On the meaning of the Vakhitov-Kolokolov stability criterion for the nonlinear Dirac equation

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Abstract

We consider the spectral stability of solitary wave solutions $\phi(x)e^{-i\omega t}$ to the nonlinear Dirac equation in any dimension. This equation is well-known to theoretical physicists as the Soler model (or, in one dimension, the Gross-Neveu model), and attracted much attention for many years. We show that, generically, at the values of ω where the Vakhitov-Kolokolov stability criterion breaks down, a pair of real eigenvalues (one positive, one negative) appears from the origin, leading to the linear instability of corresponding solitary waves.

As an auxiliary result, we state the Virial identities ("Pohozhaev theorem") for the nonlinear Dirac equation. We also show that $\pm 2\omega i$ are the eigenvalues of the nonlinear Dirac equation linearized at $\phi(x)e^{-i\omega t}$, which are embedded into the essential spectrum as long as $|\omega| > m/3$. This result holds for the nonlinear Dirac equation with any nonlinearity of the Soler form ("scalar-scalar interaction") and in any dimension.

As an illustration of the spectral stability methods, we revisit Derrick's theorem and sketch the Vakhitov-Kolokolov stability criterion for the nonlinear Schrödinger equation.

1 Introduction

Field equations with nonlinearities of local type are natural candidates for developing tools which are then used for the analysis of systems of interacting equations. Equations with local nonlinearities have been appearing in the Quantum Field Theory starting perhaps since fifties [Sch51a, Sch51b], in the context of the classical nonlinear meson theory of nuclear forces. The nonlinear version of the Dirac equation is known as the Soler model [Sol70]. The existence of standing waves in this model was proved in [Sol70, CV86]. Existence of localized solutions to the Dirac-Maxwell system was addressed in [Wak66, Lis95] and finally was proved in [EGS96] (for $\omega \in (-m, 0)$) and [Abe98] (for $\omega \in (-m, m)$). The local well-posedness of the Dirac-Maxwell system was considered in [Bou96]. The local and global well-posedness of the Dirac equation was further addressed in [EV97] (semilinear Dirac equation in $n = 3$), [Bou00] (Dirac – Klein-Gordon system in $n = 1$), and in [MNNO05] (nonlinear Dirac equation in $n = 3$). The question of stability of solitary wave solutions to the nonlinear Dirac equation attracted much attention for many years, but only partial numerical results were obtained; see e.g. [AC81, AKV83, AS83, AS86, Chu07]. The analysis of stability with respect to dilations is performed in [SV86, CKMS10].

Understanding the linear stability is the first step in the study of stability properties of solitary waves. Absence of an eigenvalue with a positive real part will be referred to as the spectral stability, while its absence as the spectral (or linear) instability. After the spectrum of the linearized problem for the nonlinear Schrödinger equation [VK73] was understood, the linearly unstable solitary waves can be proved to be ("nonlinearly", or "dynamically") unstable [Gri88, GO10], while the linearly stable solitary waves of the nonlinear Schrödinger and Klein-Gordon equations [Sha83, SS85, Wei86] and more general $\mathbf{U}(1)$-invariant systems [GSS87] were proved to be orbitally stable. The tools used to prove orbital stability break down for the Dirac equation since the corresponding energy functional is sign-indefinite. On the other hand, one can hope to use the dispersive estimates for the linearized equation to prove the asymptotic stability of the standing waves, similarly to how it is being done for the nonlinear Schrödinger equation [Wei85], [SW92], [BP93], [SW99], and [Cuc01]. The first results on asymptotic stability for the nonlinear Dirac equation are already appearing [PS10, BC11], with the assumptions on the spectrum of the linearized equation playing a crucial role.

In this paper, we study the spectrum of the nonlinear Dirac equation linearized at a solitary wave, concentrating on bifurcation of real eigenvalues from $\lambda = 0$.

Derrick's theorem

As a warm-up, let us consider the linear instability of stationary solutions to a nonlinear wave equation,

$$-\ddot{\psi} = -\Delta\psi + g(\psi), \qquad \psi = \psi(x,t) \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 1. \tag{1.1}$$

We assume that the nonlinearity $g(s)$ is smooth. Equation (1.1) is a Hamiltonian system, with the Hamiltonian $E(\psi,\pi) = \int_{\mathbb{R}^n}\left(\frac{\pi^2}{2} + \frac{|\nabla\psi|^2}{2} + G(\psi)\right)dx$, where $G(s) = \int_0^s g(s')\,ds'$.

There is a well-known result [Der64] about non-existence of stable localized stationary solutions in dimension $n \geq 3$ (known as *Derrick's Theorem*). If $u(x,t) = \theta(x)$ is a localized stationary solution to the Hamiltonian equations $\dot{\pi} = -\delta_\psi E$, $\dot{\psi} = \delta_\pi E$, then, considering the family $\theta_\lambda(x) = \theta(\lambda x)$, one has $\partial_\lambda|_{\lambda=1} E(\phi_\lambda) = 0$, and then it follows that $\partial_\lambda^2|_{\lambda=1} E(\phi_\lambda) < 0$ as long as $n \geq 3$. That is, $\delta^2 E < 0$ for a variation corresponding to the uniform stretching, and the solution $\theta(x)$ is to be unstable. Let us modify Derrick's argument to show the linear instability of stationary solutions in any dimension.

Lemma 1.1 (Derrick's theorem for $n \geq 1$)**.** *For any $n \geq 1$, a smooth finite energy stationary solution $\theta(x)$ to the nonlinear wave equation is linearly unstable.*

Proof. Since θ satisfies $-\Delta\theta + g(\theta) = 0$, we also have $-\Delta\partial_{x_1}\theta + g'(\theta)\partial_{x_1}\theta = 0$. Due to $\lim\limits_{|x|\to\infty}\theta(x) = 0$, $\partial_{x_1}\theta$ vanishes somewhere. According to the minimum principle, there is a nowhere vanishing smooth function $\chi \in H^\infty(\mathbb{R}^n)$ (due to Δ being elliptic) which corresponds to some smaller (hence negative) eigenvalue of $\mathrm{L} = -\Delta + g'(\theta)$, $\mathrm{L}\chi = -c^2\chi$, with $c > 0$. Taking $\psi(x,t) = \theta(x) + r(x,t)$, we obtain the linearization at θ, $-\ddot{r} = -\mathrm{L}r$, which we rewrite as $\partial_t \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\mathrm{L} & 0 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$. The matrix in the right-hand side has eigenvectors $\begin{bmatrix} \chi \\ \pm c\chi \end{bmatrix}$, corresponding to the eigenvalues $\pm c \in \mathbb{R}$; thus, the solution θ is linearly unstable.

Let us also mention that $\partial_\tau^2|_{\tau=0} E(\theta + \tau\chi) < 0$, showing that $\delta^2 E(\theta)$ is not positive-definite. □

Remark 1.2. A more general result on the linear stability and (nonlinear) instability of stationary solutions to (1.1) is in [KS07]. In particular, it is shown there that the linearization at a stationary solution may be spectrally stable when this particular stationary solution is not from H^1 (such examples exist in higher dimensions).

Vakhitov-Kolokolov stability criterion for the nonlinear Schrödinger equation

To get a hold of stable localized solutions, Derrick suggested that *elementary particles might correspond to stable, localized solutions which are periodic in time, rather than time-independent.* Let us consider how this works for the (generalized) nonlinear Schrödinger equation in one dimension,

$$i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi + g(|\psi|^2)\psi, \qquad \psi = \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{1.2}$$

where $g(s)$ is a smooth function with $m := g(0) > 0$. One can easily construct solitary wave solutions $\phi(x)e^{-i\omega t}$, for some $\omega \in \mathbb{R}$ and $\phi \in H^1(\mathbb{R})$: $\phi(x)$ satisfies the stationary equation $\omega\phi = -\frac{1}{2}\phi'' + g(\phi^2)\phi$, and can be chosen strictly positive, even, and monotonically decaying away from $x = 0$. The value of ω can not exceed m. We consider the Ansatz $\psi(x,t) = (\phi(x) + \rho(x,t))e^{-i\omega t}$, with $\rho(x,t) \in \mathbb{C}$. The linearized equation on ρ is called the linearization at a solitary wave:

$$\partial_t \mathbf{R} = \mathbf{JLR}, \qquad \mathbf{R}(x,t) = \begin{bmatrix} \operatorname{Re}\rho(x,t) \\ \operatorname{Im}\rho(x,t) \end{bmatrix}, \tag{1.3}$$

with

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} \mathrm{L}_+ & 0 \\ 0 & \mathrm{L}_- \end{bmatrix}, \qquad \mathrm{L}_- = -\frac{1}{2}\partial_x^2 + g(\phi^2) - \omega, \qquad \mathrm{L}_+ = \mathrm{L}_- + 2g'(\phi^2)\phi^2. \tag{1.4}$$

Note that since $\mathrm{L}_- \neq \mathrm{L}_+$, the action of $\mathbf{L}$ on ρ considered as taking values in $\mathbb{C}$ is $\mathbb{R}$-linear but not $\mathbb{C}$-linear. Since $\lim\limits_{|x|\to\infty} \phi(x) = 0$, the essential spectrum of L_- and L_+ is $[m-\omega, +\infty)$.

First, let us note that the spectrum of $\mathbf{JL}$ is located on the real and imaginary axes only: $\sigma(\mathbf{JL}) \subset \mathbb{R} \cup i\mathbb{R}$. To prove this, we consider $(\mathbf{JL})^2 = -\begin{bmatrix} \mathrm{L}_-\mathrm{L}_+ & 0 \\ 0 & \mathrm{L}_+\mathrm{L}_- \end{bmatrix}$. Since L_- is positive-definite ($\phi \in \ker \mathrm{L}_-$, being nowhere zero, corresponds to its smallest eigenvalue), we can define the selfadjoint root of L_-; then

$$\sigma_d((\mathbf{JL})^2)\backslash\{0\} = \sigma_d(\mathrm{L}_-\mathrm{L}_+)\backslash\{0\} = \sigma_d(\mathrm{L}_+\mathrm{L}_-)\backslash\{0\} = \sigma_d(\mathrm{L}_-^{1/2}\mathrm{L}_+\mathrm{L}_-^{1/2})\backslash\{0\} \subset \mathbb{R},$$

with the inclusion due to $\mathrm{L}_-^{1/2}\mathrm{L}_+\mathrm{L}_-^{1/2}$ being selfadjoint. Thus, any eigenvalue $\lambda \in \sigma_d(\mathbf{JL})$ satisfies $\lambda^2 \in \mathbb{R}$.

Given the family of solitary waves, $\phi_\omega(x)e^{-i\omega t}$, $\omega \in \Omega \subset \mathbb{R}$, we would like to know at which ω the eigenvalues of the linearized equation with $\operatorname{Re}\lambda > 0$ appear. Since $\lambda^2 \in \mathbb{R}$, such eigenvalues can only be located on the real axis, having bifurcated from $\lambda = 0$. One can check that $\lambda = 0$ belongs to the discrete spectrum of $\mathbf{JL}$, with

$$\mathbf{JL}\begin{bmatrix} 0 \\ \phi_\omega \end{bmatrix} = 0, \qquad \mathbf{JL}\begin{bmatrix} -\partial_\omega\phi_\omega \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_\omega \end{bmatrix},$$

for all ω which correspond to solitary waves. Thus, if we will restrict our attention to functions which are even in x, the dimension of the generalized null space of $\mathbf{JL}$ is at least two. Hence, the bifurcation follows the jump in the dimension of the generalized null space of $\mathbf{JL}$. Such a jump happens at a particular value of ω if one can solve the equation $\mathbf{JL}\alpha = \begin{bmatrix} \partial_\omega\phi_\omega \\ 0 \end{bmatrix}$. This leads to the condition that $\begin{bmatrix} \partial_\omega\phi_\omega \\ 0 \end{bmatrix}$ is orthogonal to the null space of the adjoint to $\mathbf{JL}$, which contains the vector $\begin{bmatrix} \phi_\omega \\ 0 \end{bmatrix}$; this results in $\langle \phi_\omega, \partial_\omega\phi_\omega \rangle = \partial_\omega \|\phi_\omega\|_{L^2}^2/2 = 0$. A slightly more careful analysis [CP03] based on construction of the moving frame in the generalized eigenspace of $\lambda = 0$ shows that there are two real eigenvalues $\pm\lambda \in \mathbb{R}$ that have emerged from $\lambda = 0$ when ω is such that $\partial_\omega \|\phi_\omega\|_{L^2}^2$ becomes positive, leading to a linear instability of the corresponding solitary wave. The opposite condition,

$$\partial_\omega \|\phi_\omega\|_{L^2}^2 < 0, \tag{1.5}$$

is the Vakhitov-Kolokolov stability criterion which guarantees the absence of nonzero real eigenvalues for the nonlinear Schrödinger equation. It appeared in [VK73, Sha83, GSS87] in relation to linear and orbital stability of solitary waves. The above approach fails for the nonlinear Dirac equation since L_- is no longer positive-definite.

For the completeness, let us present a more precise form of the Vakhitov-Kolokolov stability criterion [VK73].

Lemma 1.3 (Vakhitov-Kolokolov stability criterion)**.** *There is $\lambda \in \sigma_p(\mathbf{JL})$, $\lambda > 0$, where $\mathbf{JL}$ is the linearization* (1.3) *at the solitary wave $\phi_\omega(x)e^{-i\omega t}$, if and only if $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 > 0$ at this value of ω.*

Proof. We follow [VK73]. Assume that there is $\lambda \in \sigma_d(\mathbf{JL})$, $\lambda > 0$. The relation $(\mathbf{JL} - \lambda)\Xi = 0$ implies that $\lambda^2\Xi_1 = -\mathrm{L}_-\mathrm{L}_+\Xi_1$. It follows that Ξ_1 is orthogonal to the kernel of the selfadjoint operator L_- (which is spanned by ϕ_ω):

$$\langle \phi, \Xi_1 \rangle = -\frac{1}{\lambda^2}\langle \phi, -\mathrm{L}_-\mathrm{L}_+\Xi_1 \rangle = -\frac{1}{\lambda^2}\langle \mathrm{L}_-\phi, -\mathrm{L}_+\Xi_1 \rangle = 0,$$

hence there is $\eta \in L^2(\mathbb{R}, \mathbb{C})$ such that $\Xi_1 = \mathrm{L}_-\eta$ and $\lambda^2\eta = -\mathrm{L}_+\Xi_1$. Thus, the inverse to L_- can be applied: $\lambda^2\mathrm{L}_-^{-1}\Xi_1 = -\mathrm{L}_+\Xi_1$. Then

$$\lambda^2\langle \eta, \mathrm{L}_-\eta \rangle = -\langle \Xi_1, \mathrm{L}_+\Xi_1 \rangle.$$

Since L_- is positive-definite and $\eta \notin \ker \mathrm{L}_-$, it follows that $\langle \eta, \mathrm{L}_-\eta \rangle > 0$. Since $\lambda > 0$, $\langle \Xi_1, \mathrm{L}_+\Xi_1 \rangle < 0$, therefore the quadratic form $\langle \cdot, \mathrm{L}_+ \cdot \rangle$ is not positive-definite on vectors orthogonal to ϕ_ω. According to Lagrange's principle, the function r corresponding to the minimum of $\langle r, \mathrm{L}_+ r \rangle$ under conditions $\langle r, \phi_\omega \rangle = 0$ and $\langle r, r \rangle = 1$ satisfies

$$\mathrm{L}_+ r = \alpha r + \beta\phi_\omega, \qquad \alpha,\ \beta \in \mathbb{R}. \tag{1.6}$$

Since $\langle r, \mathrm{L}_+ r \rangle = \alpha$, we need to know whether α could be negative. Since $\mathrm{L}_+\partial_x\phi_\omega = 0$, one has $\lambda_1 = 0 \in \sigma_p(\mathrm{L}_+)$. Due to $\partial_x\phi_\omega$ vanishing at one point ($x = 0$), there is exactly one negative eigenvalue of L_+, which we denote by

$\lambda_0 \in \sigma_p(\mathsf{L}_+)$. (This eigenvalue corresponds to some non-vanishing eigenfunction.) Note that $\beta \ne 0$, or else α would have to be equal to λ_0, with r the corresponding eigenfunction of L_+, but then r, having to be nonzero, could not be orthogonal to ϕ_ω. Denote $\lambda_2 = \inf(\sigma(\mathsf{L}_+) \cap \mathbb{R}_+) > 0$. Let us consider $f(z) = \langle \phi_\omega, (\mathsf{L}_+ - z)^{-1}\phi_\omega \rangle$, which is defined and is smooth for $z \in (\lambda_0, \lambda_2)$. (Note that $f(z)$ is defined for $z = \lambda_1 = 0$ since the corresponding eigenfunction $\partial_x \phi_\omega$ is odd while ϕ_ω is even.) If $\alpha < 0$, then, by (1.6), we would have $f(\alpha) = \langle \phi_\omega, (\mathsf{L}_+ - \alpha)^{-1}\phi_\omega \rangle = \frac{1}{\beta}\langle \phi_\omega, r \rangle = 0$, and since $f'(z) > 0$, one has $f(0) > 0$. On the other hand, $f(0) = \langle \phi_\omega, \mathsf{L}_+^{-1}\phi_\omega \rangle = \langle \phi_\omega, \partial_\omega \phi_\omega \rangle = \frac{1}{2}\frac{d}{d\omega}\int_{\mathbb{R}} |\phi_\omega(x)|^2\,dx$. Therefore, the linear instability leads to $\alpha < 0$, which results in $\frac{d}{d\omega}\int_{\mathbb{R}} |\phi_\omega(x)|^2\,dx > 0$.

Alternatively, let $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 > 0$. We consider the function $f(z) = \langle \phi_\omega, (\mathsf{L}_+ - z)^{-1}\phi_\omega \rangle$, $z \in \rho(\mathsf{L}_+)$. Since $f(0) = \langle \phi_\omega, \mathsf{L}_+^{-1}\phi_\omega \rangle > 0$, $f'(z) > 0$, and $\lim\limits_{z \to \lambda_0 +} f(z) = -\infty$ (where $\lambda_0 < 0$ is the smallest eigenvalue of L_+), there is $\alpha \in (\lambda_0, 0) \subset \rho(\mathsf{L}_+)$ such that $f(\alpha) = \langle \phi_\omega, (\mathsf{L}_+ - \alpha)^{-1}\phi_\omega \rangle = 0$. Then we define $r = (\mathsf{L}_+ - \alpha)^{-1}\phi_\omega$. Since $\langle \phi_\omega, r \rangle = f(\alpha) = 0$, there is η such that $r = \mathsf{L}_-\eta$. It follows that the quadratic form $\mathsf{L}_-^{1/2}\mathsf{L}_+\mathsf{L}_-^{1/2}$ is not positive definite:

$$\langle \mathsf{L}_-^{\frac{1}{2}}\eta, (\mathsf{L}_-^{\frac{1}{2}}\mathsf{L}_+\mathsf{L}_-^{\frac{1}{2}})\mathsf{L}_-^{\frac{1}{2}}\eta \rangle = \langle r, \mathsf{L}_+ r \rangle = \langle r, (\alpha r + \phi_\omega) \rangle = \alpha \langle r, r \rangle < 0.$$

Thus, there is $\lambda > 0$ such that $-\lambda^2 \in \sigma(\mathsf{L}_-^{1/2}\mathsf{L}_+\mathsf{L}_-^{1/2})$; then also $-\lambda^2 \in \sigma(\mathsf{L}_-\mathsf{L}_+)$. Let ξ be the corresponding eigenvector, $\mathsf{L}_-\mathsf{L}_+\xi = -\lambda^2\xi$; then $\begin{bmatrix} 0 & \mathsf{L}_- \\ -\mathsf{L}_+ & 0 \end{bmatrix}\begin{bmatrix} \xi \\ -\frac{1}{\lambda}\mathsf{L}_+\xi \end{bmatrix} = \lambda \begin{bmatrix} \xi \\ -\frac{1}{\lambda}\mathsf{L}_+\xi \end{bmatrix}$, hence $\lambda \in \sigma(\mathbf{JL})$. □

Our conclusions:

1. Point eigenvalues of the linearized Dirac equation may bifurcate (as ω changes) from the origin, when the dimension of the generalized null space jumps up (when the Vakhitov-Kolokolov criterion breaks down).

2. Since the spectrum of the linearization does not have to be a subset of $\mathbb{R} \cup i\mathbb{R}$, there may also be point eigenvalues which bifurcate from the imaginary axis into the complex plane. (We do not know particular examples of such behavior for the nonlinear Dirac equation.)

3. Moreover, there may be point eigenvalues already present in the spectra of linearizations at arbitrarily small solitary waves. Formally, we could say that these eigenvalues bifurcate from the essential spectrum of the free Dirac operator (divided by i), which can be considered as the linearization of the nonlinear Dirac equation at the zero solitary wave.

In the present paper we investigate the first scenario. The main result (Lemma 4.1) states that if the Vakhitov-Kolokolov breaks down at some point ω_*, then, generically, the solitary waves with ω from an open one-sided neighborhood of ω_* are linearly unstable.

We also demonstrate the presence of the eigenvalue $\pm 2\omega i$ in the spectrum of the linearized operator (Corollary 2.8) and obtain Virial identities, or Pohozhaev theorem, for the nonlinear Dirac equation (Lemma 3.2), which we need for the analysis of the zero eigenvalue of the linearized operator.

2 Linearization of the nonlinear Dirac equation

The nonlinear Dirac equation in $\mathbb{R}^n$ has the form

$$i\partial_t \psi = -i\sum_{j=1}^{n} \alpha_j \partial_j \psi + g(\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $\partial_j = \frac{\partial}{\partial x^j}$, N is even and g smooth, with $m := g(0) > 0$. The Dirac matrices α_j and β satisfy the relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_N, \qquad \alpha_j\beta + \beta\alpha_j = 0, \qquad \beta^2 = I_N, \qquad 1 \le j,k \le n,$$

where I_N is an $N \times N$ unit matrix. We will always assume that $\beta = \begin{bmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{bmatrix}$. In the case $n = 1$, we assume $\alpha_1 = -\sigma_2$; in the case $n = 3$, one could take $\alpha_j = \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix}$, where σ_j are the standard Pauli matrices. Equation

(2.1), usually with $g(s) = 1 - s$, is called the Soler model [Sol70], which has been receiving a lot of attention in theoretical physics in relation to classical models of elementary particles.

Remark 2.1. In terms of the Dirac γ-matrices, equation (2.1) takes the explicitly relativistically-invariant form $i\gamma^\mu\partial_\mu\psi = g(\psi^*\beta\psi)\psi$, where $\gamma^0 = \beta,\ \gamma^j = \beta\alpha_j,\ \partial_0 = \partial_t,\ \partial_j = \partial_{x_j}$.

Definition 2.2. The solitary waves are solutions to (2.1) of the form $\phi_\omega(x)e^{-i\omega t}$, $\phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$, $\omega \in \mathbb{R}$.

Below, we assume that there are solitary waves for ω from some nonempty set $\Omega \subset \mathbb{R}$:

$$\phi_\omega e^{-i\omega t}, \qquad \omega \in \Omega \subset \mathbb{R}, \qquad \phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N), \tag{2.2}$$

with ϕ_ω smoothly depending on ω.

We will not indicate the dependence on ω explicitly, and will write ϕ instead of ϕ_ω.

The profile ϕ of a stationary wave satisfies the stationary nonlinear Dirac equation

$$\mathcal{L}_-\phi := (-i\sum_{j=1}^n \alpha_j\partial_j - \omega + g(\phi^*\beta\phi)\beta)\phi = 0. \tag{2.3}$$

The energy and charge functionals corresponding to the nonlinear Dirac equation (2.1) are given by

$$E(\psi) = \int_{R^n} \left(-i\sum_{j=1}^n \psi^*\alpha_j\partial_j\psi + G(\psi^*\beta\psi) \right) d^n x, \qquad Q(\psi) = \int_{\mathbb{R}^n} \psi^*\psi\, d^n x,$$

where $G(s)$ is the antiderivative of $g(s)$ which satisfies $G(0) = 0$. $Q(\psi)$ is the charge functional which is (formally) conserved for solutions to (2.1) due to the $\mathbf{U}(1)$-invariance. The nonlinear Dirac equation (2.1) can be written in the Hamiltonian form as $\partial_t \operatorname{Im}\psi = -\frac{1}{2}\delta_{\operatorname{Re}\psi}E$, $\partial_t \operatorname{Re}\psi = \frac{1}{2}\delta_{\operatorname{Im}\psi}E$, or simply $\dot\psi = -i\delta_{\psi^*}E$. The relation (2.3) satisfied by the profile of the solitary wave $\phi(x)e^{-i\omega t}$ can be written as

$$E'(\phi) = \omega Q'(\phi), \tag{2.4}$$

where the primes denote the Fréchét derivative of the functionals $E(\psi)$, $Q(\psi)$ with respect to $(\operatorname{Re}\psi, \operatorname{Im}\psi)$.

Let us write the solution in the form $\psi(x,t) = (\phi(x) + \rho(x,t))e^{-i\omega t}$, $\rho(x,t) \in \mathbb{C}^N$. The linearized equation on ρ is given by

$$\dot\rho = \mathcal{J}\mathcal{L}\rho, \tag{2.5}$$

where $\mathcal{J}$ corresponds to a multiplication by $1/i$ and

$$\mathcal{L}\rho = \mathcal{L}_-\rho + 2g'(\phi^*\beta\phi)\beta\phi \operatorname{Re}(\phi^*\beta\rho).$$

Note that, because of the presence of $\operatorname{Re}(\phi^*\beta\rho)$, the action of $\mathcal{L}$ on $\rho \in \mathbb{C}^N$ is $\mathbb{R}$-linear but not $\mathbb{C}$-linear. Because of this, it is convenient to write it as an operator $\mathbf{L}$ acting on vectors from $\mathbb{R}^{2N}$; then (2.5) takes the following form:

$$\partial_t\mathbf{R} = \mathbf{J}\mathbf{L}\mathbf{R}, \qquad \mathbf{L} = \mathbf{J}\mathbf{A}_j\partial_j - \omega + g\mathbf{B} + 2g'\mathbf{B}\Phi\langle\mathbf{B}\Phi, \cdot\rangle_{\mathbb{R}^{2N}}, \qquad \mathbf{R}(x,t) = \begin{bmatrix}\operatorname{Re}\rho\\ \operatorname{Im}\rho\end{bmatrix}, \tag{2.6}$$

where $g = g(\phi^*\beta\phi)$, $g' = g'(\phi^*\beta\phi)$ and

$$\Phi = \begin{bmatrix}\operatorname{Re}\phi\\ \operatorname{Im}\phi\end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix}0 & I_N\\ -I_N & 0\end{bmatrix}, \qquad \mathbf{A}_j = \begin{bmatrix}\operatorname{Re}\alpha_j & -\operatorname{Im}\alpha_j\\ \operatorname{Im}\alpha_j & \operatorname{Re}\alpha_j\end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix}\beta & 0\\ 0 & \beta\end{bmatrix}. \tag{2.7}$$

Note that $\mathbf{J}$, $\mathbf{A}_j$, and $\mathbf{B}$ correspond to multiplication by $-i$, α_j, and β under the $\mathbb{C}^N \leftrightarrow \mathbb{R}^{2N}$ correspondence.

Remark 2.3. When $n = 1$, one can take $\alpha_1 = -\sigma_2$ (so that $N = 2$). Then $\mathbf{L}$ has a particularly simple form since $\phi \in \mathbb{C}^2$ can be chosen valued in $\mathbb{R}^2$: $\mathbf{L} = \begin{bmatrix}\mathrm{L}_+ & 0\\ 0 & \mathrm{L}_-\end{bmatrix}$, with $\mathrm{L}_- = i\sigma_2\partial_x - \omega + g\beta$, $\mathrm{L}_+ = \mathrm{L}_- + 2g'\beta\phi\,\langle\beta\phi, \cdot\rangle_{\mathbb{C}^2}$. The numerical and analytical study of spectra of L_-, L_+ in this case is contained in [BC09].

Lemma 2.4. $\sigma_{ess}(\mathbf{L}) = \sigma_{ess}(\mathcal{L}) = \mathbb{R}\backslash(-m-\omega, m-\omega)$; $\sigma_{ess}(\mathbf{JL}) = i(\mathbb{R}\backslash(-m+\omega, m-\omega))$.

Proof. One has $\sigma(-i\alpha_j\partial_j + \beta m) = \mathbb{R}\backslash(-m, m)$. Note also that $(-i\alpha_j\partial_j + \beta m)^2 = -\Delta + m^2$ has a spectrum $[m^2, \infty)$. Taking into account that the symbol of $\mathcal{L}$ at $|x| \to \infty$ is $\alpha_j\xi_j + \beta m - \omega$, one concludes that $\sigma_{ess}(\mathbf{L}) = \sigma_{ess}(\mathcal{L}) = \mathbb{R}\backslash(-m-\omega, m-\omega)$. Since the eigenvalues of $\mathbf{J}$ are $\pm i$, corresponding to clock- and counterclockwise rotations in $\mathbb{C}$, one deduces that $\sigma_{ess}(\mathbf{JL}) = i(\mathbb{R}\backslash(-m+\omega, m-\omega))$. □

Lemma 2.5. *The null space of* $\mathbf{JL}$ *is given by* $\ker \mathbf{JL} = \operatorname{Span}\{\mathbf{J}\Phi,\ \partial_k\Phi;\ 1 \le k \le n\}$.

Proof. Recall that

$$\mathcal{L}_- = -i\sum_{j=1}^{n}\alpha_j\partial_j - \omega + g(\phi^*\beta\phi)\beta, \qquad \mathcal{L} = -i\sum_{j=1}^{n}\alpha_j\partial_j - \omega + g(\phi^*\beta\phi)\beta + 2g'(\phi^*\beta\phi)\beta\phi\operatorname{Re}(\phi^*\beta\ \cdot).$$

Since $\phi(x) \in \mathbb{C}^N$ satisfies the stationary nonlinear Dirac equation (2.3), we get:

$$\mathcal{L}(-i\phi) = \mathcal{L}_-(-i\phi) + 2\operatorname{Re}(\phi^*\beta(-i\phi)) = 0. \tag{2.8}$$

Taking the derivative of (2.3) with respect to x_k yields

$$-i\sum_{j=1}^{n}\alpha_j\partial_j(\partial_k\phi) + g(\phi^*\beta\phi)\beta\partial_k\phi + g'(\phi^*\beta\phi)(\partial_k\phi^*\beta\phi + \phi^*\beta\partial_k\phi)\beta\phi - \omega\partial_k\phi = \mathcal{L}\partial_k\phi = 0. \tag{2.9}$$

□

Lemma 2.6. *Let* α_0 *be an hermitian matrix anticommuting with* α_j*,* $1 \le j \le n$*, and with* β*. Then* $\alpha_0\phi$ *is an eigenfunction of* $\mathcal{L}_-$ *and of* $\mathcal{L}$*, corresponding to the eigenvalue* $\lambda = -2\omega$*.*

Remark 2.7. If $n = 3$, one can take $\alpha_0 = \alpha_1\alpha_2\alpha_3\beta$.

Proof. Since α_0 anticommutes with α_j ($1 \le j \le n$) and with β, and taking into account (2.3), we have:

$$\mathcal{L}_-\alpha_0\phi = (-i\sum_{j=1}^{n}\alpha_j\partial_j - \omega + g(\phi^*\beta\phi)\beta)\alpha_0\phi = \alpha_0(i\sum_{j=1}^{n}\alpha_j\partial_j - \omega - g(\phi^*\beta\phi)\beta)\phi = \alpha_0(-\mathcal{L}_- - 2\omega)\phi = -2\omega\alpha_0\phi.$$

Since α_0 and β are Hermitian, $2\operatorname{Re}[\phi^*\beta\alpha_0\phi] = \phi^*\beta\alpha_0\phi + \overline{\phi^*\beta\alpha_0\phi} = \phi^*\{\beta, \alpha_0\}\phi = 0$; therefore, one also has $\mathcal{L}\alpha_0\phi = \mathcal{L}_-\alpha_0\phi = -2\omega\alpha_0\phi$. □

It follows that the linearization operator has an eigenvalue $2\omega i$:

$$2\omega i \in \sigma_p(\mathcal{J}\mathcal{L}) = \sigma_p(\mathbf{JL}).$$

Since $\sigma(\mathbf{JL})$ is symmetric with respect to $\mathbb{R}$ and $i\mathbb{R}$, for any $g(s)$ in (2.1) and in any dimension $n \ge 1$, we have:

Corollary 2.8. $\pm 2\omega i$ *are* L^2 *eigenvalues of* $\mathbf{JL}$*.*

Remark 2.9. For $|\omega| > m/3$, the eigenvalues $\pm 2\omega i$ are embedded in the essential spectrum. This is in contradiction with the Hypothesis (H:6) in [BC11] on the absence of eigenvalues embedded in the essential spectrum, although we hope that this difficulty could be dealt with using a minor change in the proof.

Remark 2.10. The result of Corollary 2.8 takes place for any nonlinearity $g(\psi^*\beta\psi)$ and in any dimension. The spatial dimension n and the number of components of ψ could be such that there is no matrix α_0 which anticommutes with α_j, $1 \le j \le n$, and with β; then the eigenvector corresponding to $\pm 2\omega i$ can be constructed using the spatial reflections.

3 Virial identities

When studying the bifurcation of eigenvalues from $\lambda = 0$, we will need some conclusions about the generalized null space of the linearized operator. We will draw these conclusions from the Virial identities, which are also known as the Pohozhaev theorem [Poh65]. In the context of the nonlinear Dirac equations, similar results were presented in [ES95].

Lemma 3.1. *For a differentiable family $\lambda \mapsto \phi_\lambda \in H^1(\mathbb{R}^n)$, $\phi_\lambda|_{\lambda=1} = \phi$, one has $\partial_\lambda E(\phi_\lambda)|_{\lambda=1} = \omega \partial_\lambda Q(\phi_\lambda)|_{\lambda=1}$.*

Proof. This immediately follows from (2.4). □

We split the Hamiltonian $E(\psi)$ into $E(\psi) = T(\psi) + V(\psi) = \sum_{j=1}^{n} T_j(\psi) + V(\psi)$, where

$$T_j(\psi) = -i \int_{\mathbb{R}^n} \psi^* \alpha_j \partial_j \psi \, d^n x \quad \text{(no summation in } j\text{)}, \qquad T(\psi) = \sum_{j=1}^{n} T_j(\psi), \qquad V(\psi) = \int_{\mathbb{R}^n} G(\psi^* \beta \psi) \, d^n x.$$

Lemma 3.2 (Pohozhaev Theorem for the nonlinear Dirac equation)**.** *For each solitary wave $\phi(x) e^{-i\omega t}$, there are the following relations:*

$$\frac{n-1}{n} T(\phi) + V(\phi) = \omega Q(\phi), \qquad T_j(\phi) = \frac{1}{n} T(\phi) = \int_{\mathbb{R}^n} \big(G(\phi^* \beta \phi) - \phi^* \beta \phi \, g(\phi^* \beta \phi) \big) \, dx, \quad 1 \le j \le n.$$

Proof. We set $\phi_\lambda(x) = \phi(x_1/\lambda, x_2, \dots, x_n)$. Since $T_1(\phi_\lambda) = T_1(\phi)$, $T_j(\phi_\lambda) = \lambda T_j(\phi)$ for $2 \le j \le n$, $V(\phi_\lambda) = \lambda V(\phi)$, $Q(\phi_\lambda) = \lambda Q(\phi)$, we can use Lemma 3.1 to obtain the following relation ("Virial theorem"):

$$\sum_{j=2}^{n} T_j(\phi) + V(\phi) = \partial_\lambda|_{\lambda=1} \Big(\sum_{j=1}^{n} T_j(\phi_\lambda) + V(\phi_\lambda) \Big) = \omega \partial_\lambda|_{\lambda=1} Q(\phi_\lambda) = \omega Q(\phi). \tag{3.1}$$

Similarly, rescaling in x_j, $1 \le j \le n$, we conclude that $T_1(\phi) = \ldots = T_n(\phi) = \frac{1}{n} T(\phi)$, and (3.1) gives

$$\frac{n-1}{n} T(\phi) + V(\phi) = \omega Q(\phi). \tag{3.2}$$

Moreover, the relation (2.3) yields $T(\phi) + \int_{\mathbb{R}^n} \phi^* \beta \phi \, g(\phi^* \beta \phi) \, dx = \omega Q(\phi)$. Together with (3.2), this gives the desired relation $\frac{1}{n} T(\phi) = \int_{\mathbb{R}^n} \big(G(\phi^* \beta \phi) - \phi^* \beta \phi \, g(\phi^* \beta \phi) \big) \, dx$. □

Remark 3.3. For all nonlinearities for which the existence of solitary wave solutions is proved in [ES95], one has $G(s) - sG'(s) > 0$ for all $s \in \mathbb{R}$ except finitely many points (e.g. $s = 0$); hence for these solitary waves one has $T(\phi) > 0$. In particular, for $G(s) = s - \frac{s^2}{2}$, one has $T(\phi) = n \int_{\mathbb{R}^n} \frac{|\phi^* \beta \phi|^2}{2} \, dx > 0$.

4 Bifurcations from $\lambda = 0$

Lemma 4.1. *Assume that the nonlinearity satisfies the following inequality (see Remark 3.3):*

$$\frac{n+1}{n} G(s) - sG'(s) > 0, \quad s \in \mathbb{R}, \quad s \ne 0. \tag{4.1}$$

Further, assume that $\phi^* \phi$ and $\phi^* \beta \phi$ are spherically symmetric and that

$$\mathscr{N}(\mathbf{L}) = \operatorname{Span} \{ \mathbf{J} \varPhi, \ \partial_k \varPhi; 1 \le k \le n \}; \qquad \dim \mathscr{N}(\mathbf{L}) = n + 1.$$

If $\partial_\omega Q(\phi) \ne 0$, then the generalized null space of $\mathbf{JL}$ is given by

$$\mathscr{N}_g(\mathbf{JL}) = \operatorname{Span} \{ \mathbf{J} \varPhi, \ \partial_\omega \varPhi, \ \partial_k \varPhi, \ \mathbf{A}_k \varPhi; 1 \le k \le n \}; \qquad \dim \mathscr{N}_g(\mathbf{JL}) = 2n + 2.$$

If $\partial_\omega Q(\phi)$ vanishes at ω_*, then $\dim \mathscr{N}_g(\mathbf{JL}|_{\omega_*}) \ge 2n + 4$. Moreover, generically, there is an eigenvalue $\lambda \in \sigma_d(\mathbf{JL})$ with $\operatorname{Re} \lambda > 0$ for ω from an open one-sided neighborhood of ω_*.

Remark 4.2. The assumption that $\phi^*\phi$ and $\phi^*\beta\phi$ are spherically symmetric is satisfied by the ansatz

$$\phi(x) = \begin{bmatrix} g(r)\begin{bmatrix}1\\0\end{bmatrix} \\ if(r)\begin{bmatrix}\cos\theta\\ e^{i\phi}\sin\theta\end{bmatrix}\end{bmatrix}$$

used in e.g. [Wak66, Sol70, ES95, EGS96].

Proof. Taking the derivative of (2.3) with respect to ω, we get

$$-i\sum_{j=1}^{n}\alpha_j\partial_j\partial_\omega\phi + g(\phi^*\beta\phi)\beta\partial_\omega\phi + g'(\phi^*\beta\phi)(\partial_\omega\phi^*\beta\phi + \phi^*\beta\partial_\omega\phi)\beta\phi - \omega\partial_\omega\phi - \phi = \mathcal{L}\partial_\omega\phi - \phi = 0. \tag{4.2}$$

Since $\phi^*\beta\alpha_k\phi + (\alpha_k\phi)^*\beta\phi = \phi^*\{\beta,\alpha\}\phi = 0$, we have

$$\mathcal{L}(\alpha_k\phi) = \mathcal{L}_-(\alpha_k\phi) = -2i\partial_k\phi - 2\omega\alpha_k\phi - \alpha_k\mathcal{L}_-\phi = -2i\partial_k\phi - 2\omega\alpha_k\phi. \tag{4.3}$$

Similarly, since $\phi^*\beta(ix_k\phi) + (ix_k\phi)^*\beta\phi = 0$,

$$\mathcal{L}(ix_k\phi) = \mathcal{L}_-(ix_k\phi) = \alpha_k\phi + ix_k\mathcal{L}_-\phi = \alpha_k\phi. \tag{4.4}$$

Using (4.3) and (4.4), we have

$$\mathcal{L}(\alpha_k\phi + 2\omega i x_k\phi) = -2i\partial_k\phi. \tag{4.5}$$

Until the end of this section, it will be more convenient for us to work in terms of $\Phi \in \mathbb{R}^{2N}$ (see (2.7)). We summarize the above relations (2.8), (2.9), (4.2), (4.3), (4.4), and (4.5) as follows:

$$\mathbf{L}\mathbf{J}\Phi = 0, \qquad \mathbf{L}\partial_k\Phi = 0, \qquad \mathbf{L}\partial_\omega\Phi = \Phi, \qquad \mathbf{L}(-x_k\mathbf{J}\Phi) = \mathbf{A}_k\Phi, \qquad \mathbf{L}(\mathbf{A}_k\Phi - 2\omega x_k\mathbf{J}\Phi) = 2\mathbf{J}\partial_k\Phi,$$

where

$$\mathbf{L} = \mathbf{J}\mathbf{A}_j\partial_j - \omega + g\mathbf{B} + 2g'\mathbf{B}\Phi\langle\Phi, \mathbf{B}\,\cdot\rangle_{\mathbb{R}^{2N}}, \qquad \mathbf{L}_0 = \mathbf{J}\mathbf{A}_j\partial_j - \omega + g\mathbf{B},$$

with $g = g(\phi^*\beta\phi) = g(\Phi^*\mathbf{B}\Phi)$, $g' = g'(\phi^*\beta\phi) = g'(\Phi^*\mathbf{B}\Phi)$. There are no F_k such that $\mathbf{J}\mathbf{L}F_k = \mathbf{A}_k\Phi - 2\omega x_k\mathbf{J}\Phi$. Indeed, checking the orthogonality of $\mathbf{A}_k\Phi - 2\omega x_k\mathbf{J}\Phi$ with respect to $\mathscr{N}((\mathbf{J}\mathbf{L})^*) = \operatorname{Span}\{\Phi, \mathbf{J}\partial_k\Phi\}$, we have:

$$\langle\mathbf{A}_k\Phi - 2\omega x_k\mathbf{J}\Phi, \mathbf{J}\partial_k\Phi\rangle = \int_{\mathbb{R}^n}\phi^*(-i\sum_{k=1}^{n}\alpha_k\partial_k)\phi\,dx + \omega\int_{\mathbb{R}^n}\phi^*\phi\,dx = \frac{T(\phi)}{n} + \omega Q(\phi). \tag{4.6}$$

Note that there is no summation in k in (4.6). By Lemma 3.2, the right-hand side of (4.6) is equal to

$$\frac{T}{n} + \omega Q = \frac{T}{n} + \frac{(n-1)T}{n} + V = T + V = \int_{\mathbb{R}^n}(n(G(\rho) - \rho G'(\rho)) + G(\rho))\,dx, \tag{4.7}$$

where $\rho(x) = \phi^*(x)\beta\phi(x)$. By (4.1), the right-hand side of (4.7) is strictly positive.

Due to Lemma 2.4, we can choose a small counterclockwise-oriented circle γ centered at $\lambda = 0$ such that at $\omega = \omega_*$ the only part of the spectrum $\sigma(\mathbf{J}\mathbf{L})$ inside γ is the eigenvalue $\lambda = 0$. Assume that $\mathcal{O}$ is an open neighborhood of ω_* small enough so that $\sigma(\mathbf{J}\mathbf{L})$ does not intersect γ for $\omega \in \mathcal{O}$. Define

$$P_0 = \frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda - \mathbf{J}\mathbf{L}}, \qquad \omega \in \mathcal{O}. \tag{4.8}$$

For each $\omega \in \mathcal{O}$, P_0 is a projection onto a finite-dimensional vector space $\mathscr{X} := \operatorname{Range}(P_0) \subset L^2(\mathbb{R}^n)$. The operator $P_0\mathbf{J}\mathbf{L}$ is bounded (since P_0 is smoothing of order one) and with the finite-dimensional range. Applying the Fredholm alternative to $P_0\mathbf{J}\mathbf{L}$, we conclude that there is $\boldsymbol{E}_3$ such that $P_0\mathbf{J}\mathbf{L}\boldsymbol{E}_3 = \partial_\omega\Phi$ (hence $\mathbf{J}\mathbf{L}\boldsymbol{E}_3 = \partial_\omega\Phi$) if and only if $\partial_\omega Q(\phi) = 0$. Indeed, one can check that it is precisely in this case that $\partial_\omega \Phi$ is orthogonal to $\mathscr{N}((\mathbf{JL})^*) = \operatorname{Span}\{\Phi, \mathbf{J}\partial_k\Phi\}$:

$$\langle \partial_\omega \Phi, \Phi \rangle = \frac{1}{2}\partial_\omega \int \phi^* \phi \, dx = \frac{1}{2}\partial_\omega Q(\phi),$$

$$2\langle \partial_\omega \Phi, \mathbf{J}\partial_k \Phi \rangle = \langle \partial_\omega \Phi, \mathbf{L}(\mathbf{A}_k \Phi - 2\omega x_k \mathbf{J}\Phi) \rangle = \langle \mathbf{L}\partial_\omega \Phi, (\mathbf{A}_k \Phi - 2\omega x_k \mathbf{J}\Phi) \rangle$$
$$= \langle \Phi, (\mathbf{A}_k \Phi - 2\omega x_k \mathbf{J}\Phi) \rangle = \langle \Phi, \mathbf{A}_k \Phi \rangle - 2\omega \langle \Phi, x_k \mathbf{J}\Phi \rangle = \int_{\mathbb{R}^n} \phi^* \alpha_k \phi \, dx = 0.$$

The right-hand side vanishes since it is the kth component of the (zero) momentum of the standing solitary wave.

Remark 4.3. Using (2.3), one can explicitly compute

$$2\omega \langle \phi, \alpha_k \phi \rangle = \langle \phi, \alpha_k(-i\alpha_j \partial_j + g\beta)\phi \rangle + \langle (-i\alpha_j \partial_j + g\beta)\phi, \alpha_k \phi \rangle = -2i \langle \phi, \partial_k \phi \rangle = -i \int_{\mathbb{R}^n} \partial_k(\phi^* \phi) \, dx = 0.$$

Once there is $\boldsymbol{E}_3$ such that $\mathbf{JL}\boldsymbol{E}_3 = \partial_\omega \Phi$, there is also $\boldsymbol{E}_4$ such that $\mathbf{JL}\boldsymbol{E}_4 = \boldsymbol{E}_3$, since $\boldsymbol{E}_3$ is orthogonal to the null space $\mathscr{N}((\mathbf{JL})^*) = \operatorname{Span}\{\Phi, \mathbf{J}\partial_k \Phi\}$:

$$\langle \boldsymbol{E}_3, \Phi \rangle = \langle \boldsymbol{E}_3, \mathbf{L}\partial_\omega \Phi \rangle = \langle \mathbf{L}\boldsymbol{E}_3, \partial_\omega \Phi \rangle = -\langle \mathbf{J}\partial_\omega \Phi, \partial_\omega \Phi \rangle = 0,$$

$$2\langle \boldsymbol{E}_3, \mathbf{J}\partial_k \Phi \rangle = \langle \boldsymbol{E}_3, \mathbf{L}(\mathbf{A}_k \Phi - 2\omega x_k \mathbf{J}\Phi) \rangle = \langle \mathbf{L}\boldsymbol{E}_3, \mathbf{A}_k \Phi - 2\omega x_k \mathbf{J}\Phi \rangle = -\langle \mathbf{J}\partial_\omega \Phi, \mathbf{A}_k \Phi - 2\omega x_k \mathbf{J}\Phi \rangle = 0.$$

To check that the right-hand side of the second line is indeed equal to zero, one needs to take into account the following:

$$\langle \mathbf{J}\partial_\omega \Phi, \omega x_k \mathbf{J}\Phi \rangle = \omega \langle \partial_\omega \Phi, x_k \Phi \rangle = \frac{\omega}{2}\partial_\omega \int_{\mathbb{R}^n} x_k \phi^* \phi \, dx = 0, \tag{4.9}$$

$$\begin{aligned} &\langle \mathbf{J}\partial_\omega \Phi, \mathbf{A}_k \Phi \rangle = \langle \mathbf{J}\partial_\omega \Phi, \mathbf{L}(-x_k \mathbf{J}\Phi) \rangle = \langle \mathbf{J}\partial_\omega \Phi, \mathbf{L}_0(-x_k \mathbf{J}\Phi) \rangle = -\langle \mathbf{L}_0 \partial_\omega \Phi, x_k \Phi \rangle \\ &= -\langle \Phi - 2g'(\phi^* \beta \phi)\mathbf{B}\Phi \langle \Phi, \mathbf{B}\partial_\omega \Phi \rangle_{\mathbb{R}^{2N}}, x_k \Phi \rangle = -\langle \Phi, x_k \Phi \rangle + \langle 2g'(\phi^* \beta \phi)\mathbf{B}\Phi \langle \Phi, \mathbf{B}\partial_\omega \Phi \rangle_{\mathbb{R}^{2N}}, x_k \Phi \rangle \\ &= -\int_{\mathbb{R}^n} x_k \phi^* \phi \, dx + \partial_\omega \int_{\mathbb{R}^n} x_k K(\phi^* \beta \phi) \, dx = 0. \end{aligned} \tag{4.10}$$

Above, $K(s)$ is the antiderivative of $sg'(s)$ such that $K(0) = 0$. The integrals in the right-hand sides of (4.9) and (4.10) are equal to zero due to our assumption on the symmetry properties of $\phi^*\phi$ and $\phi^*\beta\phi$. Note that $\mathcal{L}_-$ is $\mathbb{C}$-linear, hence commutes with a multiplication by i, and therefore $\mathbf{J}$ and $\mathbf{L}_0$ commute; we used this when deriving (4.10).

We will assume that

$$\langle \boldsymbol{E}_3, \mathbf{L}\boldsymbol{E}_3 \rangle \neq 0. \tag{4.11}$$

Then there is no $\boldsymbol{E}_5$ such that $\mathbf{JL}\boldsymbol{E}_5 = \boldsymbol{E}_4$, since $\boldsymbol{E}_4$ is not orthogonal to the null space $\mathcal{N}((\mathbf{JL})^*) \ni \Phi$:

$$\langle \boldsymbol{E}_4, \Phi \rangle = \langle \boldsymbol{E}_4, \mathbf{L}\partial_\omega \Phi \rangle = \langle \boldsymbol{E}_4, \mathbf{LJL}\boldsymbol{E}_3 \rangle = -\langle \boldsymbol{E}_3, \mathbf{L}\boldsymbol{E}_3 \rangle \neq 0. \tag{4.12}$$

Remark 4.4. If (4.11) is not satisfied, then the dimension of the generalized null space of $\mathbf{JL}$ at ω_* may jump by more than two; this means that there are more than two eigenvalues colliding at $\lambda = 0$ as ω passes through ω_*. We expect that generically this scenario does not take place.

We will break the finite-dimensional vector space $\mathscr{X} = \operatorname{Range}(P_0)$ into a direct sum

$$\mathscr{X} = \mathscr{Y} \oplus \mathscr{Z}, \qquad \mathscr{Z} = \operatorname{Span}\{\partial_j \Phi, \mathbf{A}_j \Phi; 1 \le j \le n\},$$

so that both $\mathscr{Y}$ and $\mathscr{Z}$ are invariant with respect to the action of $\mathbf{JL}$. (Let us mention that P_0, $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$, and $\mathbf{JL}$ depend on ω.) Set

$$\boldsymbol{e}_4(\omega) = P_0 \boldsymbol{E}_4 \in \mathscr{Y}, \qquad \boldsymbol{e}_3(\omega) = \mathbf{JL}\boldsymbol{e}_4 \in \mathscr{Y}. \tag{4.13}$$

Since P_0 continuously depends on ω, $\boldsymbol{e}_3(\omega)$ and $\boldsymbol{e}_4(\omega)$ are continuous functions of ω. $\{\boldsymbol{e}_1(\omega), \boldsymbol{e}_2(\omega), \boldsymbol{e}_3(\omega), \boldsymbol{e}_4(\omega)\}$, $\omega \in \mathcal{O}$, is a frame in the space $\mathscr{Y}$. For some continuous functions $\sigma_1(\omega)$, $\sigma_2(\omega)$, $\sigma_3(\omega)$, and $\sigma_4(\omega)$, there is the relation

$$\mathbf{JL}\boldsymbol{e}_3(\omega) = \sigma_1(\omega)\boldsymbol{e}_1(\omega) + \sigma_2(\omega)\boldsymbol{e}_2(\omega) + \sigma_3(\omega)\boldsymbol{e}_3(\omega) + \sigma_4(\omega)\boldsymbol{e}_4(\omega), \qquad \omega \in \mathcal{O}. \tag{4.14}$$

Evaluating (4.14) at ω_*, we conclude that

$$\sigma_1(\omega_*) = 0, \qquad \sigma_2(\omega_*) = 1, \qquad \sigma_3(\omega_*) = 0, \qquad \sigma_4(\omega_*) = 0. \tag{4.15}$$

In the frame $\{\boldsymbol{e}_1(\omega),\ \boldsymbol{e}_2(\omega),\ \boldsymbol{e}_3(\omega),\ \boldsymbol{e}_4(\omega)\}$, the operator $\mathbf{JL}$ restricted onto $\mathscr{Y}$ is represented by the matrix

$$M = \begin{bmatrix} 0 & 1 & \sigma_1(\omega) & 0 \\ 0 & 0 & \sigma_2(\omega) & 0 \\ 0 & 0 & \sigma_3(\omega) & 1 \\ 0 & 0 & \sigma_4(\omega) & 0 \end{bmatrix}, \qquad \omega \in \mathcal{O}, \tag{4.16}$$

which is a 4×4 Jordan block at ω_*. Let us investigate its entries. Pairing (4.14) with $\Phi = \mathbf{J}^{-1}\boldsymbol{e}_1$ and taking into account that $\left\langle \mathbf{J}^{-1}\boldsymbol{e}_1, (\mathbf{JL})^2\boldsymbol{e}_4(\omega)\right\rangle = -\left\langle \mathbf{L}\boldsymbol{e}_1, \mathbf{JL}\boldsymbol{e}_4(\omega)\right\rangle = 0$, $\langle \mathbf{J}^{-1}\boldsymbol{e}_1, \boldsymbol{e}_1\rangle = 0$, and $\langle \mathbf{J}^{-1}\boldsymbol{e}_1, \boldsymbol{e}_3\rangle = \langle \mathbf{J}^{-1}\boldsymbol{e}_1, \mathbf{JL}\boldsymbol{e}_4\rangle = -\langle \mathbf{L}\boldsymbol{e}_1, \boldsymbol{e}_4\rangle = 0$, we get:

$$0 = \sigma_2(\omega)\left\langle \mathbf{J}^{-1}\boldsymbol{e}_1, \boldsymbol{e}_2\right\rangle + \sigma_4(\omega)\left\langle \mathbf{J}^{-1}\boldsymbol{e}_1, \boldsymbol{e}_4\right\rangle. \tag{4.17}$$

Define the function

$$\mu(\omega) := -\langle \Phi, \boldsymbol{e}_4\rangle, \qquad \omega \in \mathcal{O}. \tag{4.18}$$

Since

$$\mu(\omega_*) = -\langle \Phi, \boldsymbol{e}_4\rangle|_{\omega_*} = -\langle \Phi|_{\omega_*}, \boldsymbol{E}_4\rangle = \langle \boldsymbol{E}_3, \mathbf{L}\boldsymbol{E}_3\rangle \neq 0 \tag{4.19}$$

by (4.12), we may take the open neighborhood $\mathcal{O}$ of ω_* to be sufficiently small so that $\mu(\omega)$ does not vanish for $\omega \in \mathcal{O}$. Then

$$\sigma_4(\omega) = \frac{\sigma_2(\omega)\partial_\omega Q(\phi)}{\mu(\omega)}, \qquad \omega \in \mathcal{O}. \tag{4.20}$$

Let us show that $\sigma_3(\omega)$ is identically zero in an open neighborhood of ω_*. Applying $(\mathbf{JL})^2$ to (4.14), we get:

$$(\mathbf{JL})^3\boldsymbol{e}_3(\omega) = \sigma_3(\omega)(\mathbf{JL})^2\boldsymbol{e}_3(\omega) + \sigma_4(\omega)(\mathbf{JL})^2\boldsymbol{e}_4(\omega). \tag{4.21}$$

Coupling with $\mathbf{J}^{-1}\boldsymbol{e}_4$ and using $\langle \mathbf{J}^{-1}\boldsymbol{e}_4, (\mathbf{JL})^3\boldsymbol{e}_3\rangle = \langle \boldsymbol{e}_3, \mathbf{LJL}\boldsymbol{e}_3\rangle = 0$ and $\langle \mathbf{J}^{-1}\boldsymbol{e}_4, (\mathbf{JL})^2\boldsymbol{e}_4\rangle = -\langle \boldsymbol{e}_4, \mathbf{LJL}\boldsymbol{e}_4\rangle = 0$ (due to anti-selfadjointness of $\mathbf{LJL}$), we have

$$\sigma_3(\omega)\langle \mathbf{J}^{-1}\boldsymbol{e}_4, (\mathbf{JL})^2\boldsymbol{e}_3\rangle = 0. \tag{4.22}$$

The factor at $\sigma_3(\omega)$ is nonzero. Indeed, using (4.15),

$$\langle \mathbf{J}^{-1}\boldsymbol{e}_4, (\mathbf{JL})^2\boldsymbol{e}_3\rangle|_{\omega_*} = \langle \mathbf{J}^{-1}\boldsymbol{e}_4,\ \sigma_2\boldsymbol{e}_1 + \sigma_3\mathbf{JL}\boldsymbol{e}_3 + \sigma_4\boldsymbol{e}_3\rangle|_{\omega_*} = \langle \mathbf{J}^{-1}\boldsymbol{e}_4, \boldsymbol{e}_1\rangle|_{\omega_*} = -\langle \boldsymbol{e}_4, \Phi\rangle|_{\omega_*},$$

which is nonzero due to (4.12). We conclude that $\sigma_3(\omega)$ is identically zero in an open neighborhood of ω_*. We take $\mathcal{O}$ small enough so that $\sigma_3|_{\mathcal{O}} \equiv 0$.

Near $\omega = \omega_*$, the eigenvalues of $\mathbf{JL}$ which are located inside a small contour around $\lambda = 0$ coincide with the eigenvalues of the matrix M, defined in (4.16). Since σ_3 is identically zero in $\mathcal{O}$, these eigenvalues satisfy

$$\lambda^2(\lambda^2 - \sigma_4(\omega)) = 0, \qquad \omega \in \mathcal{O}. \tag{4.23}$$

By (4.20), if $\partial_\omega Q(\phi)$ changes sign at ω_*, so does $\sigma_4(\omega)$, hence in a one-sided open neighborhood of ω_* there are two real eigenvalues of $\mathbf{JL}$, one positive (indicating the linear instability) and one negative. □

Remark 4.5. This argument is slightly longer than a similar computation in [CP03] since we do not assume that ϕ could be chosen purely real (allowing for a common ansatz used in [ES95] in the context of the nonlinear Dirac equation), and consequently we could not take $\boldsymbol{e}_j$ to be "imaginary" for j odd and "real" for j even, and enjoy the vanishing of $\langle \mathbf{J}^{-1}\boldsymbol{e}_j, \boldsymbol{e}_k\rangle$ for $j + k$ is even.

5 Concluding remarks

In the conclusion, let us make several observations.

Remark 5.1. In the case of the nonlinear Schrödinger equation, the function $\mu(\omega)$ is strictly positive (this is due to positive-definiteness of L_- in (1.4), which results in positivity of (4.19); see [VK73, CP03]), so that the collisions of eigenvalues at $\lambda = 0$ always happen according to the following scenario: when $dQ/d\omega$ changes from negative to positive (no matter whether this happens as ω increases or decreases), there is a pair of eigenvalues on the imaginary axis colliding at $\lambda = 0$ and proceeding along the real axis. Then, further, if $dQ/d\omega$ changes from positive to negative, this pair of real eigenvalues return to $\lambda = 0$ and then retreat onto the imaginary axis. For the Dirac equation, we can not rule out that $\mu(\omega)$ changes the sign (becoming negative). If this were the case, vanishing of $dQ/d\omega$ would be accompanied with the reversed bifurcation mechanism: as $dQ/d\omega$ goes from positive to negative, another pair of imaginary eigenvalues collide at $\lambda = 0$ and proceed along the real axis. We do not have examples of particular nonlinearities which lead to such a scenario.

Remark 5.2. In one dimension, for the nonlinearity $g(s) = 1 - s^k$, $k \in \mathbb{N}$, using the asymptotics as in [Gua08], one can derive that, as $\omega \to 1-$, the function $\mu(\omega)$ defined in (4.18) has the asymptotics

$$\mu(\omega) = O((1 - \omega^2)^{\frac{1}{k} - \frac{7}{2}}).$$

There is the same asymptotics in the case of the nonlinear Schrödinger equation (1.2) with the same nonlinearity $g(s)$. In particular, $\lim\limits_{\omega\to 1-} \mu(\omega) = +\infty$, hence $\mu(\omega)$ remains positive for ω sufficiently close to 1 (precisely as for the nonlinear Schrödinger equation, when $\mu(\omega) > 0$ for all ω), suggesting that the bifurcation scenario for the nonlinear Dirac equation near $\omega = 1$ is the same as for the nonlinear Schrödinger equation.

Remark 5.3. For the nonlinear Dirac equation in one dimension with the nonlinearity $g(s) = 1 - s$ (“Soler model”), the solitary waves exist for $\omega \in (0, 1)$. One can explicitly compute that the charge is given by $Q(\omega) = \int_{\mathbb{R}} |\psi|^2\,dx = \frac{2\sqrt{1-\omega^2}}{\omega}$ (see e.g. [LG75]) so that $dQ/d\omega < 0$ for $\omega \in (0, 1)$, implying that there are no eigenvalues of $\mathbf{JL}$ colliding at $\lambda = 0$ for any $\omega \in (0, 1)$. According to our numerical results [BC09], we expect that in this model the solitary waves are spectrally stable, at least for ω sufficiently close to 1.

Remark 5.4. Even if $dQ/d\omega$ never vanishes, so that there are no bifurcations of nonzero real eigenvalues from $\lambda = 0$, there may be nonzero real eigenvalues present in the spectrum of *all* solitary waves. We expect that the Vakhitov-Kolokolov criterion could again be useful here when applied in the nonrelativistic limit $\omega \to m$, when the properties of the nonlinear Dirac equation are similar to properties of the nonlinear Schrödinger equation. This idea has been mentioned in [CKMS10]. Our preliminary results indicate that in one dimension, for ω sufficiently close to 1, the nonlinearity with $g(s) = 1 - s^k + o(s^k)$ with $k = 1$ and $k = 2$ does not produce nonzero real eigenvalues, while for $k \geq 3$ there are two real eigenvalues, one positive (leading to linear instability) and one negative.

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