

**A LINK BETWEEN BLACK HOLES
AND THE GOLDEN RATIO**

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Abstract

We consider a variational formalism to describe black holes solution in higher dimensions. Our procedure clarifies the arbitrariness of the radius parameter and, in particular, the meaning of the event horizon of a black hole. Moreover, our formalism enables us to find a surprising link between black holes and the golden ratio.

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It is well known that the ansatz for a spherically symmetric static black hole solution in a d -dimensional space-time M^d can be written as (see Refs. [1]-[5] and references therein)

$$\gamma_{\mu\nu} = \begin{pmatrix} -e^{f(r)} & 0 & 0 \\ 0 & e^{h(r)} & 0 \\ 0 & 0 & \varphi^2(r)\tilde{\gamma}_{ij}(\xi^k) \end{pmatrix}, \quad (1)$$

where $\tilde{\gamma}_{ij}(\xi^k)$ determines a maximally spherically symmetric space in $(d-2)$ dimensions. From this metric one finds that the only non-vanishing Ricci tensor components are (see Ref. [6]-[8] and references therein)

$$R_{11} = e^{f-h} \left(\frac{\ddot{f}}{2} + \frac{\dot{f}^2}{4} - \frac{\dot{f}\dot{h}}{4} + \frac{(d-2)}{2} \dot{f} \frac{\dot{\varphi}}{\varphi} \right), \quad (2)$$

$$R_{22} = -\frac{\ddot{f}}{2} - \frac{\dot{f}^2}{4} + \frac{\dot{f}\dot{h}}{4} + \frac{(d-2)}{2} \dot{h} \frac{\dot{\varphi}}{\varphi} - (d-2) \frac{\ddot{\varphi}}{\varphi}, \quad (3)$$

$$R_{ij} = e^{-h} \left\{ \frac{(\dot{h} - \dot{f})\varphi\dot{\varphi}}{2} - \varphi\ddot{\varphi} - (d-3)\dot{\varphi}^2 \right\} \tilde{\gamma}_{ij} + k(d-3)\tilde{\gamma}_{ij}. \quad (4)$$

Here, we used the notation $\dot{A} \equiv \frac{dA}{dr}$ for any function $A = A(r)$. Moreover, we have $k = \{-1, 0, 1\}$. Thus, the Ricci scalar $R = \gamma^{\mu\nu} R_{\mu\nu}$ becomes

$$R = e^{-h} \left\{ -\ddot{f} - \frac{\dot{f}^2}{2} + \frac{\dot{f}\dot{h}}{2} + (d-2)(\dot{h} - \dot{f}) \frac{\dot{\varphi}}{\varphi} - 2(d-2) \frac{\ddot{\varphi}}{\varphi} - (d-3)(d-2) \frac{\dot{\varphi}^2}{\varphi^2} \right\} + k(d-3)(d-2)\varphi^{-2}. \quad (5)$$

On the other hand, we have

$$\sqrt{-\gamma} = e^{\frac{f+h}{2}\varphi^{(d-2)}} \sqrt{\tilde{\gamma}}, \quad (6)$$

where γ and $\tilde{\gamma}$ denote the determinant of $\gamma_{\mu\nu}$ and $\tilde{\gamma}_{ij}$, respectively. Consequently, up to total derivative, the higher dimensional Einstein-Hilbert action

$$S = \frac{1}{2} \int_{M^d} \sqrt{-\gamma} R, \quad (7)$$

gives

$$S = \frac{(d-2)}{2} \int_{M^d} \sqrt{\tilde{\gamma}} \left[\Omega^{-1} \left\{ (\varphi^{(d-2)} \mathcal{F}) \left((d-3) \frac{\dot{\varphi}^2}{\varphi^2} + 2 \frac{\dot{\mathcal{F}}}{\mathcal{F}} \frac{\dot{\varphi}}{\varphi} \right) \right\} + \Omega \{ k(d-3) \mathcal{F} \varphi^{(d-4)} \} \right]. \quad (8)$$

Here, we used the notation $\mathcal{F} \equiv e^{\frac{f}{2}}$ and $\Omega \equiv e^{\frac{h}{2}}$. Note that the case $d = 2$ is exceptional. Similar conclusion can be obtained in the case of $d = 3$. For our purpose it turns out convenient to assume that $d - 2 \neq 0$ and $d - 3 \neq 0$. Observe that in (8) Ω acts as auxiliary field.

Performing variations of the action (8) with respect to \mathcal{F} , Ω and φ one derives the following equations

$$\frac{2\ddot{\varphi}}{\varphi} + (d-3)\frac{\dot{\varphi}^2}{\varphi^2} - 2\frac{\dot{\Omega}}{\Omega}\frac{\dot{\varphi}}{\varphi} - k(d-3)\Omega^2\varphi^{-2} = 0, \quad (9)$$

$$(d-3)\frac{\dot{\varphi}^2}{\varphi^2} + 2\frac{\dot{\mathcal{F}}}{\mathcal{F}}\frac{\dot{\varphi}}{\varphi} - k(d-3)\Omega^2\varphi^{-2} = 0 \quad (10)$$

and

$$\begin{aligned} \frac{(d-4)(d-3)}{2}\frac{\dot{\varphi}^2}{\varphi^2} - \frac{\dot{\mathcal{F}}}{\mathcal{F}}\frac{\dot{\Omega}}{\Omega} + \frac{\dot{\mathcal{F}}}{\mathcal{F}} - (d-3)\left(\left(\frac{\dot{\Omega}}{\Omega} - \frac{\dot{\mathcal{F}}}{\mathcal{F}}\right)\frac{\dot{\varphi}}{\varphi} - \frac{\ddot{\varphi}}{\varphi}\right) \\ - \frac{k(d-4)(d-3)}{2}\Omega^2\varphi^{-2} = 0, \end{aligned} \quad (11)$$

respectively. Using (1)-(4) and considering the gravitational field equations

$$R_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R = 0,$$

which can be obtained from the action (7), one can verify that the formulae (9)-(11) are consistent. In particular one can combine (9) and (10) to obtain

$$\frac{\dot{\mathcal{F}}}{\mathcal{F}} + \frac{\dot{\Omega}}{\Omega} = \frac{\ddot{\varphi}}{\dot{\varphi}}. \quad (12)$$

One recognizes in this formula the typical relation between \mathcal{F} , Ω and φ obtained after setting $R_{11} = 0$ and $R_{22} = 0$ in (2) and (3) and making the combination $e^{-f+h}R_{11} + R_{22} = 0$.

We shall assume that one may associate with (8) the expression (see Refs. [1]-[5] and references therein)

$$\begin{aligned} \mathcal{L} = \frac{1}{2}[\Omega^{-1}(\varphi^{(d-2)}\mathcal{F})\{(d-3)\frac{\dot{\varphi}^2}{\varphi^2} + 2\frac{\dot{\mathcal{F}}}{\mathcal{F}}\frac{\dot{\varphi}}{\varphi}\} \\ + \Omega\{k(d-3)\mathcal{F}\varphi^{(d-4)}\}]. \end{aligned} \quad (13)$$

The object \mathcal{L} is a kind of Lagrangian, but one should keep in mind that there is not time dependence in (13). So, rigorously speaking, (13) is not a Lagrangian. For this reason, and since it was derived from the action (8), let us call \mathcal{L} in (13) ‘Lagravity’, from ‘Lagrangian and gravity’ relation. We observe that the

variable Ω plays the role of Lagrange multiplier. This suggests that we seek a possible analogue of (13) with constraint Hamiltonian formulation. (in order to be consistent we shall call the kind of Hamiltonian \mathcal{H} associated with \mathcal{L} ‘Hagravity’.) For this purpose let us first introduce the redefinition

$$\lambda \equiv \varphi^{-(d-2)} \mathcal{F}^{-1} \Omega, \quad (14)$$

of the Lagrange multiplier Ω . In terms of λ the Lagravity, (13) reads

$$\mathcal{L} = \frac{1}{2} [\lambda^{-1} ((d-3) \frac{\dot{\varphi}^2}{\varphi^2} + 2 \frac{\dot{\mathcal{F}}}{\mathcal{F}} \frac{\dot{\varphi}}{\varphi}) + \lambda (k(d-3) \mathcal{F}^2 \varphi^{2(d-3)})]. \quad (15)$$

This can be simplified further by introducing the two coordinates q^1 and q^2 in the following form:

$$\varphi \equiv e^{q^1} \quad (16)$$

and

$$\mathcal{F} \equiv e^{q^2}. \quad (17)$$

In fact, in terms q^1 and q^2 , one sees that (15) can be written as

$$\mathcal{L} = \frac{1}{2} [\lambda^{-1} ((d-3) (\dot{q}^1)^2 + 2(\dot{q}^1)(\dot{q}^2)) + \lambda m_0^2], \quad (18)$$

where

$$m_0^2 = k(d-3) e^{2(d-3)q^1} e^{2q^2}. \quad (19)$$

Before proceeding further, it is convenient to verify that we are in the right route, by first writing the ‘Euler-Lagrange’ equations associated with (18). For the coordinate q^1 we have the equation

$$\frac{d}{dr} (\lambda^{-1} ((d-3) \dot{q}^1 + \dot{q}^2) - \lambda (d-3) m_0^2) = 0, \quad (20)$$

and for q^2 we get

$$\frac{d}{dr} (\lambda^{-1} \dot{q}^1) - \lambda m_0^2 = 0. \quad (21)$$

While for λ we find

$$\lambda^{-2} ((d-3) (\dot{q}^1)^2 + 2\dot{q}^1 \dot{q}^2) - m_0^2 = 0. \quad (22)$$

The idea now is to show that from these equations one can derive (9)-(11). For this purpose, one first note that from (20) and (21) one finds

$$\frac{d}{dr}(\lambda^{-1}\dot{q}^2) = 0. \quad (23)$$

Thus, by writing (21), (22) and (23) in terms \mathcal{F}, Ω and φ we obtain

$$(d-3)\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\ddot{\varphi}}{\varphi} + \frac{\dot{\mathcal{F}}}{\mathcal{F}}\frac{\dot{\varphi}}{\varphi} - \frac{\dot{\Omega}}{\Omega}\frac{\dot{\varphi}}{\varphi} - k(d-3)\Omega^2\varphi^{-2} = 0, \quad (24)$$

$$(d-3)\frac{\dot{\varphi}^2}{\varphi^2} + \frac{2\dot{\mathcal{F}}}{\mathcal{F}}\frac{\dot{\varphi}}{\varphi} - k(d-3)\Omega^2\varphi^{-2} = 0 \quad (25)$$

and

$$(d-2)\frac{\dot{\mathcal{F}}}{\mathcal{F}}\frac{\dot{\varphi}}{\varphi} - \frac{\dot{\mathcal{F}}}{\mathcal{F}}\frac{\dot{\Omega}}{\Omega} + \frac{\ddot{\mathcal{F}}}{\mathcal{F}} = 0, \quad (26)$$

respectively. We first note that (25) is just (10). Now, multiplying (24) by $(d-3)$ and combining the resultant formula with (25) and (26) one sees that (11) follows. Finally, it is not difficult to obtain (9) from (24) and (25).

For our goal it turns out convenient to solve directly the equations (21)-(23). Combining (21) and (22) one formally get

$$\lambda\frac{d}{dr}(\lambda^{-1}\dot{q}^1) = (d-3)(\dot{q}^1)^2 + 2\dot{q}^1\dot{q}^2 \quad (27)$$

or

$$-\frac{\dot{\lambda}}{\lambda}\dot{q}^1 + \ddot{q}^1 = (d-3)(\dot{q}^1)^2 + 2\dot{q}^1\dot{q}^2. \quad (28)$$

This expression suggests that we define the quantity

$$\Omega \equiv e^{(d-2)q^1} e^{q^2} \lambda, \quad (29)$$

which is, of course, consistent with (14). In terms of Ω , (28) gives

$$(d-2)(\dot{q}^1)^2 + \dot{q}^1\dot{q}^2 - \frac{\dot{\Omega}}{\Omega}\dot{q}^1 + \ddot{q}^1 = (d-3)(\dot{q}^1)^2 + 2\dot{q}^1\dot{q}^2. \quad (30)$$

Simplifying this equation, one sees that (30) is reduced to

$$(\dot{q}^2 + \frac{\dot{\Omega}}{\Omega})\dot{q}^1 = \ddot{q}^1 + (\dot{q}^1)^2 \quad (31)$$

or

$$(\dot{q}^2 + \frac{\dot{\Omega}}{\Omega}) = \frac{\ddot{q}^1}{\dot{q}^1} + \dot{q}^1. \quad (32)$$

But since $q^1 = \ln \varphi$ and $q^2 = \ln \mathcal{F}$ one discovers that (32) can be rewritten as (12). The solution of the formula (32) is

$$(q^2 + \ln \Omega) = \ln \dot{q}^1 + q^1 + \ln \alpha, \quad (33)$$

where α is a constant. Notice that (33) can also be written as

$$\mathcal{F}\Omega = \alpha\dot{\varphi}. \quad (34)$$

On the other that from (23) we must have

$$\lambda = \beta\dot{q}^2, \quad (35)$$

where β is another constant. In terms of Ω , \mathcal{F} and φ the equation (35) becomes

$$\Omega = \beta\varphi^{(d-2)}\dot{\mathcal{F}}. \quad (36)$$

Therefore, using (36) the equation (34) leads to

$$\mathcal{F}\dot{\mathcal{F}} = \gamma\frac{\dot{\varphi}}{\varphi^{(d-2)}}, \quad (37)$$

where $\gamma = \frac{\alpha}{\beta}$. The expression (37) implies the equation

$$\frac{d}{dr}(\mathcal{F}^2) = \frac{d}{dr}\left(-\frac{2\gamma}{(d-3)\varphi^{(d-3)}}\right), \quad (38)$$

whose solution is

$$\mathcal{F}^2 = a - \frac{b}{\varphi^{(d-3)}}. \quad (39)$$

Here, a is a constant and $b = \frac{2\gamma}{(d-3)}$. Moreover, one can verify that (39) satisfies (25) if $a = k\alpha^2$. In the Newtonian limit one can set $a = k$ and $b = \frac{G_d M}{2(d-3)c^2}$, where G_d is the d -dimensional Newton gravitational constant, M is the mass associated with the black hole and c is the light velocity.

Considering (39) one can also determine Ω . In fact, from (34) we see that

$$\Omega^2 = \frac{\alpha^2\dot{\varphi}^2}{\mathcal{F}^2} = \frac{\alpha^2\dot{\varphi}^2}{a - \frac{b}{\varphi^{(d-3)}}}. \quad (40)$$

Writing $\mathcal{F} = e^{\frac{t}{2}}$ and $\Omega = e^{\frac{h}{2}}$ one recognizes in (39) and (40) the traditional solutions of a black hole. What is interesting about our formalism is that we obtained these solutions by using the Lagravity (18) rather than the Einstein-Hilbert field equations.

Another advantage of our formalism is that we can now shed some light on the meaning of the event horizon. Suppose we set $\varphi = r$. From (40) one sees that Ω is singular when $a - \frac{b}{r^{(d-3)}} = 0$. This surface defines the so called event horizon. It has been always argued that this is not true singularity because one can find another set of coordinates (in particular the Kruskal-Szekeres coordinates) that avoids such a singularity. From our formalism, however, Ω (or λ) plays a role of a Lagrange multiplier and therefore from (40) one sees that setting $\varphi = r$ gives $\dot{\varphi}^2 = 1$ and this is equivalent to fix the gauge associated with Ω . So, from our perspective the event horizon does not determine a true singularity because, in relation with the parameter r , such a surface is gauge dependent.

Now, let us compute the analogue of the ‘canonical momenta’ p_1 and p_2 associated with the coordinates q^1 and q^2 , respectively. From (18) one gets

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}^1} = \lambda^{-1}((d-3)\dot{q}^1 + \dot{q}^2) \quad (41)$$

and

$$p_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}^2} = \lambda^{-1}\dot{q}^1. \quad (42)$$

Considering these results one can now show that (18) can be obtained from the first order Lagravity

$$\mathcal{L} = \dot{q}^1 p_1 + \dot{q}^2 p_2 - \frac{\lambda}{2}(-(d-3)(p_2)^2 + 2p_1 p_2 - m_0^2). \quad (43)$$

The Lagravity (18) can also be written as

$$\mathcal{L} = \frac{1}{2}[\lambda^{-1}(\dot{q}^a \dot{q}^b \xi_{ab}) + \lambda m_0^2], \quad (44)$$

where the metric ξ_{ab} is given by

$$\xi_{ab} = \begin{pmatrix} (d-3) & 1 \\ 1 & 0 \end{pmatrix}. \quad (45)$$

Similarly, we can write (43) in the following form:

$$\mathcal{L} = \dot{q}^a p_a - \frac{\lambda}{2} \mathcal{H}. \quad (46)$$

Here,

$$\mathcal{H} = \xi^{ab} p_a p_b - m_0^2 \quad (47)$$

with

$$\xi^{ab} = \begin{pmatrix} 0 & 1 \\ 1 & -(d-3) \end{pmatrix}. \quad (48)$$

Note that (48) is the inverse matrix of (45). Of course, \mathcal{H} can be interpreted as ‘constraint’ Hagravity. In fact, it can be shown in straightforward way that this constraint satisfies the analogue condition of a first class constraint and therefore it can be interpreted as the gauge generator of the variable r (see Ref. [9] and references therein).

It is also interesting to write the second order Lagravity

$$\mathcal{L} = \frac{1}{2} m_0 (\dot{q}^a \dot{q}^b \xi_{ab})^{1/2}, \quad (49)$$

which, by using the corresponding ‘Euler-Lagrange’ equation for λ , can be obtained from (44). It is straightforward to see that the associated ‘action’

$$\mathcal{S} = \frac{1}{2} \int dr m_0 (\dot{q}^a \dot{q}^b \xi_{ab})^{1/2}, \quad (50)$$

is invariant under reparamitrazation $r' = r'(r)$. This explain the reason for existence of the arbitrary function λ and the Hagravity constraint \mathcal{H} .

If the cosmological constant Λ is included via the usual extended action $S = \frac{1}{2} \int_{M^d} \sqrt{-\gamma} (R - 2\Lambda)$ one can show that the Lagravity (49) also follows but with the ‘mass’ m_0^2 now given by

$$m_0^2 = k(d-3)e^{2(d-3)q^1} e^{2q^2} - \frac{2\Lambda}{d-2} e^{2(d-2)q^1} e^{2q^2}. \quad (51)$$

It is worth mentioning that, in four dimensions, that is in $d = 4$, from our formalism arises an intriguing and fascinating result. In such a case the metric (45) is reduced to

$$\xi_{ab} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \quad (52)$$

Of course, the matrix (52) can be diagonalized. We first find the eigenvalues by computing the determinant

$$\begin{vmatrix} 1 - \phi & 1 \\ 1 & -\phi \end{vmatrix} = \phi^2 - \phi - 1 = 0. \quad (53)$$

Surprisingly, one recognizes in (53) the famous formula which determines the golden ratio. In fact, the two roots of (53) are

$$\phi_1 = \frac{1 + \sqrt{5}}{2} \quad (54)$$

and

$$\phi_2 = \frac{1 - \sqrt{5}}{2}. \quad (55)$$

The formula (54) is the so called golden ratio or golden number (see Refs. [10]-[13] and references therein).

In terms of ϕ_1 and ϕ_2 the Lagravity (49) becomes

$$\mathcal{L} = \frac{1}{2(\sqrt{5})^{1/2}} m_0 (\omega^a \omega^b \eta_{ab})^{1/2}, \quad (56)$$

where

$$\begin{aligned} \omega^1 &= \phi_1 \dot{q}^1 + \dot{q}^2, \\ \omega^2 &= \phi_2 \dot{q}^1 + \dot{q}^2. \end{aligned} \quad (57)$$

Here, $\eta_{ab} = \text{diag}(1, -1)$. Thus, the expression (56) formally establishes a connection between the golden ratio and black holes. (It is worth mentioning that in Ref. [14] a different relation between the golden ratio and black holes has been found. In fact, in such a reference it is shown that rotating black holes make a phase transition when the ratio of the square of its mass to the square of its angular momentum is equal to the golden ratio.)

Summarizing we have developed a variational method that can help to clarify the meaning of the black holes solution in any dimension. Our procedure is based in an analogy with the constraint Hamiltonian formalism. As a reward for our efforts we found an interesting link between two fascinating concepts: black holes and the golden ratio.

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