Im(\lambda)|}.$$

The Paley-Wiener theory for the Fourier transform then implies that

$$h \in C^\infty(\mathbb{R}, H^s(\mathbb{S})) \text{ and } supp(h) \subset [-2,2] \times \mathbb{S}.$$

Standard theory for wave equation show that the equation (12) has a solution $V \in C^\infty([-2,\infty), H^s(\mathbb{B}))$. Extending the solution by zero for $t \leq -2$, we obtain $V \in C^\infty(\mathbb{R}, H^s(\mathbb{B}))$. We now prove that V is even in t. In order to do so, we translate the equation to an equation with homogeneous boundary value. Let $W = V - E(h)$, where $E(h)(.,t)$ is the harmonic extension of $h(.,t)$ to $\overline{\mathbb{B}}$. We arrive to

$$\begin{cases} W_{tt}(x,t) - \Delta W(x,t) = P(x,t), (x,t) \in \mathbb{B} \times \mathbb{R}, \\ W(x,t) = 0, \ x \in \mathbb{S}, \\ W(x,-2) = W_t(x,-2) = 0, \ x \in \mathbb{B}. \end{cases} \tag{13}$$

Here,

$$P(x,t) = \partial_t^2 E(g)(x,t) = E(h_{tt})(x,t)$$

is the harmonic extension of h_{tt}. Let $\{\varphi_k\}_k$ be an orthonormal basis of $L^2(\mathbb{B})$ consisting of eigenvectors of the Dirichlet Laplace-Beltrami operator Δ_D. We now expand the functions W and P in terms of φ_k:

$$W(x,t) = \sum_k \omega_k(t)\varphi_k(x), \quad P(x,t) = \sum_k p_k(t)\varphi_k(x).$$

Equation (13) reduces to

$$\begin{cases} \omega_k''(t) + \lambda_k^2 \omega_k(t) = p_k(t), \\ \omega_k(-2) = \omega_k'(-2) = 0. \end{cases} \tag{14}$$

Here, $-\lambda_k^2$ is the eigenvalue of Δ_D corresponding to φ_k. The function p_k is determined by

$$p_k(x) = \int_B P(x,t)\varphi_k(x) = \frac{-1}{\lambda_k^2} \int_B P(x,t)\Delta\varphi_k(x)dx.$$

Taking integration by parts, we obtain

$$\begin{aligned} p_k(x) &= \frac{-1}{\lambda_k^2} \left\{ \int_S \left[P(x,t)\partial_\nu \varphi_k(x) - \partial_n P(x,t)\varphi_k(x)\right] d\sigma(x) + \int_B \Delta P(x,t)\varphi_k(x)dx \right\} \\ &= \frac{-1}{\lambda_k^2} \int_S P(x,t)\partial_\nu\varphi_k(x)d\sigma(x). \end{aligned}$$

Since $P(x,t)$ is the (harmonic) extension of $h_{tt}(x,t)$:

$$p_k(t) = \frac{-1}{\lambda_k^2}\int\limits_S h_{tt}(x,t)\partial_\nu \varphi_k(x) d\sigma(x). \tag{15}$$

Let $\alpha = \alpha(t)$ be a function such that

$$(-\lambda^2 + \lambda_k^2)\widetilde{\alpha}(\lambda) = \widetilde{p}_k(\lambda).$$

Here, $\widetilde{\alpha}$ and $\widetilde{p}_k$ are the Fourier transforms of α and p_k. Since $p_k \in C^\infty(\mathbb{R})$ and $supp(p_k) \subset [-2,2]$, $\widetilde{p}_k$ is an even function and of Paley-Wiener type $R = 2$. From (15):

$$\widetilde{p}_k(\lambda) = \frac{\lambda^2}{\lambda_k^2}\int\limits_S \widetilde{h}(x,\lambda)\partial_\nu \varphi_k(x) d\sigma(x) = \frac{\lambda^2}{\lambda_k^2}\int\limits_S \widehat{g}(x,\lambda)\partial_\nu \varphi_k(x) d\sigma(x).$$

The last equation is due to the fact that $h(x,t)$ is the inverse Fourier transform of $\widehat{g}$. Due to the orthogonality condition (9), $\widetilde{p}_k(\lambda_k) = 0$. Therefore, $\widetilde{\alpha} = \frac{\widetilde{p}_k(\lambda)}{\lambda^2-\lambda_k^2}$ is also an even function and of the Paley-Wiener type $R = 2$. Hence, $\alpha(t)$ is even and supported inside $[-2,2]$. It is easy to see that α solves the equation (14). Hence, $\omega_k = \alpha$ due to the uniqueness of the solution of the equation. This implies that ω_k is even in t for all k. Therefore, $W(x,t)$ is even in t. We then conclude that there is a solution $V \in C^\infty(\overline{\mathbb{B}} \times \mathbb{R})$ of (11) satisfying V is even in t and $supp(V) \subset \overline{\mathbb{B}} \times [-2,2]$.

We now carry out the second step. Let us consider the equation

$$\begin{cases} [\partial_r^2 + (n-1)\coth(r)\partial_r - \Delta]U(x,r) = 0, \\ U(x,t)|_{S\times[0,2]} = g(x,t). \end{cases}$$

It suffices to prove that there is a solution $U \in C^\infty(\overline{\mathbb{B}} \times \mathbb{R})$ satisfying U is even in r and $supp(U) \subset \overline{\mathbb{B}} \times [-2,2]$. This solution, in indeed, is obtained from V by taking the Fourier transform and then the inverse of the Fourier-Bessel transform. Due to Lemma 2.2, U is even, $supp(U) \subset \overline{\mathbb{B}} \times [-2,2]$, and $U \in C^\infty(\mathbb{R}, H^s(\mathbb{B}))$. Since the last property is true for all s, $U \in C^\infty(\overline{\mathbb{B}} \times [-2,2])$. To finish the proof of Theorem 2.1, we let $f^* = U(x,0)$. □

2.2. **Vanishing on the boundary.** In the previous section, we show that there is a function $f^* \in C^\infty(\overline{\mathbb{B}})$ such that the equation:

$$\begin{cases} [\partial_r^2 + (n-1)\coth(r)\partial_r - \Delta]U(x,r) = 0, \; (x,t) \in \mathbb{B}\times\mathbb{R}, \\ U(x,r)|_{\mathbb{S}\times\mathbb{R}} = g(x,r), \quad (x,r) \in \mathbb{S}\times\mathbb{R}, \\ U(x,0) = f^*(x), \; U_r(x,0) = 0, \; \forall x \in \mathbb{B}, \end{cases} \tag{16}$$

has a solution $U \in C^\infty(\overline{\mathbb{B}} \times \mathbb{R}_+)$ even in r satisfying $supp(U) \in \overline{\mathbb{B}} \times [-2,2]$. We now prove that:

Theorem 2.3. *The function f^* vanishes up to infinite order on $\mathbb{S}$.*

Let us look at the spherical harmonics expansion of U and f^*:

$$U(x,t) = \sum_{m=0}^{\infty}\sum_{i=1}^{l_m} U_{m,i}(s,t)Y_l^m(\theta), \quad f^*(x) = \sum_{m=0}^{\infty}\sum_{i=1}^{l_m} f_{m,i}(s)Y_l^m(\theta).$$

Here, Y_i^m is a spherical harmonics of degree m, $s = d_{\mathbb{H}^n}(x,0)$, and $\theta = \frac{x}{|x|} \in \mathbb{S}_E$. It suffices to prove that $f_{m,i}$ vanishes up to infinite order at $s = 1$ for all m,i. For the sake of convenience, from now on, we will drop the un-relevant index i in $f_{m,i}$. Let us define the operator

$$\mathcal{D}_m = d_s^2 + (n-1)\coth(s) d_s - \frac{m(m+n-2)}{\sinh^2 s}.$$

Since $\Delta_\theta Y^m(\theta) = -m(m+n-2)Y^m(\theta)$, for any function $\alpha = \alpha(s)$:

$$\Delta[\alpha(s) Y^m(\theta)] = (\mathcal{D}_k \alpha)(s) Y^m(\theta).$$

From equation (16), we obtain $U(r,s)$ is even in each variable and:

$$\begin{cases} (\mathcal{D}_{0,r} - \mathcal{D}_{m,s}) U_m(s,r) = 0, \text{ in } [-1,1] \times \mathbb{R}, \\ U_m(\pm 1, r) = g_m(r), \quad r \in \mathbb{R}. \\ U_m(s,0) = f_m(s), \ \partial_r U_m(s,0) = 0, \ \forall s \in [-1,1]. \end{cases} \tag{17}$$

Here, $\mathcal{D}_{0,r}$ is the operator $\mathcal{D}_0$ applying to the variable r and $\mathcal{D}_{m,s}$ is $\mathcal{D}_m$ applying to s.

Lemma 2.4. *For any $l \geq 0$, one has*

$$\left[\mathcal{D}_m^l f_m\right](1) = 0. \tag{18}$$

Proof. By iterating equation (17) l times, we obtain

$$\mathcal{D}_{m,s}^l U_m(s,r) = \mathcal{D}_{0,r}^l U_m(s,r).$$

At $(r,s) = (1,0)$, we obtain

$$\mathcal{D}_m^l f_m(1) = \mathcal{D}_0^l g_m(0).$$

Since $g_m(r)$ vanishes up to infinite order at $r = 0$, the above equation gives $\mathcal{D}_m^l f_m(1) = 0$. □

Lemma 2.5. *Let $\Gamma_k = d_s + (n+k-2)\coth(s)$ and*

$$\mathcal{Q}_m = \prod_{k=1}^m \Gamma_k.$$

Then

$$\left[d_s^l \mathcal{Q}_m(f_m)\right](1) = 0, \quad \forall l \geq 0. \tag{19}$$

This result is similar to [AN, Lemma 4.1, i)] for the Euclidean spaces. The later result was proved by applying the argument in [EK93]. That is to use the formulas of projections on the spaces on spherical harmonics of certain degree. In principle, we can apply similar argument for the hyperbolic spaces, due to the conformal mapping between $\mathbb{H}^n$ and the unit ball in $\mathbb{R}^n$. However, we present here a different proof, which comes directly from the equation (17). Let us state here the following result, whose proof is provided in Appendix.

Proposition 2.6. *We have the following identity*

$$\Gamma_k \mathcal{D}_k = \mathcal{D}_{k-1} \Gamma_k.$$

Proof of Lemma 2.5. Due to equation (16) and the fact that $U(x,t) = U_r(x,t) = 0$ for all $x \in \overline{\mathbb{B}}$ and $t \geq 2$, the domain of dependence argument implies $U(0,t) = 0$ for all $t \geq 0$. This, in turn, gives

$$U_m(0,t) = 0, \quad \forall t \geq R. \tag{20}$$

Recall from equation (17):

$$\mathcal{D}_{m,s}U_m(s,r) = \mathcal{D}_{0,r}U_m(s,r).$$

Now apply the operator Γ_m (with respect to variable s) to this equation we obtain

$$\Gamma_m\mathcal{D}_{m,s}U_m(s,r) = \Gamma_m\mathcal{D}_{0,r}U_m(s,r).$$

Due to Proposition 2.6, arrive to

$$\mathcal{D}_{m-1,s}\Gamma_m U_m(s,r) = \mathcal{D}_{0,r}\Gamma_m U_m(s,r).$$

By induction, we obtain

$$\mathcal{D}_{0,s}[\mathcal{Q}_m U_m(s,r)] = \mathcal{D}_{0,r}[\mathcal{Q}_m U_m(s,r)]. \tag{21}$$

Let us recall the following result from [Hel59]:

Proposition 2.7. *Let $v = v(s,r) \in C^\infty(\mathbb{R}\times\mathbb{R})$ be even with respect to each variable. If $\mathcal{D}_{0,s}v(s,t) = \mathcal{D}_{0,r}v(s,t)$ for all $(s,r)\in\mathbb{R}\times\mathbb{R}$, then $v(s,r) = v(r,s)$ for all $(s,r)\in\mathbb{R}\times\mathbb{R}$.*

Let us finish the proof of Lemma 2.5. Due to equation (21) and Proposition 2.7, we obtain

$$[\mathcal{Q}_m U_m](0,s) = [\mathcal{Q}_m U_m](s,0) = [\mathcal{Q}_m f_m](s).$$

Due to (20), we obtain

$$d_s^l[\mathcal{Q}_m f_m](1) = d_s^l[\mathcal{Q}_m U_m](0,1) = 0.$$

□

The main ingredient for proof of Theorem 2.3 is the following lemma:

Lemma 2.8. *Let f_m be a function defined on a neighborhood of $s=1$ such that for all $l = 0,..,m-1$:*

$$[d_s^l\mathcal{Q}_m f_m](1) = [\mathcal{D}_m^l f_m](1) = 0.$$

Then, $f_m^{(i)}(1) = 0$, for all $i = 0,..,2m-1$.

The proof of this lemma is quite involved, which constitutes the most difficult part of this paper. It will be presented in the next section. Here we use it to prove Theorem 2.3.

Proof of Theorem 2.3. Due to Lemmas 2.4, 2.5, and 2.8, we obtain $f_m^{(i)}(1) = 0$ for all $i = 0, 2m-1$. Choosing $l = m$ in (19), we obtain that $f_m^{(2m)}(1) = 0$. By choosing $l = m+1, ...$ and using the induction argument, we obtain $f_m^{(i)}(1) = 0$ for all $i \geq 0$. Therefore, due to the uniform convergence,

$$f(x) = \sum_{m=0}^{\infty} f_{m,i}(s)Y_i^m(\theta)$$

vanishes up to infinite order at $x\in\mathbb{S}$. □

2.3. **Finishing the proof of Theorem 1.1.** We now complete the proof of Theorem 1.1. Extending f^* by zero outside $\mathbb{B}$ to a function f, we obtain $f \in C_0^\infty(\mathbb{H}^n)$. We now prove that $g = \mathcal{R}_{\mathbb{S}}(f)$. Indeed, let $G := \mathcal{R}(f) \in C^\infty(\mathbb{H}^n \times [0,\infty))$. Then, G satisfies the equation [Ole44]

$$\begin{cases} [\partial_r^2 + (n-1)\coth(r)\partial_r - \Delta]G(x,r) = 0, \ (x,t) \in \mathbb{H}^n \times \mathbb{R}_+. \\ G(x,0) = f(x), G_r(x,0) = 0, \quad x \in \mathbb{H}^n. \end{cases}$$

We recall that the following equation

$$\begin{cases} [\partial_r^2 + (n-1)\coth(r)\partial_r - \Delta]U(x,r) = 0, \text{ in } \mathbb{B} \times \mathbb{R}_+, \\ U(x,r)|_{S\times[0,2]} = g(x,r), \quad (x,r) \in \mathbb{S} \times \mathbb{R}_+. \\ U(x,0) = f^*(x), \ U_r(x,0) = 0, \ \forall x \in \mathbb{B}, \end{cases}$$

has a unique solution $U \in C^\infty(\overline{\mathbb{B}} \times \mathbb{R}_+)$ satisfying $U(x,t) = 0$ for $t \geq 2$.

Let $H(x,t) = G(x,t) - U(x,t)$, then:

$$\begin{cases} [\partial_r^2 + (n-1)\coth(r)\partial_r - \Delta]H(x,r) = 0, \ (x,t) \in \mathbb{B} \times \mathbb{R}_+. \\ H(x,0) = 0, \ H_r(x,0) = 0, \quad x \in \mathbb{B}. \end{cases}$$

The domain of dependence argument then shows that $U(x,t) = 0$ in the downward cone

$$\mathcal{K}_- = \{(x,t) : t \geq 0, d_{\mathbb{H}^n}(x,0) + t \leq 1\}$$

On the other hand, $f \in C_0^\infty(\overline{\mathbb{B}})$ implies that $G(x,r) = 0$ for all $x \in \mathbb{B}$ and $r \geq 2$. Therefore, H satisfies the following time-reversed equation

$$\begin{cases} [\partial_r^2 + (n-1)\coth(r)\partial_r - \Delta]H(x,r) = 0, \quad (x,t) \in \mathbb{B} \times (0,2], \\ H(x,2) = 0, \ H_r(x,2) = 0, \end{cases}$$

The domain of dependence argument shows that $H(x,t) = 0$ in the upward cone

$$\mathcal{K}_+ = \{(x,t) : t \leq 2, t - d_{\mathbb{H}^n}(x,0) \geq 1\}.$$

Therefore, $U(x,t) = 0$ in $\mathcal{K}_+ \cup \mathcal{K}_-$. This, in particular, implies $D_x^\alpha H(x,t) = 0$ for all $(x,t) \in \{0\} \times \mathbb{R}_+$. Now we apply the transform $\mathcal{F}$ with respect to the variable t to H:

$$\lambda^2 \widehat{H}(x,\lambda) + \Delta \widehat{H}(x,\lambda) = 0, \ x \in \mathbb{B}.$$

This shows that $\widehat{H}(\lambda,.)$ is analytic in $\mathbb{B}$. Since $D_x^\alpha H(t,0) = 0$ for all $t \in \mathbb{R}_+$, $D_x^\alpha \widehat{H}(\lambda,0) = 0$ for any α. Therefore, $\widehat{H}(\lambda,x) = 0$ for all $x \in \mathbb{B}$. Now taking the inverse Fourier-Bessel transform we obtain $H(x,t) = 0$ for all $x \in \mathbb{B}, t \in \mathbb{R}_+$. Now the continuity of H implies $H(x,t) = 0$ for all $\overline{\mathbb{B}} \times \mathbb{R}_+$. This, in particular, shows $\mathcal{R}_{\mathbb{S}}(x,t) = G(x,t) = U(x,t) = g(x,t)$ for all $(x,t) \in \mathbb{S} \times \mathbb{R}_+$.

3. Proof of Lemma 2.8

We now prove Lemma 2.8, which is the cornerstone of this article. We first state some auxiliary results:

Proposition 3.1. *For all $i = 0,..,m-1$, let $u_i(s) = \cosh^i(s)\sinh^{-n-m+2}(s)$. Then,*

$$(\mathcal{Q}_m u_i)(1) = 0.$$

Proof. We have

$$\begin{aligned}\Gamma_k \left[\cosh^i(s)\sinh^{-k-n+2}(s)\right] &= [d_s + (n+k-2)\coth(s)]\left[\cosh^i(s)\sinh^{-n-k+2}(s)\right] \\ &= i\cosh^{i-1}(s)\sinh^{-n-(k-1)+2}(s).\end{aligned}$$

Therefore,

$$\mathcal{Q}_m \left[\cosh^i(s)\sinh^{-k-n+2}(s)\right] = \left[\prod_{j=1}^{m-i-1}\Gamma_j\right]\Gamma_{m-i}...\Gamma_m\left[\cosh^i(s)\sinh^{-n-m+2}(s)\right]$$

$$= i\left[\prod_{j=1}^{m-i-1}\Gamma_j\right]\Gamma_{m-i}...\Gamma_{m-1}\left[\cosh^{i-1}(s)\sinh^{-n-(m-1)+2}(s)\right]$$

$$= i!\left[\prod_{j=1}^{m-i-1}\Gamma_j\right]\Gamma_{m-i}\left[\sinh^{-n-(m-i)+2}(s)\right] = 0.$$

□

Proposition 3.2. *For all $i = 0,..,m-1$:*

$$(\mathcal{D}_m - \kappa_i)u_i(s) = -i(i-1)u_{i-2},$$

where $\kappa_i = (m-i-1)(m+n-2-i)$.

A proof of this proposition will be provided in Appendix. We now prove Lemma 2.8.

Proof of Lemma 2.8. For $l = 0,..,m-1$, we can write

$$\left[d_s^l Q_m f_m\right](1) = \sum_{i=0}^{2m-1} A_{l,i} f_m^{(i)}(1),$$

and

$$[\mathcal{D}_m^l f_m](1) = \sum_{i=0}^{2m-1} B_{l,i} f_m^{(i)}(1).$$

We denote by A_l, B_l the corresponding row vectors $A_l = (A_{l,i})_{i=0,2m-1}$, $B_l = (B_{l,i})_{i=0,2m-1}$. It suffices to prove that $\{A_l, B_l\}_{l=0}^{m-1}$ is linearly independent. Indeed, assume that $\{\alpha_l, \beta_l\}_{l=0}^{m-1}$ is such that

$$\sum_{l=0}^{m-1}\alpha_l A_l + \sum_{l=0}^{m-1}\beta_l B_l = 0. \tag{22}$$

We now prove that $\alpha_l = \beta_l = 0$ for all $l = 0,..,m-1$. Let

$$P_1(x) = \sum_{l=0}^{m-1}\alpha_l x^l, \quad P_2(x) = \sum_{l=0}^{m-1}\beta_l x^l.$$

From (22), we obtain

$$[P_1(d_s)\mathcal{Q}_m u](1) + [P_2(\mathcal{D}_m)u](1) = 0,$$

for any function u smooth at $s = 1$. Let $u = u_i$, we obtain

$$[P_1(d_s)\mathcal{Q}_m u_i](1) + [P_2(\mathcal{D}_m)u_i](1) = 0.$$

Due to Propositions 3.1, the first term of the left hand side is zero. We, thus, obtain

$$[P_2(\mathcal{D}_m)u_i]\,(1) = 0, \quad \forall i = 0,..,m-1. \tag{23}$$

We now prove that

$$P_2(x) = Q(x)\prod_{i=0}^{m-1}(x-\kappa_i),$$

where Q is a polynomial and $\kappa_i's$ are defined in Proposition 3.2. Indeed, let $i=0$ in equation (23). Due to Proposition 3.2, we deduce

$$P_2(\kappa_0)u_0(1) = 0.$$

Since $u_0(1) \neq 0$, we obtain $P_2(\kappa_0) = 0$. The same argument then shows that $P_2(\kappa_1) = 0$. We now prove that P_2 is divisible by $(x-\kappa_i)$ for all $2 \leq i \leq m-1$ by induction. Indeed, assume that it is true for all $k \leq i-1$. We obtain

$$P_2(x) = Q(x)\prod_{p=1}^{\lfloor \frac{i}{2} \rfloor}(x-\kappa_{i-2p}).$$

Here, $\lfloor \frac{i}{2} \rfloor$ is the the integer part of $\frac{i}{2}$, which is the biggest integer less than or equal to $\frac{i}{2}$. Let us write $Q(x) = R(x)(x-\kappa_i) + C$, where C is a constant. We arrive at

$$P_2(x) = R(x)\prod_{p=0}^{\lfloor \frac{i}{2} \rfloor}(x-\kappa_{i-2p}) + C\prod_{p=1}^{\lfloor \frac{i}{2} \rfloor}(x-\kappa_{i-2p}).$$

Therefore,

$$P_2(\mathcal{D}_m)u_i = R(\mathcal{D}_m)\prod_{p=0}^{\lfloor \frac{i}{2} \rfloor}(\mathcal{D}_m-\kappa_{i-2p})u_i + C\prod_{p=1}^{\lfloor \frac{i}{2} \rfloor}(\mathcal{D}_m-\kappa_{i-2p})u_i. \tag{24}$$

Due to Proposition 3.2, we deduce

$$\begin{aligned}\prod_{p=0}^{\lfloor \frac{i}{2} \rfloor}(\mathcal{D}_m-\kappa_{i-2p})u_i &= \left[\prod_{p=1}^{\lfloor \frac{i}{2} \rfloor}(\mathcal{D}_m-\kappa_{i-2p})\right][(\mathcal{D}_m-\kappa_i)u_i] \\ &= i(i-1)\left[\prod_{p=1}^{\lfloor \frac{i}{2} \rfloor}(\mathcal{D}_m-\kappa_{i-2p})\right]u_{i-2}.\end{aligned}$$

Continuing the argument, we obtain

$$\prod_{p=0}^{\lfloor \frac{i}{2} \rfloor}(\mathcal{D}_m-\kappa_{i-2p})u_i = \begin{cases} i!(\mathcal{D}_m-\kappa_1)u_1, & \text{if } i \text{ is odd,} \\ i!(\mathcal{D}_m-\kappa_0)u_0, & \text{if } i \text{ is even.}\end{cases}$$

Applying Proposition 3.2 once more, we conclude

$$\prod_{p=0}^{\lfloor \frac{i}{2} \rfloor}(\mathcal{D}_m-\kappa_{i-2p})u_i = 0.$$

Hence, equation (24) gives:

$$P_2(\mathcal{D}_m)u_i = C\prod_{p=1}^{\lfloor \frac{i}{2} \rfloor}(\mathcal{D}_m-\kappa_{i-2p})u_i = (-1)^{\lfloor \frac{i}{2} \rfloor}C\prod_{p=1}^{\lfloor \frac{i}{2} \rfloor}(\kappa_{i-2p}-\mathcal{D}_m)u_i.$$

Due to (23), we arrive to

$$C\left[\prod_{p=1}^{\lfloor\frac{i}{2}\rfloor}(\kappa_{i-2p}-\mathcal{D}_m)u_i\right](1)=0. \tag{25}$$

Let us recall that Proposition 3.2 gives

$$(\kappa_j-\mathcal{D}_m)u_k=(\kappa_j-\kappa_k)u_k+k(k-1)u_{k-2}.$$

Therefore,

$$\begin{aligned}\prod_{p=1}^{\lfloor\frac{i}{2}\rfloor}(\kappa_{i-2p}-\mathcal{D}_m)u_i &= \left[\prod_{p=2}^{\lfloor\frac{i}{2}\rfloor}(\kappa_{i-2p}-\mathcal{D}_m)\right][(\kappa_{(i-2)}-\mathcal{D}_m)u_i]\\ &= \left[\prod_{p=2}^{\lfloor\frac{i}{2}\rfloor}(\kappa_{i-2p}-\mathcal{D}_m)\right][(\kappa_{i-2}-\kappa_i)u_i+i(i-1)u_{i-2}]\\ &= (\kappa_{i-2}-\kappa_i)\left[\prod_{p=2}^{\lfloor\frac{i}{2}\rfloor}(\kappa_{i-2p}-\mathcal{D}_m)\right]u_i\\ &+ i(i-1)\left[\prod_{p=2}^{\lfloor\frac{i}{2}\rfloor}(\kappa_{i-2p}-\mathcal{D}_m)\right]u_{i-2}.\end{aligned}$$

We notice here that κ_i strictly decreases in i. Hence, the coefficient in front of the next to last bracket is positive and the one in front of the last bracket is nonnegative. Continuing the expansion for $p=2,...,\lfloor\frac{i}{2}\rfloor$, we obtain

$$\prod_{i=1}^{\lfloor\frac{i}{2}\rfloor}(\kappa_{i-2p}-\mathcal{D}_m)u_i=\sum_{p=0}^{[\frac{i}{2}]}c_{i-2p}u_{i-2p},$$

where $c_i>0$ and all other coefficients $c_k's$ are nonnegative. Since $u_j(1)>0$ for all $j=0,..,i$, we obtain then

$$\left[\prod_{i=1}^{\lfloor\frac{i}{2}\rfloor}(\kappa_{i-2p}-\mathcal{D}_m)u_i\right](1)=\sum_{p=0}^{\lfloor\frac{i}{2}\rfloor}c_{i-2p}u_{i-2p}(1)>0.$$

From (25), we then derive $C=0$. Thefore, P_2 is divisible by $(x-\kappa_i)$, for $i=0,..,m-1$. Since κ_i are pairwise different, we obtain the factorization

$$P_2(x)=Q(x)\prod_{i=0}^{m-1}(x-\kappa_i).$$

We now recall that P_2 is a polynomial of degree at most $m-1$. This shows $P_2=0$, and so $\beta_l=0$ for all $i=0,..,m-1$. From (22), we arrive at

$$\sum_l \alpha_l A_l=0.$$

It is easy to observe that $A_{l,m+l}=1$ and $A_{l,j}=0$ for all $j>m+l$. Hence, the above equation gives $\alpha_l=0$ for all $l=0,..,m-1$. This shows the linear independence of $\{A_l,B_l\}_{l=0}^{m-1}$. It also finishes the proof of the lemma. □

Appendix

3.1. **Proof of Proposition 2.6.**

Proof. Let

$$\begin{aligned}
A &= d_s\mathcal{D}_k = d_s\left[d_s^2 + (n-1)\coth(s)d_s - \frac{k(k+n-2)}{\sinh^2 s}\right],\\
B &= (n+k-2)\coth(s)\mathcal{D}_k\\
&= (n+k-2)\coth(s)\left[d_s^2 + (n-1)\coth(s)d_s - \frac{k(k+n-2)}{\sinh^2 s}\right],\\
C &= \mathcal{D}_{k-1}d_s\\
&= \left[d_s^2 + (n-1)\coth(s)d_s - \frac{(k-1)(k+n-3)}{\sinh^2 s}\right]d_s,\\
D &= \mathcal{D}_{k-1}(n+k-2)\coth(s)\\
&= (n+k-2)\left[\partial_s^2 + (n-1)\coth(s)d_s - \frac{(k-1)(k+n-3)}{\sinh^2 s}\right]\coth(s).
\end{aligned}$$

We have

$$\begin{aligned}
A-C &= d_s\left[d_s^2 + (n-1)\coth(s)d_s - \frac{k(k+n-2)}{\sinh^2 s}\right],\\
&- \left[d_s^2 + (n-1)\coth(s)d_s - \frac{(k-1)(k+n-3)}{\sinh^2 s}\right]d_s\\
&= -(n-1)\frac{1}{\sinh^2 s}d_s + 2k(k+n-2)\sinh^{-3}(s)\cosh(s)d_s\\
&+ \left[-k(k+n-2) + (k-1)(k+n-3)\right]\frac{1}{\sinh^2 s}\\
&= -2(n+k-2)\frac{1}{\sinh^2 s}d_s + 2k(k+n-2)\sinh^{-3}(s)\cosh(s).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
D-B &= (n+k-2)\left[d_s^2 + (n-1)\coth(s)d_s - \frac{(k-1)(k+n-3)}{\sinh^2 s}\right]\coth(s)\\
&- (n+k-2)\coth(s)\left[d_s^2 + (n-1)\coth(s)d_s - \frac{k(k+n-2)}{\sinh^2 s}\right].\\
&= (n+k-2)\left[d^2\coth(s) + 2\partial_s\coth(s)d_s + (n-1)\coth(s)d_s\coth(s)\right]\\
&+ (n+k-2)\left[k(n+k-2) - (k-1)(n+k-3)\right]\sinh^{-2}(s)\coth(s)\\
&= (n+k-2)\left[-2\sinh^{-2}(s)d_s + 2k\sinh^{-3}(s)\cosh(s)\right].
\end{aligned}$$

We conclude that

$$A - C = D - B.$$

This proves the proposition. ∎

3.2. **Proof of Proposition 3.2.**

Proof. We have

$$\begin{aligned}
d_s[\cosh^i(s)\sinh^l(s)] &= i\cosh^{i-1}(s)\sinh^{l+1}(s) + l\cosh^{i+1}(s)\sinh^{l-1}(s)\\
&= (i+l)\cosh^{i-1}(s)\sinh^{l+1} + l\cosh^{i-1}\sinh^{l-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}(n-1)\coth(s)d_s[\cosh^i(s)\sinh^l(s)] &= (n-1)(i+l)\cosh^i(s)\sinh^l \\ &+ (n-1)l\cosh^i(s)\sinh^{l-2}(s).\end{aligned}$$

$$\begin{aligned}d_s^2[\cosh^i(s)\sinh^l(s)] &= d_s^2(\cosh^i(s))\sinh^l(s) + 2d_s\cosh^i(s)d_s\sinh^l(s) \\ &+ \cosh^i(s)d_s^2\sinh^l(s) \\ &= [i^2\cosh^i(s) - i(i-1)\cosh^{i-2}(s)]\sinh^l(s) \\ &+ 2il\cosh^i\sinh^l(s) \\ &+ \cosh^i(s)[l^2\sinh^l(s) + l(l-1)\sinh^{l-2}(s)] \\ &= (i+l)^2\cosh^i(s)\sinh^l(s) - i(i-1)\cosh^{i-2}(s)\sinh^l(s) \\ &+ l(l-1)\cosh^i(s)\sinh^{l-2}(s).\end{aligned}$$

Therefore,

$$\begin{aligned}&[d_s^2 + (n-1)\coth(s)d_s](\cosh^i(s)\sinh^l(s)) \\ =\; &(i+l+n-1)(i+l)\cosh^i(s)\sinh^l(s) \\ -\; &i(i-1)\cosh^{i-2}(s)\cosh^l(s) + (n-2+l)l\cosh^i(s)\sinh^{l-2}(s)\end{aligned}$$

Since $u_i = \cosh^i(s)\sinh^{-n-m+2}$, we obtain

$$\begin{aligned}\mathcal{D}_m u_i(s) &= \left[d_s^2 + (n-1)\coth(s)d_s - \frac{m(m+n-2)}{\sinh^2(s)}\right]u_i(s) \\ &= (m-i-1)(m+n-2-i)\cosh^i(s)\sinh^{-m-n+2}(s) \\ &- i(i-1)\cosh^{i-2}(s)\cosh^{-m-n+2}(s) \\ &= \kappa_i u_i(s) - i(i-1)u_{i-2}.\end{aligned}$$

□

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