

DIMENSION SPECTRUM FOR A NONCONVENTIONAL ERGODIC AVERAGE

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ABSTRACT. We compute the dimension spectrum of certain nonconventional averages, namely, the Hausdorff dimension of the set of $0, 1$ sequences, for which the frequency of the pattern 11 in positions $k, 2k$ equals a given number $\theta \in [0, 1]$.

1. INTRODUCTION

For a dynamical system (X, T) (say, a continuous self-map of a compact metric space), the dimension spectrum of ordinary Birkhoff averages is defined as the function

$$\theta \mapsto \dim_H \left\{ x \in X : \lim_{n \rightarrow \infty} \frac{S_n f(x)}{n} = \theta \right\}.$$

where $S_n f(x) = \sum_{k=1}^n f(T^k x)$ and f is a function on X . It has been widely investigated in Multifractal Analysis, see e.g. [2]. The most basic example of such analysis goes back to Besicovitch [4] and Eggleston [7] who proved that

$$\dim_H \left\{ (x_k)_1^\infty \in \{0, 1\}^{\mathbb{N}} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_k = \theta \right\} = H(\theta), \quad \theta \in [0, 1], \quad (1.1)$$

where $H(\theta) = -\theta \log_2 \theta - (1-\theta) \log_2 (1-\theta)$ is the entropy function. Throughout the paper, $\{0, 1\}^{\mathbb{N}} = \Sigma_2$ is the symbolic space, with the usual metric $\varrho((x_k), (y_k)) = 2^{-\min\{n: x_n \neq y_n\}}$. For dimension purposes, this is equivalent to $[0, 1]$ with the standard metric, since for any set $A \subset \Sigma_2$, its image under the binary representation map has the same dimension as A , see [8, Section 2.4].

Furstenberg [10] was the first to consider *multiple Birkhoff averages*, and their study has become a very active area of research, see e.g. Bourgain [6], Host and Kra [11], and others. For a system (X, T) one considers

$$\frac{1}{n} S_n(f_1, \dots, f_\ell)(x) := \frac{1}{n} \sum_{k=1}^n f_1(T^k x) f_2(T^{2k} x) \cdots f_\ell(T^{\ell k} x)$$

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for some bounded functions f_1, \dots, f_ℓ . Very recently, Yu. Kifer [16] and A.-H. Fan, L. Liao, J. Ma [9] initiated the study of the dimension spectrum for such averages (in [16] more general “nonconventional averages” are considered as well). Multifractal analysis of this kind appears to be very complicated, so it is natural to start with the simplest situation, namely, the shift map T on the symbolic space and the functions f_1, \dots, f_ℓ depending only on the first digit x_1 , for $\ell \geq 2$. Specializing even further, to $\ell = 2$ and $f_1(x) \equiv f_2(x) = x_1$ leads to the sets

$$A_\theta := \left\{ (x_k)_1^\infty \in \Sigma_2 : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_k x_{2k} = \theta \right\}, \quad \theta \in [0, 1]. \quad (1.2)$$

The question about the dimension of A_θ was raised in [9]. Note that this directly generalizes the Besicovitch-Eggleston set-up from $\ell = 1$ to $\ell = 2$.

Motivated by this problem, A.-H. Fan, L. Liao, J. Ma, and J. Schmeling [private communication in August 2010] computed the Minkowski (box-counting) dimension of another set

$$X_G := \left\{ (x_k)_1^\infty \in \Sigma_2 : x_k x_{2k} = 0 \text{ for all } k \right\}$$

and asked what is its Hausdorff dimension. It is obvious that $X_G \subset A_0$, and in fact, it is easy to see that $\dim_H(X_G) = \dim_H(A_0)$.

In joint work with R. Kenyon, we computed the Hausdorff dimension of X_G and a large class of similarly defined sets, putting it into the context of *subshifts invariant under the semi-group of multiplicative integers* [14, 15]. Here we adapt the techniques of [14, 15] to compute the full dimension spectrum $\dim_H(A_\theta)$.

Theorem 1.1. *Let A_θ be given by (1.2). For $\theta \in (0, 1)$ we have*

$$\dim_H(A_\theta) = f(\theta) := -\log_2(1-p) - \frac{\theta}{2} \log_2 \left[\frac{(1-q)(1-p)}{qp} \right], \quad (1.3)$$

where

$$p^2 q = (1-p)^3, \quad 0 < p < 1, \quad 0 < q < 1, \quad (1.4)$$

$$\theta = \frac{2p(1-q)}{1+p+q}. \quad (1.5)$$

We have $\dim_H(A_0) = \lim_{\theta \rightarrow 0} f(\theta) = -\log_2(1-p)$, with $p^2 = (1-p)^3$, and $\dim_H(A_1) = \lim_{\theta \rightarrow 1} f(\theta) = 0$.

The meaning of p and q will be explained in the next section. Of course, it is easy to eliminate q from (1.3) and (1.5). For a given θ , we get an algebraic equation of degree 4 for p . Solving the equation numerically yields the graph in Figure 1.

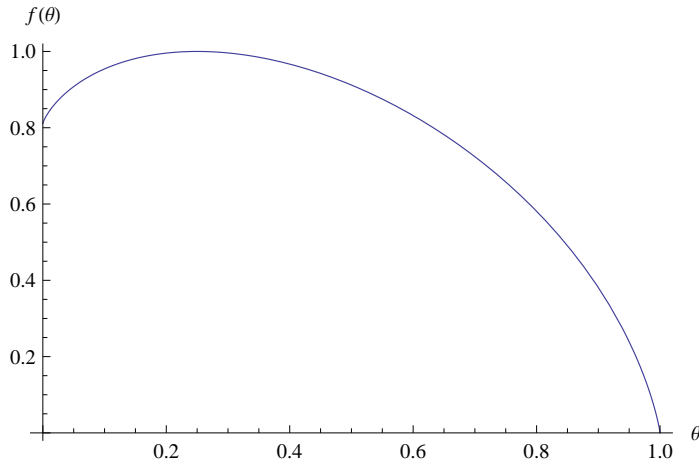


FIGURE 1. Dimension of A_θ

Remarks.

1. As already mentioned, the formula for $\dim_H(A_0)$ easily follows from [14]. Note that in [14, 15] notation was slightly different, so that p in those papers is $1 - p$ here.

2. It is immediate that A_1 is contained in the set of 0-1 sequences which have frequency of 1's equal to 1. Thus, $\dim_H(A_1) = 0$ by (1.1), and we assume $\theta < 1$ for the rest of the paper.

3. By the Strong Law of Large Numbers for weakly correlated random variables (see [18]), for a.e. sequence (x_k) with respect to the Bernoulli $(\frac{1}{2}, \frac{1}{2})^\mathbb{N}$ measure, $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n x_j x_{2j} = 1/4$. This agrees with our result: for $\theta = 1/4$ we get $\dim_H(A_\theta) = 1$ and $p = q = 1/2$.

4. In [9] it is proved that $\dim_H(B_\theta) = 1 - \frac{1}{\ell} + \frac{1}{\ell} H\left(\frac{1+\theta}{2}\right)$ for $\theta \in [-1, 1]$ and $\ell \geq 2$, where $B_\theta := \{(x_k)_1^\infty \in \{-1, 1\}^\mathbb{N} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_k x_{2k} \cdots x_{\ell k} = \theta\}$, using the techniques of Riesz products. It is further pointed out in [9] that the problem becomes drastically different if one takes the digits 0,1 (which reduces to A_θ for $\ell = 2$) instead of $-1, 1$.

5. Yu. Kifer [16] considered a slightly different question: he studied the Hausdorff dimension of sets defined by the frequencies of all ℓ -tuples of digits i_1, \dots, i_ℓ in positions $k, 2k, \dots, \ell k$. However, he was able to compute the dimensions only under the assumption that such frequencies are of the form $p_{i_1, \dots, i_\ell} = p_{i_1} \cdots p_{i_\ell}$.

6. As pointed out in [15], there are some parallels between the multiplicative shifts of finite type and self-affine carpets [3, 19]; we should also add here self-affine sponges [13]. The present paper may similarly be compared to the work on multifractal self-affine

carpets and sponges, see e.g. [17, 20, 1, 12]; however, we do not see any way to directly transfer the results.

2. PRELIMINARIES AND THE SCHEME OF THE PROOF.

The dimension of A_θ is computed with the help of the following lemma which goes back to Billingsley [5]. We write $[u]$ for the cylinder set of sequences starting with a finite word u and $x_1^n := x_1 \dots x_n$.

Lemma 2.1 (see Prop.4.9 in [8]). *Let E be a Borel set in Σ_2 and let ν be a finite Borel measure on Σ_2 .*

- (i) *If $\nu(E) > 0$ and $\liminf_{n \rightarrow \infty} \frac{-\log_2 \nu[x_1^n]}{n} \geq s$ for ν -a.e. $x \in E$, then $\dim_H(E) \geq s$.*
- (ii) *If $\liminf_{n \rightarrow \infty} \frac{-\log_2 \nu[x_1^n]}{n} \leq s$ for all $x \in E$, then $\dim_H(E) \leq s$.*

Following [14, 15], for a probability measure μ on Σ_2 , we define another measure \mathbb{P}_μ on Σ_2 by

$$\mathbb{P}_\mu[u] := \prod_{i \leq n, i \text{ odd}} \mu[u|_{J(i)}], \quad \text{where } J(i) = \{2^r i\}_{r=0}^\infty \quad (2.1)$$

and $u|_{J(i)}$ is the subsequence of u (viewed as a finite sequence) along the geometric progression $J(i)$. The new measure \mathbb{P}_μ is invariant under the action of the *multiplicative semigroup of odd positive numbers*:

$$(x_k)_{k=1}^\infty \mapsto (x_{ik})_{k=1}^\infty \quad \text{for odd } i.$$

We consider Markov measures $\mu_{\mathbf{p},P}$ on Σ_2 , with the initial probability distribution $\mathbf{p} = (1-p, p)$ (so that p is the probability of initial 1), and the stochastic transition matrix $P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$. Note that our Markov measures are not stationary; instead, their initial distribution coincides with the first row of the transition matrix.

Next we indicate the scheme of the proof of Theorem 1.2. Recall that $\theta \in [0, 1)$. In view of Lemma 2.1(i), the lower bound for $\dim_H(A_\theta)$ will be established once we prove the following.

Lemma 2.2. *Fix $p \in (0, 1)$, $q \in [0, 1)$, and let \mathbb{P}_μ , with $\mu = \mu_{\mathbf{p},P}$, be defined by (2.1).*

- (i) *If p, q satisfy (1.5), then $\mathbb{P}_\mu(\Sigma_2 \setminus A_\theta) = 0$. For $\theta = 0$ we take $q = 1$.*
- (ii) *For any p, q we have*

$$\lim_{n \rightarrow \infty} \frac{-\log \mathbb{P}_\mu[x_1^n]}{\log n} = s(p, q) := \frac{H(p)(1+q) + pH(q)}{1+p+q} \quad \text{for } \mathbb{P}_\mu\text{-a.e. } x.$$

(iii) The maximum of $s(p, q)$, subject to (1.5), is achieved when $p^2q = (1-p)^3$, and it equals

$$f(\theta) = -\log_2(1-p) - \frac{\theta}{2} \log_2 \left[\frac{(1-q)(1-p)}{qp} \right] \quad (2.2)$$

The upper bound in Theorem 1.1 will follow from Lemma 2.1(ii), once we prove the following

Lemma 2.3. *Let $\mu = \mu_{\mathbf{p}, P}$ be the Markov measure with initial probability vector $\mathbf{p} = (1-p, p)$ and transition matrix $P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$, where $p^2q = (1-p)^3$ and (1.5) holds, and let \mathbb{P}_μ be the corresponding multiplicative invariant measure. Then*

$$\liminf_{n \rightarrow \infty} \frac{-\log_2 \mathbb{P}_\mu[x_1^n]}{n} \leq f(\theta) \quad \text{for all } x \in A_\theta. \quad (2.3)$$

3. PROOF OF THE LOWER BOUND.

Lemma 3.1. *Let μ be a Markov measure on Σ_2 , with the initial probability (row) vector \mathbf{p} and transition matrix P . Then for \mathbb{P}_μ -a.e. $x \in \Sigma_2$ we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n x_j x_{2j} = \sum_{k=0}^{\infty} 2^{-(k+1)} (\mathbf{p}P^k)_1 \cdot P(1, 1). \quad (3.1)$$

Proof. Fix $\ell \in \mathbb{N}$ and consider $n = 2^\ell r$. Denote $J_i^n := J(i) \cap [1, n]$ (recall that $J(i)$ is the geometric progression with ratio 2 starting at i .) Let

$$S_n(x) = \sum_{j=1}^n x_j x_{2j} \quad \text{and} \quad S'_n(x) := \sum_{j \in J_i^n, \text{ odd } i > r} x_j x_{2j}.$$

We have

$$S'_n(x) \leq S_n(x) \leq S'_n(x) + r(\ell + 1), \quad (3.2)$$

since

$$\begin{aligned} \sum_{\text{odd } i \leq r} |J_i^n| &= n - \sum_{\text{odd } i > r} |J_i^n| \\ &= n - \sum_{k=1}^{\ell} \frac{nk}{2^{k+1}} \\ &= n - n[\ell \cdot 2^{-\ell-1} - (\ell + 1)2^{-\ell} + 1] \\ &< n(\ell + 1)2^{-\ell} = r(\ell + 1). \end{aligned}$$

Now, denote

$$\mathbf{x}_i^{(k)} := x_i x_{2i} \dots x_{2^k i}.$$

We have

$$S'_n(x) = \sum_{k=1}^{\ell} \sum_{\substack{\frac{n}{2^k} < i \leq \frac{n}{2^{k-1}} \\ i \text{ odd}}} L_{11}(\mathbf{x}_i^{(k)}),$$

where $L_{11}(u)$ is the number of subwords 11 in the word u (counting all, possibly overlapping, occurrences). We consider the sequence of words $(\mathbf{x}_i^{(k)})$ (of length $k+1$) for odd $i \in (\frac{n}{2^k}, \frac{n}{2^{k-1}}]$ and $n = 2^\ell r$, as $r \rightarrow \infty$. By the definition of the measure \mathbb{P}_μ , this sequence is i.i.d. with the distribution induced by μ on the set of words of length $k+1$. By the Strong Law of Large Numbers, for any $k \leq \ell$,

$$\frac{1}{n/2^{k+1}} \sum_{\substack{\frac{n}{2^k} < i \leq \frac{n}{2^{k-1}} \\ i \text{ odd}}} L_{11}(\mathbf{x}_i^{(k)}) \rightarrow \mathbb{E}_\mu[L_{11}(u)] \text{ as } n = 2^\ell r \rightarrow \infty, \text{ for } \mathbb{P}_\mu\text{-a.e. } x,$$

where $|u| = k+1$. Here we used the fact that there are $n/2^{k+1}$ odd numbers in $(\frac{n}{2^k}, \frac{n}{2^{k-1}}]$. By the definition of μ as a Markov measure, writing the expectation as the sum of probabilities of seeing 11 at all possible locations,

$$\mathbb{E}_\mu[L_{11}(u)] = \sum_{j=0}^{k-1} (\mathbf{p}P^j)_1 \cdot P(1, 1).$$

It follows that for \mathbb{P}_μ -a.e. x ,

$$\begin{aligned} \frac{1}{n} S'_n(x) &\rightarrow \sum_{k=1}^{\ell} 2^{-k-1} \sum_{j=0}^{k-1} (\mathbf{p}P^j)_1 \cdot P(1, 1) \\ &= \sum_{j=0}^{\ell-1} (\mathbf{p}P^j)_1 \cdot P(1, 1) \sum_{k=j+1}^{\ell} 2^{-k-1} \\ &= \sum_{j=0}^{\ell-1} (2^{-(j+1)} - 2^{-(\ell+1)}) \cdot (\mathbf{p}P^j)_1 \cdot P(1, 1) \text{ as } n = 2^\ell r \rightarrow \infty. \end{aligned}$$

Combining this with (3.2), we obtain

$$\liminf_{n \rightarrow \infty} \frac{S_n}{n} = \liminf_{r \rightarrow \infty} \frac{S_{2^\ell r}}{2^\ell r} \geq \sum_{k=0}^{\ell-1} (2^{-(k+1)} - 2^{-(\ell+1)}) \cdot (\mathbf{p}P^k)_1 \cdot P(1, 1)$$

and

$$\limsup_{n \rightarrow \infty} \frac{S_n}{n} = \limsup_{r \rightarrow \infty} \frac{S_{2^\ell r}}{2^\ell r} \leq \sum_{k=0}^{\ell-1} (2^{-(k+1)} - 2^{-(\ell+1)}) \cdot (\mathbf{p}P^k)_1 \cdot P(1, 1) + \frac{\ell+1}{2^\ell},$$

so letting $\ell \rightarrow \infty$ yields (3.1). \square

Proof of Lemma 2.2(i). The right-hand side of (3.1) is easy to compute explicitly. The matrix $P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$ has left eigenvectors $\boldsymbol{\pi} = (\frac{q}{p+q}, \frac{p}{p+q})$, $\boldsymbol{\tau} = (1, -1)$ corresponding to the eigenvalues 1, $1-p-q$, respectively. Recall that $\mathbf{p} = (1-p, p)$. Thus $\mathbf{p} = \boldsymbol{\pi} + \frac{p(1-p-q)}{p+q}\boldsymbol{\tau}$, hence

$$(\mathbf{p}P^k)_0 = \frac{q}{p+q} + \frac{p}{p+q}(1-p-q)^{k+1}, \quad (\mathbf{p}P^k)_1 = \frac{p}{p+q} - \frac{p}{p+q}(1-p-q)^{k+1}. \quad (3.3)$$

Therefore, it follows from (3.1), in view of $P(1,1) = 1-q$, that for \mathbb{P}_μ -a.e. $x \in \Sigma_2$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n x_j x_{2j} &= \frac{p(1-q)}{p+q} \sum_{k=0}^{\infty} 2^{-(k+1)} [1 - (1-p-q)^{k+1}] \\ &= \frac{p(1-q)}{p+q} \left[1 - \frac{1-p-q}{2(1-\frac{1-p-q}{2})} \right] = \frac{2p(1-q)}{1+p+q} = \theta, \end{aligned}$$

whenever (1.5) holds, as desired. \square

For a probability measure μ on Σ_2 let

$$s(\mu) = \sum_{k=1}^{\infty} \frac{H^\mu(\alpha_k)}{2^{k+1}}, \quad (3.4)$$

where α_k is the partition of Σ_2 into cylinders of length k and H^μ is the Shannon entropy of a partition, using \log_2 .

Lemma 3.2 (see Prop.2.3 in [15]). *Let μ be a probability Borel measure on Σ_2 and \mathbb{P}_μ the corresponding measure given by (2.1). Then*

$$\lim_{n \rightarrow \infty} \frac{-\log \mathbb{P}_\mu[x_1^n]}{\log n} = s(\mu) \quad \text{for } \mathbb{P}_\mu\text{-a.e. } x,$$

where $x_1^n := x_1 \dots x_n$.

Proof of Lemma 2.2(ii). In view of Lemma 3.2, it remains to verify the formula

$$s(\mu) = s(p, q) = \frac{H(p)(1+q) + pH(q)}{1+p+q}. \quad (3.5)$$

By the definition of $s(\mu)$ given in (3.4), and the properties of Shannon entropy, we have

$$s(\mu) = \frac{H^\mu(\alpha_1)}{2} + \sum_{k=1}^{\infty} \frac{H^\mu(\alpha_{k+1}|\alpha_k)}{2^{k+1}}.$$

Now, $H^\mu(\alpha_1) = H(p)$ by assumption, and

$$\begin{aligned} H^\mu(\alpha_{k+1}|\alpha_k) &= - \sum_{u: |u|=k-1} (\mu[u0]H(p) + \mu[u1]H(q)) \\ &= (\mathbf{p}P^{k-1})_0 H(p) + (\mathbf{p}P^{k-1})_1 H(q), \end{aligned}$$

since $(\mathbf{p}P^{k-1})_i$ is the probability of having $x_k = i$ according to the measure μ . Therefore,

$$s(\mu) = \frac{H(p)}{2} + \sum_{k=0}^{\infty} 2^{-(k+2)} [(\mathbf{p}P^k)_0 H(p) + (\mathbf{p}P^k)_1 H(q)].$$

Now, using (3.3) and summing the geometric series yields (3.5). \square

In view of the above, we have that

$$\dim_H(A_\theta) \geq s(\mu) = s(p, q) = \frac{H(p)(1+q) + pH(q)}{1+p+q}, \quad \text{where } \theta = \frac{2p(1-q)}{1+p+q}. \quad (3.6)$$

Thus, we should find the constrained maximum of $s(p, q)$ on $[0, 1]^2$. This is a straightforward exercise, but we include it for the record and in order to explain where the formula (1.4) comes from. It is actually not needed for the proof, since we could have just produced the answer.

Proof of Lemma 2.2(iii). We use the method of Lagrange multipliers. Differentiating $s(p, q)$ yields

$$\begin{aligned} (1+p+q)^2 \frac{\partial s(p, q)}{\partial p} &= (1+q)[(1+p+q) \log_2\left(\frac{1-p}{p}\right) - H(p) + H(q)] \\ &= (1+q)[(2+q) \log_2(1-p) - (1+q) \log_2 p + H(q)], \\ (1+p+q)^2 \frac{\partial s(p, q)}{\partial q} &= p[(1+p+q) \log_2\left(\frac{1-q}{q}\right) + H(p) - H(q)] \\ &= p[(2+p) \log_2(1-q) - (1+p) \log_2 q + H(p)]. \end{aligned}$$

Differentiating the constraint $g(p, q) = \theta(1+p+q) - 2p(1-q) = 0$ yields

$$\nabla g(p, q) = (\theta - 2(1-q), \theta + 2p) = \left(\frac{-2(1-q)(1+q)}{1+p+q}, \frac{2p(2+p)}{1+p+q} \right).$$

At the point of constrained maximum we have $\nabla s(p, q) = \lambda \nabla g(p, q)$, which reduces to

$$\begin{aligned} &(2+p)[(2+q) \log_2(1-p) - (1+q) \log_2 p + H(q)] \\ &= -(1-q)[(2+p) \log_2(1-q) - (1+p) \log_2 q + H(p)]. \end{aligned}$$

The latter becomes, after collecting the terms:

$$3(1+p+q) \log_2(1-p) = 2(1+p+q) \log_2 p + (1+p+q) \log_2 q,$$

so $p^2q = (1-p)^3$, as claimed.

It remains to verify the formula (2.2). We have

$$\begin{aligned} f(\theta) &= -\log_2(1-p) - \frac{\theta}{2} \log_2 \left[\frac{(1-q)(1-p)}{qp} \right] \\ &= -\log_2(1-p) - \frac{p(1-q)}{1+p+q} \log_2 \left[\frac{(1-q)(1-p)}{qp} \right]. \end{aligned}$$

Comparing the latter with

$$s(p, q) = -\frac{(p \log_2 \frac{1-p}{p} - \log_2(1-p))(1+q) + p(q \log_2 \frac{1-q}{q} - \log_2(1-q))}{1+p+q}$$

results in $(1+p+q)(f(\theta) - s(p, q)) = p[2 \log_2 \frac{1-p}{p} + \log_2 \frac{1-q}{q} + \log_2(1-p) - \log_2(1-q)] = 0$, whenever $p^2q = (1-p)^3$, as desired. \square

4. PROOF OF THE UPPER BOUND.

Proof of Lemma 2.3. Fix an even integer n . Denote

$$N_{1,\text{odd}} = N_{1,\text{odd}}(x_1^n) := \{k \leq n : k \text{ odd}, x_k = 1\},$$

and similarly define $N_{1,\text{even}}(x_1^n)$. Further, for $i, j \in \{0, 1\}$ let

$$N_{ij} = N_{ij}(x_1^n) := \{k \leq n/2 : x_k = i, x_{2k} = j\}.$$

By the definition of $\mu = \mu_{\mathbf{p}, P}$ and \mathbb{P}_μ we have, for any x and even n :

$$\mathbb{P}_\mu[x_1^n] = p^{N_{1,\text{odd}}} (1-p)^{n/2 - N_{1,\text{odd}}} (1-p)^{N_{00}} p^{N_{01}} q^{N_{10}} (1-q)^{N_{11}}.$$

Note that $N_{00} + N_{01} + N_{10} + N_{11} = n/2$, so

$$\begin{aligned} \mathbb{P}_\mu[x_1^n] &= p^{N_{1,\text{odd}} + N_{01}} (1-p)^{n - N_{1,\text{odd}} - N_{01} - N_{10} - N_{11}} q^{N_{10}} (1-q)^{N_{11}} \\ &= (1-p)^n \left(\frac{p}{1-p} \right)^{N_{1,\text{odd}} + N_{01}} \left(\frac{q}{1-q} \right)^{N_{10}} \left(\frac{1-q}{1-p} \right)^{N_{10} + N_{11}}. \end{aligned}$$

Observe that

$$N_{10} + N_{11} = N_1(x_1^{n/2}) \quad \text{and} \quad N_{1,\text{odd}} = N_1(x_1^n) - N_{1,\text{even}} = N_1(x_1^n) - N_{01} - N_{11}.$$

The equation $p^2(1-q) = (1-p)^3$ can be rewritten as $\frac{1-q}{1-p} = \frac{1-q}{q} \left(\frac{1-p}{p} \right)^2$. Combining this with the last several equalities yields

$$\begin{aligned} \mathbb{P}_\mu[x_1^n] &= (1-p)^n \left(\frac{p}{1-p} \right)^{N_1(x_1^n) - N_{11} - 2N_1(x_1^{n/2})} \left(\frac{1-q}{q} \right)^{N_{11}} \\ &= (1-p)^n \left(\frac{(1-q)(1-p)}{qp} \right)^{N_{11}} \left(\frac{p}{1-p} \right)^{N_1(x_1^n) - 2N_1(x_1^{n/2})}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{-\log_2 \mathbb{P}_\mu[x_1^n]}{n} &= -\log_2(1-p) - \frac{N_{11}}{n} \log_2\left(\frac{(1-q)(1-p)}{qp}\right) \\ &\quad + \left(\frac{N_1(x_1^n)}{n} - \frac{N_1(x_1^{n/2})}{n/2}\right) \log_2\left(\frac{p}{1-p}\right). \end{aligned} \quad (4.1)$$

Observe that $\lim_{n \rightarrow \infty} \frac{N_{11}}{n} = \theta/2$ for all $x \in A_\theta$. Now replace n by 2^ℓ for $\ell = 1, \dots, L$, and take the average over ℓ . The expression in the second line of (4.1) telescopes, so we obtain, in view of (2.2),

$$\begin{aligned} \frac{1}{L} \sum_{\ell=1}^L \left(\frac{-\log \mathbb{P}_\mu[x_1^{2^\ell}]}{2^\ell} - f(\theta) \right) &= \frac{1}{L} \sum_{\ell=1}^L \left(\frac{\theta}{2} - \frac{N_{11}(x_1^{2^\ell})}{2^\ell} \right) \log_2\left(\frac{(1-q)(1-p)}{qp}\right) \\ &\quad + \frac{1}{L} \log_2\left(\frac{p}{1-p}\right) \left(\frac{N_1(x_1^{2^L})}{2^L} - \frac{N_1(x_1^2)}{2} \right). \end{aligned}$$

It follows that

$$\liminf_{\ell \rightarrow \infty} \frac{-\log \mathbb{P}_\mu[x_1^{2^\ell}]}{2^\ell} \leq f(\theta) \quad \text{for all } x \in A_\theta,$$

and the proof of (2.3) is complete. \square

5. CONCLUDING REMARKS

1. It is not hard to verify that, under the conditions (1.4) and (1.5) we have $\frac{(1-q)(1-p)}{qp} < 1$ if and only if $\theta < 1/4$. Therefore, by the argument in the last section, it immediately follows that

$$\dim_H(A_\theta^+) = f(\theta) \quad \text{for } \theta \in (0, 1/4), \quad \dim_H(A_\theta^-) = f(\theta) \quad \text{for } \theta \in (1/4, 1),$$

where

$$\begin{aligned} A_\theta^+ &:= \left\{ (x_k)_1^\infty \in \Sigma_2 : \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_k x_{2k} \leq \theta \right\}, \\ A_\theta^- &:= \left\{ (x_k)_1^\infty \in \Sigma_2 : \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_k x_{2k} \geq \theta \right\}. \end{aligned}$$

2. We extended the result of Theorem 1.1 to the case of arbitrary functions f_1, f_2 on the shift Σ_2 depending on the first digit x_1 . The method is the same, but the calculations are more involved, so we only state the result.

Theorem 5.1. *For $\beta, \gamma \in \mathbb{R}$, let*

$$A_\theta(\beta, \gamma) := \left\{ (x_k)_1^\infty \in \Sigma_2 : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (x_k + \beta)(x_{2k} + \gamma) = \theta \right\}$$

We have

$$\dim_H(A_\theta(\beta, \gamma)) = -\frac{1}{2} \log_2[p_0(1-p)] - \frac{\theta}{2} \log_2\left[\frac{(1-q)(1-p)}{qp}\right],$$

where

$$\frac{1-p}{1-q} = \left(\frac{q}{1-q}\right)^{1+2\beta+\gamma} \left(\frac{p}{1-p}\right)^{2+2\beta+\gamma}, \quad \frac{1-p_0}{p_0} = \left(\frac{q}{1-q}\right)^\beta \left(\frac{p}{1-p}\right)^{1+\beta},$$

and

$$\theta = \beta\gamma + \frac{(1+\beta+\gamma-q)(1+p-p_0) + \beta(p_0(p+q) - q)}{1+p+q}.$$

The appropriate measure is \mathbb{P}_μ , with μ Markov, having the initial distribution $(p_0, 1-p_0)$ and the transition matrix $\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$.

3. After this work was essentially completed, we were informed that A.-H. Fan, J. Schmeling, and M. Wu have computed the dimension of A_θ (in a different, but equivalent form) and other sets of this type, independently, but also building on [15].

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