On the Local Well-posedness of a 3D Model for Incompressible Navier-Stokes Equations with Partial Viscosity

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Abstract

In this short note, we study the local well-posedness of a 3D model for incompressible Navier-Stokes equations with partial viscosity. This model was originally proposed by Hou-Lei in [4]. In a recent paper, we prove that this 3D model with partial viscosity will develop a finite time singularity for a class of initial condition using a mixed Dirichlet Robin boundary condition. The local well-posedness analysis of this initial boundary value problem is more subtle than the corresponding well-posedness analysis using a standard boundary condition because the Robin boundary condition we consider is non-dissipative. We establish the local well-posedness of this initial boundary value problem by designing a Picard iteration in a Banach space and proving the convergence of the Picard iteration by studying the well-posedness property of the heat equation with the same Dirichlet Robin boundary condition.

1 Introduction

In this short note, we prove the local well-posedness of the 3D model with partial viscosity. The 3D model with partial viscosity has the following form:

$$\left\{\begin{array}{rcl} u_t &=& 2u\psi_z \\ \omega_t &=& (u^2)_z + \nu\Delta\omega \\ -\Delta\psi &=& \omega \end{array}\right. , \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, \infty), \tag{1}$$

where $\Omega_{\mathbf{x}} = (0, a) \times (0, a)$. Let $\Gamma = \{(\mathbf{x}, z) \mid \mathbf{x} \in \Omega_{\mathbf{x}},\ z = 0\}$. The initial and boundary conditions for (1) are given as following:

$$\omega|_{\partial\Omega\setminus\Gamma} = 0, \quad (\omega_z + \gamma\omega)\,|_\Gamma = 0, \tag{2}$$

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \quad (\psi_z + \beta\psi)\,|_\Gamma = 0, \tag{3}$$

$$\omega|_{t=0} = \omega_0(\mathbf{x}, z), \quad u|_{t=0} = u_0(\mathbf{x}, z). \tag{4}$$

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This 3D model with viscosity in both u and ω components was first proposed by Hou and Lei in [4]. The only difference between this 3D model and the reformulated Navier-Stokes equations is that convection term is neglected in the model. If one adds the convection term back to the 3D model, one would recover the full Navier-Stokes equations. This model preserves almost all the properties of the full 3D Navier-Stokes equations. Despite the striking similarity at the theoretical level between the 3D model and the Navier-Stokes equations, the former seems to have a very different behavior from the full Navier-Stokes equations. In a recent paper [5], we prove that the above 3D model with partial viscosity develops a finite time singularity for a class of initial condition using a mixed Dirichlet Robin boundary condition.

The analysis of finite time singularity formation of the 3D model [5] uses the local well-posedness result of the 3D model. The local well-posedness of the 3D model can be proved by using a standard energy estimate and a mollifier if there is no boundary or if the boundary condition is a standard one, see e.g. [6]. For the mixed Dirichlet Robin boundary condition we consider here, the analysis is a bit more complicated since the mixed Dirichlet Robin condition gives rise to a growing eigenmode.

There are two key ingredients in our local well-posedness analysis. The first one is to design a Picard iteration for the 3D model. The second one is to show that the mapping that generates the Picard iteration is a contraction mapping and the Picard iteration converges to a fixed point of the Picard mapping by using the Contraction Mapping Theorem. To establish the contraction property of the Picard mapping, we need to use the well-posedness property of the heat equation with the same Dirichlet Robin boundary condition as ω. The well-posedness analysis of the heat equation with a mixed Dirichelet Robin boundary has been studied in the literature. The case of $\gamma > 0$ is more subtle because there is a growing eigenmode. Nonetheless, we prove that all the essential regularity properties of the heat equation are still valid for the mixed Dirichlet Robin boundary condition with $\gamma > 0$.

2 The main result

The local existence result of our 3D model with partial viscosity is stated in the following theorem.

Theorem 2.1 *Assume that* $u_0 \in H^{s+1}(\Omega)$, $\omega_0 \in H^s(\Omega)$ *for some* $s > 3/2$, $u_0|_{\partial\Omega} = u_{0z}|_{\partial\Omega} = 0$ *and* ω_0 *satisfies (2). Moreover, we assume that* $\beta \in S_\infty$ *(or* S_b*) as defined in Lemma 2.1. Then there exists a finite time* $T = T\left(\|u_0\|_{H^{s+1}(\Omega)}, \|\omega_0\|_{H^s(\Omega)}\right) > 0$ *such that the system (1) with boundary condition (2),(3) and initial data (4) has a unique solution,* $u \in C([0,T], H^{s+1}(\Omega))$, $\omega \in C([0,T], H^s(\Omega))$ *and* $\psi \in C([0,T], H^{s+2}(\Omega))$.

The local well-posedness analysis relies on the following local well-posedness of the heat equation and the elliptic equation with mixed Dirichlet and Robin boundary conditions. First, the local well-posedness of the elliptic equation with the mixed Dirichlet and Robin boundary condition is given by the following lemma [5]:

Lemma 2.1 *There exists a unique solution $v \in H^s(\Omega)$ to the boundary value problem:*

$$-\Delta v = f, \quad (\mathbf{x}, z) \in \Omega, \tag{5}$$

$$v|_{\partial\Omega\setminus\Gamma} = 0, \quad (v_z + \beta v)|_\Gamma = 0, \tag{6}$$

if $\beta \in S_\infty \equiv \{\beta \mid \beta \neq \frac{\pi|k|}{a}$ for all $k \in \mathbb{Z}^2\}$, $f \in H^{s-2}(\Omega)$ with $s \geq 2$ and $f|_{\partial\Omega\setminus\Gamma} = 0$. Moreover we have

$$\|v\|_{H^s(\Omega)} \leq C_s \|f\|_{H^{s-2}(\Omega)}, \tag{7}$$

where C_s is a constant depending on s, $|k| = \sqrt{k_1^2 + k_2^2}$.

Definition 2.1 *Let $\mathcal{K} : H^{s-2}(\Omega) \to H^s(\Omega)$ be a linear operator defined as following:*

for all $\quad f \in H^{s-2}(\Omega), \quad \mathcal{K}(f)$ is the solution of the boundary value problem (5)-(6).

It follows from Lemma 2.1 that for any $f \in H^{s-2}(\Omega)$, we have

$$\|\mathcal{K}(f)\|_{H^s(\Omega)} \leq C_s \|f\|_{H^{s-2}(\Omega)}. \tag{8}$$

For the heat equation with the mixed Dirichlet and Robin boundary condition, we have the following result.

Lemma 2.2 *There exists a unique solution $\omega \in C([0,T]; H^s(\Omega))$ to the initial boundary value problem:*

$$\omega_t = \nu \Delta \omega, \quad (\mathbf{x}, z) \in \Omega, \tag{9}$$

$$\omega|_{\partial\Omega\setminus\Gamma} = 0, \quad (\omega_z + \gamma\omega)|_\Gamma = 0, \tag{10}$$

$$\omega|_{t=0} = \omega_0(\mathbf{x}, z). \tag{11}$$

for $\omega_0 \in H^s(\Omega)$ with $s > 3/2$. Moreover we have the following estimates in the case of $\gamma > 0$

$$\|\omega(t)\|_{H^s(\Omega)} \leq C(\gamma, s) e^{\nu\gamma^2 t} \|\omega_0\|_{H^s(\Omega)}, \quad t \geq 0, \tag{12}$$

and

$$\|\omega(t)\|_{H^s(\Omega)} \leq C(\gamma, s, t) \|\omega_0\|_{L^2(\Omega)}, \quad t > 0. \tag{13}$$

Remark 2.1 *We remark that the growth factor $e^{\nu\gamma^2 t}$ in (12) is absent in the case of $\gamma \leq 0$ since there is no growing eigenmode in this case.*

Proof First, we prove the solution of the system (9)-(11) is unique. Let ω_1, $\omega_2 \in H^s(\Omega)$ be two smooth solutions of the heat equation for $0 \leq t < T$ satisfying the same initial condition and the Dirichlet Robin boundary condition. Let $\omega = \omega_1 - \omega_2$. We will prove that $\omega = 0$ by using an energy estimate and the Robin boundary condition at Γ:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \omega^2 d\mathbf{x}dz &= \nu \int_\Omega \omega \Delta \omega d\mathbf{x}dz \\
&= -\nu \int_\Omega |\nabla \omega|^2 d\mathbf{x}dz - \nu \int_\Gamma \omega \omega_z d\mathbf{x} \\
&= -\nu \int_\Omega |\nabla \omega|^2 d\mathbf{x}dz + \nu\gamma \int_\Gamma \omega^2 d\mathbf{x} \\
&= -\nu \int_\Omega |\nabla \omega|^2 d\mathbf{x}dz - \nu\gamma \int_\Gamma \int_z^\infty \left(\omega^2\right)_z dz d\mathbf{x} \\
&= -\nu \int_\Omega |\nabla \omega|^2 d\mathbf{x}dz - 2\nu\gamma \int_\Gamma \int_z^\infty \omega \omega_z d\mathbf{x}dz \\
&\leq -\nu \int_\Omega |\nabla \omega|^2 d\mathbf{x}dz + \frac{\nu}{2}\int_\Omega |\omega_z|^2 d\mathbf{x}dz + 2\nu\gamma^2 \int_\Omega \omega^2 d\mathbf{x}dz \\
&\leq -\frac{\nu}{2}\int_\Omega |\nabla \omega|^2 d\mathbf{x}dz + 2\nu\gamma^2 \int_\Omega \omega^2 d\mathbf{x}dz, \qquad (14)
\end{aligned}
$$

where we have used the fact that the smooth solution of the heat equation ω decays to zero as $z \to \infty$. Thus, we get

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega \omega^2 d\mathbf{x}dz \leq 2\nu\gamma^2 \int_\Omega \omega^2 d\mathbf{x}dz. \qquad (15)
$$

It follows from Gronwall's inequality

$$
e^{-4\nu\gamma^2 t}\int_\Omega \omega^2 d\mathbf{x}dz \leq \int_\Omega \omega_0^2 d\mathbf{x}dz = 0, \qquad (16)
$$

since $\omega_0 = 0$. Since $\omega \in H^s(\Omega)$ with $s > 3/2$, this implies that $\omega = 0$ for $0 \leq t < T$ which proves the uniqueness of smooth solutions for the heat equation with the mixed Dirichlet Robin boundary condition.

Next, we will prove the existence of the solution by constructing a solution explicitly. Let $\eta(\mathbf{x}, z, t)$ be the solution of the following initial boundary value problem:

$$
\eta_t = \nu \Delta \eta, \quad (\mathbf{x}, z) \in \Omega, \qquad (17)
$$

$$
\eta|_{\partial\Omega} = 0, \quad \eta|_{t=0} = \eta_0(\mathbf{x}, z), \qquad (18)
$$

and let $\xi(\mathbf{x}, t)$ be the solution of the following PDE in $\Omega_\mathbf{x}$:

$$
\xi_t = \nu \Delta_\mathbf{x} \xi + \nu\gamma^2 \xi, \quad \mathbf{x} \in \Omega_\mathbf{x}, \qquad (19)
$$

$$
\xi|_{\partial\Omega_\mathbf{x}} = 0, \quad \xi|_{t=0} = \overline{\omega}_0(\mathbf{x}), \qquad (20)
$$

where $\Delta_\mathbf{x} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ and $\overline{\omega}_0(\mathbf{x}) = 2\gamma \int_0^\infty \omega_0(\mathbf{x}, z) e^{-\gamma z} dz$. From the standard theory of the heat equation, we know that η and ξ both exist globally in time.

We are interested in the case when the initial value $\eta_0(\mathbf{x}, z)$ is related to ω_0 by solving the following ODE as a function of z with $\mathbf{x}$ being fixed as a parameter:

$$-\frac{1}{\gamma}\eta_{0z} + \eta_0 = \omega_0(\mathbf{x}, z) - \overline{\omega}_0(\mathbf{x})e^{-\gamma z}, \quad \eta_0(\mathbf{x}, 0) = 0. \tag{21}$$

Define

$$\omega(\mathbf{x}, z, t) \equiv -\frac{1}{\gamma}\eta_z + \eta + \xi(\mathbf{x}, t)e^{-\gamma z}, \quad (\mathbf{x}, z) \in \Omega. \tag{22}$$

It is easy to check that ω satisfies the heat equation for $t > 0$ and the initial condition. Obviously, ω also satisfies the boundary condition on $\partial\Omega\backslash\Gamma$. To verify the boundary condition on Γ, we observe by a direct calculation that $(\omega_z + \gamma\omega)|_\Gamma = -\frac{1}{\gamma}\eta_{zz}|_\Gamma$. Since $\eta(\mathbf{x}, z)|_\Gamma = 0$, we obtain by using $\eta_t = \nu\Delta\eta$ and taking the limit as $z \to 0+$ that $\Delta\eta|_\Gamma = 0$, which implies that $\eta_{zz}|_\Gamma = 0$. Therefore, ω also satisfies the Dirichlet Robin boundary condition at Γ. This shows that ω is a solution of the system (9)-(11). By the uniqueness result that we proved earlier, the solution of the heat equation must be given by (22).

Since η and ξ are solutions of the heat equation with a standard Dirichlet boundary condition, the classical theory of the heat equation [1] gives the following regularity estimates:

$$\|\eta\|_{H^s(\Omega)} \le C\|\eta_0\|_{H^s(\Omega)}, \quad \|\xi(\mathbf{x})\|_{H^s(\Omega_\mathbf{x})} \le Ce^{\nu\gamma^2 t}\|\overline{\omega}_0(\mathbf{x})\|_{H^s(\Omega_\mathbf{x})}. \tag{23}$$

Recall that $\eta_{zz}|_\Gamma = 0$. Therefore, η_z also solves the heat equation with the same Dirichlet Robin boudary condition:

$$(\eta_z)_t = \nu\Delta\eta_z, \quad (\mathbf{x}, z) \in \Omega, \tag{24}$$

$$(\eta_z)_z|_\Gamma = 0, \quad (\eta_z)|_{\partial\Omega\backslash\Gamma} = 0, \quad (\eta_z)|_{t=0} = \eta_{0z}(\mathbf{x}, z), \tag{25}$$

which implies that

$$\|\eta_z\|_{H^s(\Omega)} \le C\|\eta_{0z}\|_{H^s(\Omega)}. \tag{26}$$

Putting all the above estimates for η, η_z and ξ together and using (22), we obtain the following estimate:

$$\begin{aligned}\|\omega\|_{H^s(\Omega)} &= \left\|-\frac{1}{\gamma}\eta_z + \eta + \xi(\mathbf{x}, t)e^{-\gamma z}\right\|_{H^s(\Omega)} \\ &\le \frac{1}{\gamma}\|\eta_z\|_{H^s(\Omega)} + \|\eta\|_{H^s(\Omega)} + \left\|\xi(\mathbf{x}, t)e^{-\gamma z}\right\|_{H^s(\Omega)} \\ &\le C(\gamma, s)\left(\|\eta_{0z}\|_{H^s(\Omega)} + \|\eta_0\|_{H^s(\Omega)} + e^{\nu\gamma^2 t}\|\overline{\omega}_0(\mathbf{x})\|_{H^s(\Omega_\mathbf{x})}\right).\end{aligned} \tag{27}$$

It remains to bound $\|\eta_{0z}\|_{H^s(\Omega)}$, $\|\eta_0\|_{H^s(\Omega)}$ and $\|\overline{\omega}_0(\mathbf{x})\|_{H^s(\Omega_\mathbf{x})}$ in terms of $\|\omega_0\|_{H^s(\Omega)}$. By solving the ODE (21) directly, we can express η in terms of ω_0 explicitly

$$\eta_0(\mathbf{x}, z) = -\gamma e^{\gamma z}\int_0^z e^{-\gamma z'}f(\mathbf{x}, z')dz' = \gamma\int_z^\infty e^{-\gamma(z'-z)}f(\mathbf{x}, z')dz', \tag{28}$$

where $f(\mathbf{x},z)=\omega_0(\mathbf{x},z)-\overline{\omega}_0(\mathbf{x})e^{-\gamma z}$ and we have used the property that

$$\int_0^\infty f(\mathbf{x},z)e^{-\gamma z}dz=0.$$

By using integration by parts, we have

$$\eta_{0z}(\mathbf{x},z)=-\gamma f(\mathbf{x},z)+\gamma^2\int_z^\infty e^{-\gamma(z'-z)}f(\mathbf{x},z')dz'=\gamma\int_z^\infty e^{-\gamma(z'-z)}f_{z'}(\mathbf{x},z')dz'. \tag{29}$$

By induction we can show that for any $\alpha=(\alpha_1,\alpha_2,\alpha_3)\geq 0$

$$D^\alpha\eta_0=\gamma\int_z^\infty e^{-\gamma(z'-z)}D^\alpha f(\mathbf{x},z')dz'. \tag{30}$$

Let $K(z)=\gamma e^{-\gamma z}\chi(z)$ and $\chi(z)$ be the characteristic function

$$\chi(z)=\begin{cases}0, & z\leq 0,\\ 1, & z>0.\end{cases} \tag{31}$$

Then $D^\alpha\eta_0$ can be written in the following convolution form:

$$D^\alpha\eta_0(\mathbf{x},z)=\int_0^\infty K(z'-z)D^\alpha f(\mathbf{x},z')dz'. \tag{32}$$

Using Young's inequality (see e.g. page 232 of [2]), we obtain:

$$\begin{aligned}\|D^\alpha\eta_0\|_{L^2(\Omega)} &\leq \|K(z)\|_{L^1(\mathbb{R}^+)}\|D^\alpha f\|_{L^2(\Omega)}\\ &\leq C(\gamma)\left\|D^\alpha\omega_0-(-\gamma)^{\alpha_3}\,e^{-\gamma z}D^{(\alpha_1,\alpha_2)}\overline{\omega}_0(\mathbf{x})\right\|_{L^2(\Omega)}\\ &\leq C(\gamma,\alpha)\left(\|D^\alpha\omega_0\|_{L^2(\Omega)}+\left\|D^{(\alpha_1,\alpha_2)}\overline{\omega}_0(\mathbf{x})\right\|_{L^2(\Omega_{\mathbf{x}})}\right).\end{aligned} \tag{33}$$

Moreover, we obtain by using the Hölder inequality that

$$\begin{aligned}\left\|D^{(\alpha_1,\alpha_2)}\overline{\omega}_0(\mathbf{x})\right\|_{L^2(\Omega_{\mathbf{x}})} &= \left(\int_{\Omega_{\mathbf{x}}}\left(\int_0^\infty e^{-\gamma z}D^{(\alpha_1,\alpha_2)}\omega_0(\mathbf{x},z)dz\right)^2 d\mathbf{x}\right)^{1/2}\\ &\leq \left(\frac{1}{2\gamma}\int_{\Omega_{\mathbf{x}}}\int_0^\infty\left(D^{(\alpha_1,\alpha_2)}\omega_0(\mathbf{x},z)\right)^2 dzd\mathbf{x}\right)^{1/2}\\ &= \frac{1}{\sqrt{2\gamma}}\left\|D^{(\alpha_1,\alpha_2)}\omega_0(\mathbf{x},z)\right\|_{L^2(\Omega)}.\end{aligned} \tag{34}$$

Substituting (34) to (33) yields

$$\|D^\alpha\eta_0\|_{L^2(\Omega)} \leq C(\gamma,\alpha)\left(\|D^\alpha\omega_0\|_{L^2(\Omega)}+\left\|D^{(\alpha_1,\alpha_2)}\omega_0\right\|_{L^2(\Omega)}\right), \tag{35}$$

which implies that

$$\|\eta_0\|_{H^s(\Omega)}\leq C(\gamma,s)\,\|\omega_0\|_{H^s(\Omega)}\,,\quad \forall\ s\geq 0. \tag{36}$$

It follows from (34) that

$$\|\overline{\omega}_0(\mathbf{x})\|_{H^s(\Omega_{\mathbf{x}})} \le C(\gamma)\,\|\omega_0\|_{H^s(\Omega)}\,, \quad \forall\, s \ge 0. \tag{37}$$

On the other hand, we obtain from the equation for η_0 (21) that

$$\|\eta_{0z}\|_{H^s(\Omega)} = \gamma\,\|f+\eta_0\|_{H^s(\Omega)} \le C(\gamma,s)\,\|\omega_0\|_{H^s(\Omega)}\,, \quad \forall\, s \ge 0. \tag{38}$$

Upon substituting (36)-(38) to (27), we obtain

$$\|\omega\|_{H^s(\Omega)} \le C(\gamma,s)e^{\nu\gamma^2 t}\,\|\omega_0\|_{H^s(\Omega)}\,, \tag{39}$$

where $C(\gamma,s)$ is a constant depending on γ and s only. This proves (12).

To prove (13), we use the classical regularity result for the heat equation with the homogeneous Dirichlet boundary condition to obtain the following estimates for $t>0$:

$$\|\eta\|_{H^s(\Omega)} \le C(t)\|\eta_0\|_{L^2(\Omega)}, \tag{40}$$

$$\|\eta_z\|_{H^s(\Omega)} \le C(s,t)\|\eta_{0z}\|_{L^2(\Omega)}, \tag{41}$$

$$\|\overline{\omega}(\mathbf{x})\|_{H^s(\Omega_{\mathbf{x}})} \le C(s,t)e^{\nu\gamma^2 t}\,\|\overline{\omega}_0(\mathbf{x})\|_{L^2(\Omega_{\mathbf{x}})}\,, \tag{42}$$

where $C(s,t)$ is a constant depending on s and t. By combining (40)-(42) with estimates (36)-(38), we obtain for any $t>0$ that

$$\begin{aligned}\|\omega\|_{H^s(\Omega)} &\le C(\gamma,s,t)\left(\|\eta_{0z}\|_{L^2(\Omega)} + \|\eta_0\|_{L^2(\Omega)} + e^{\nu\gamma^2 t}\,\|\overline{\omega}_0(\mathbf{x})\|_{L^2(\Omega_{\mathbf{x}})}\right)\\ &\le C(\gamma,s,t)\|\omega_0\|_{L^2(\Omega)},\end{aligned} \tag{43}$$

where $C(\gamma,s,t)<\infty$ is a constant depending on γ, s and t. This proves (13) and completes the proof of the Lemma. ☐

We also need the following well-known Sobolev inequality [3].

Lemma 2.3 *Let $u,v \in H^s(\Omega)$ with $s>3/2$. We have*

$$\|uv\|_{H^s(\Omega)} \le c\|u\|_{H^s(\Omega)}\|v\|_{H^s(\Omega)}. \tag{44}$$

Now, we are ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1 Let $v=u^2$. First, using the definition of the operator $\mathcal{K}$ (see Definition 2.1), we can rewrite the 3D model with partial viscosity in the following equivalent form:

$$\left\{\begin{array}{rcl} v_t &=& 4v\mathcal{K}(\omega)_z \\ \omega_t &=& v_z+\nu\Delta\omega \end{array}\right. , \quad (\mathbf{x},z)\in\Omega=\Omega_{\mathbf{x}}\times(0,\infty), \tag{45}$$

with the initial and boundary conditions given as follows:

$$\omega|_{\partial\Omega\setminus\Gamma}=0, \quad (\omega_z+\gamma\omega)\,|_{\Gamma}=0, \tag{46}$$

$$\omega|_{t=0}=\omega_0(\mathbf{x},z)\in W^s, \quad v|_{t=0}=v_0(\mathbf{x},z)\in V^{s+1}, \tag{47}$$

where $V^{s+1}=\{v\in H^{s+1}: v|_{\partial\Omega}=0, v_z|_{\partial\Omega}=0, v_{zz}|_{\partial\Omega}=0\}$ and $W^s=\{w\in H^s: w|_{\partial\Omega\backslash\Gamma}=0, (w_z+\gamma w)|_\Gamma=0\}$.

We note that the condition $u_0|_{\partial\Omega}=u_{0z}|_{\partial\Omega}=0$ implies that $v_0|_{\partial\Omega}=v_{0z}|_{\partial\Omega}=v_{0zz}|_{\partial\Omega}=0$ by using the relation $v_0=u_0^2$. Thus we have $v_0\in V^{s+1}$. It is easy to show by using the u-equation that the property $u_0|_{\partial\Omega}=u_{0z}|_{\partial\Omega}=0$ is preserved dynamically. Thus we have $v\in V^{s+1}$.

Define $U=(U_1,U_2)=(v,\omega)$ and $X=C([0,T];V^{s+1})\times C([0,T];W^s)$ with the norm

$$\|U\|_X=\sup_{t\in[0,T]}\|U_1\|_{H^{s+1}(\Omega)}+\sup_{t\in[0,T]}\|U_2\|_{H^s(\Omega)}, \quad \forall U\in X$$

and let $S=\{U\in X: \|U\|_X\le M\}$.

Now, define the map $\Phi: X\to X$ in the following way: let $\Phi(\tilde{v},\tilde{\omega})=(v,\omega)$, then for any $t\in[0,T]$,

$$v(\mathbf{x},z,t) = v_0(\mathbf{x},z,t)+4\int_0^t\tilde{v}(\mathbf{x},z,t')\mathcal{K}(\tilde{\omega})_z(\mathbf{x},z,t')dt', \tag{48}$$

$$\omega(\mathbf{x},z,t) = \mathcal{L}(\tilde{v}_z,\omega_0;\mathbf{x},z,t), \tag{49}$$

where $\omega(\mathbf{x},z,t)=\mathcal{L}(\tilde{v}_z,\omega_0;\mathbf{x},z,t)$ is the solution of the following equation:

$$\omega_t=\tilde{v}_z+\nu\Delta\omega, \quad (\mathbf{x},z)\in\Omega=\Omega_{\mathbf{x}}\times(0,\infty), \tag{50}$$

with the initial and boundary conditions:

$$\omega|_{\partial\Omega\backslash\Gamma}=0, \quad (\omega_z+\gamma\omega)|_\Gamma=0, \quad \omega|_{t=0}=\omega_0(\mathbf{x},z).$$

We use the map Φ to define a Picard iteration: $U^{k+1}=\Phi(U^k)$ with $U^0=(v_0,\omega_0)$. In the following, we will prove that there exist $T>0$ and $M>0$ such that

1. $U^k\in S$, for all k.
2. $\|U^{k+1}-U^k\|_X\le\frac{1}{2}\|U^k-U^{k-1}\|_X$, for all k.

Then by the contraction mapping theorem, there exists $U=(v,\omega)\in S$ such that $\Phi(U)=U$ which implies that U is a local solution of the system (45) in X.

First, by Duhamel's principle, we have for any $g\in C([0,T];V^s)$ that

$$\mathcal{L}(g,\omega_0;\mathbf{x},z,t)=\mathcal{P}(\omega_0;0,t)+\int_0^t\mathcal{P}(g;t',t)dt', \tag{51}$$

where $\mathcal{P}(g;t',t)=\tilde{g}(\mathbf{x},z,t)$ is defined as the solution of the following initial boundary value problem at time t:

$$\tilde{g}_t=\nu\Delta\tilde{g}, \quad (\mathbf{x},z)\in\Omega=\Omega_{\mathbf{x}}\times(0,\infty), \tag{52}$$

with the initial and boundary conditions:

$$\tilde{g}|_{\partial\Omega\backslash\Gamma}=0, \quad (\tilde{g}_z+\gamma\tilde{g})|_\Gamma=0, \quad \tilde{g}(\mathbf{x},z,t')=g(\mathbf{x},z,t'). \tag{53}$$

We observe that $g(\mathbf{x}, z, t')$ also satisfies the same boundary condition as ω for any $0 \le t' \le t$ since $g = v_z^k$ and $v^k \in V^{s+1}$.

Now we can apply Lemma 2.2 to conclude that for any $t' < T$ and $t \in [t', T]$ we have

$$\|\mathcal{P}(g; t', t)\|_{H^s(\Omega)} \le C(\gamma, s) e^{\nu\gamma^2(t-t')} \|g(\mathbf{x}, z, t')\|_{H^s(\Omega)}. \tag{54}$$

which implies the following estimate for $\mathcal{L}$: for all $t \in [0, T]$,

$$\|\mathcal{L}(g, \omega_0; \mathbf{x}, z, t)\|_{H^s(\Omega)} \le C(\gamma, s) e^{\nu\gamma^2 t} \left(\|\omega_0\|_{H^s(\Omega)} + t \sup_{t' \in [0,t]} \|g(\mathbf{x}, z, t')\|_{H^s(\Omega)} \right). \tag{55}$$

Further, by using Lemma 2.1 and the above estimate (55) for the sequence $U^k = (v^k, \omega^k)$, we get the following estimate:

$$\begin{aligned} \left\|v^{k+1}\right\|_{H^{s+1}(\Omega)} &\le \|v_0\|_{H^{s+1}(\Omega)} + 4T \sup_{t \in [0,T]} \left\|v^k(\mathbf{x}, z, t)\right\|_{H^{s+1}(\Omega)} \sup_{t \in [0,T]} \left\|\mathcal{K}(\omega^k)_z(\mathbf{x}, z, t)\right\|_{H^{s+1}(\Omega)}, \\ &\le \|v_0\|_{H^{s+1}(\Omega)} + 4T \sup_{t \in [0,T]} \left\|v^k(\mathbf{x}, z, t)\right\|_{H^{s+1}(\Omega)} \sup_{t \in [0,T]} \left\|\omega^k(\mathbf{x}, z, t)\right\|_{H^s(\Omega)}, \forall t \in [0, T] \end{aligned} \tag{56}$$

$$\begin{aligned} \left\|\omega^{k+1}\right\|_{H^s(\Omega)} &\le C(\gamma, s) e^{\nu\gamma^2 t} \left(\|\omega_0\|_{H^s(\Omega)} + t \sup_{t' \in [0,t]} \left\|v_z^k(\mathbf{x}, z, t')\right\|_{H^s(\Omega)} \right) \\ &\le C(\gamma, s) e^{\nu\gamma^2 T} \left(\|\omega_0\|_{H^s(\Omega)} + T \sup_{t \in [0,T]} \left\|v^k\right\|_{H^{s+1}(\Omega)} \right), \quad \forall t \in [0, T]. \end{aligned} \tag{57}$$

Next, we will use mathematical induction to prove that if T satisfies the following inequality:

$$8C(\gamma, s) T e^{\nu\gamma^2 T} \left(\|\omega_0\|_{H^s(\Omega)} + 2T \|v_0\|_{H^{s+1}(\Omega)} \right) \le 1 \tag{58}$$

then for all $k \ge 0$ and $t \in [0, T]$, we have that

$$\left\|v^k\right\|_{H^{s+1}(\Omega)} \le 2 \|v_0\|_{H^{s+1}(\Omega)}, \tag{59}$$

$$\left\|\omega^k\right\|_{H^s(\Omega)} \le C(\gamma, s) e^{\nu\gamma^2 T} \left(\|\omega_0\|_{H^s(\Omega)} + 2T \|v_0\|_{H^{s+1}(\Omega)} \right). \tag{60}$$

First of all, $U^0 = (v_0, \omega_0)$ satisfies (59) and (60). Assume $U^k = (v^k, \omega^k)$ has this property, then for $U^{k+1} = (v^{k+1}, \omega^{k+1})$, using (56) and (57), we have

$$\begin{aligned} \left\|v^{k+1}\right\|_{H^{s+1}(\Omega)} &\le \|v_0\|_{H^{s+1}(\Omega)} + 4T \sup_{t \in [0,T]} \left\|v^k(\mathbf{x}, z, t)\right\|_{H^{s+1}(\Omega)} \sup_{t \in [0,T]} \left\|\omega^k(\mathbf{x}, z, t)\right\|_{H^s(\Omega)} \\ &\le \|v_0\|_{H^{s+1}(\Omega)} \left(1 + 8C(\gamma, s) T e^{\nu\gamma^2 T} \left(\|\omega_0\|_{H^s(\Omega)} + 2T \|v_0\|_{H^{s+1}(\Omega)} \right)\right) \\ &\le 2 \|v_0\|_{H^{s+1}(\Omega)}, \quad \forall t \in [0, T]. \end{aligned} \tag{61}$$

$$\begin{aligned} \left\|\omega^{k+1}\right\|_{H^s(\Omega)} &\le C(\gamma, s) e^{\nu\gamma^2 T} \left(\|\omega_0\|_{H^s(\Omega)} + T \sup_{t \in [0,T]} \left\|v^k\right\|_{H^{s+1}(\Omega)} \right) \\ &\le C(\gamma, s) e^{\nu\gamma^2 T} \left(\|\omega_0\|_{H^s(\Omega)} + 2T \|v_0\|_{H^{s+1}(\Omega)} \right), \quad \forall t \in [0, T]. \end{aligned} \tag{62}$$

Then, by induction, we prove that for any $k \geq 0$, $U^k = (v^k, \omega^k)$ is bounded by (59) and (60).

We want to point that there exists $T > 0$ such that the inequality (58) is satisfied. One choice of T is given as following:

$$T_1 = \min\left\{ \left[8C(\gamma, s) e^{\nu\gamma^2} \left(\|\omega_0\|_{H^s(\Omega)} + 2\|v_0\|_{H^{s+1}(\Omega)} \right) \right]^{-1}, 1 \right\}. \tag{63}$$

Using the choice of T in (63), we can choose $M = 2\|v_0\|_{H^{s+1}(\Omega)} + C(\gamma, s) e^{\nu\gamma^2} \left(\|\omega_0\|_{H^s(\Omega)} + 2\|v_0\|_{H^{s+1}(\Omega)} \right)$, then we have $U^k \in S$, for all k.

Next, we will prove that Φ is a contraction mapping for some small $0 < T \leq T_1$.

First of all, by using Lemmas 2.1 and 2.3, we have

$$\begin{aligned}
\left\| v^{k+1} - v^k \right\|_{H^{s+1}(\Omega)} &= \left\| \int_0^t v^k(\mathbf{x}, t') \mathcal{K}(\omega^k)_z(\mathbf{x}, t') dt' - \int_0^t v^{k-1}(\mathbf{x}, t') \mathcal{K}(\omega^{k-1})_z(\mathbf{x}, t') dt' \right\|_{H^{s+1}(\Omega)} \\
&\leq \left\| \int_0^t \left(v^k - v^{k-1} \right)(\mathbf{x}, t') \mathcal{K}(\omega^k)_z(\mathbf{x}, t') dt' \right\|_{H^{s+1}(\Omega)} \\
&\quad + \left\| \int_0^t v^{k-1}(\mathbf{x}, t') \left(\mathcal{K}(\omega^k)_z - \mathcal{K}(\omega^{k-1})_z \right)(\mathbf{x}, t') dt' \right\|_{H^{s+1}(\Omega)} \\
&\leq T \sup_{t\in[0,T]} \left\| v^k - v^{k-1} \right\|_{H^{s+1}(\Omega)} \sup_{t\in[0,T]} \left\| \mathcal{K}(\omega^k)_z \right\|_{H^{s+1}(\Omega)} \\
&\quad + T \sup_{t\in[0,T]} \left\| v^{k-1} \right\|_{H^{s+1}(\Omega)} \sup_{t\in[0,T]} \left\| \mathcal{K}(\omega^k - \omega^{k-1})_z \right\|_{H^{s+1}(\Omega)} \\
&\leq MT \left(\sup_{t\in[0,T]} \left\| v^k - v^{k-1} \right\|_{H^{s+1}(\Omega)} + \sup_{t\in[0,T]} \left\| \omega^k - \omega^{k-1} \right\|_{H^s(\Omega)} \right).
\end{aligned} \tag{64}$$

On the other hand, Lemma 2.2 and (51) imply

$$\begin{aligned}
\left\| \omega^{k+1} - \omega^k \right\|_{H^s(\Omega)} &= \left\| \mathcal{L}(v_z^k, \omega_0; \mathbf{x}, t) - \mathcal{L}(v_z^{k-1}, \omega_0; \mathbf{x}, t) \right\|_{H^s(\Omega)} \\
&\leq \left\| \int_0^t \mathcal{P}(v_z^k - v_z^{k-1}; t', t) dt' \right\|_{H^s(\Omega)} \\
&\leq TC(\gamma, s) e^{\nu\gamma^2 T} \sup_{t\in[0,T]} \left\| v_z^k - v_z^{k-1} \right\|_{H^s(\Omega)} \\
&\leq TC(\gamma, s) e^{\nu\gamma^2 T} \sup_{t\in[0,T]} \left\| v^k - v^{k-1} \right\|_{H^{s+1}(\Omega)}.
\end{aligned} \tag{65}$$

Let

$$T = \min\left\{ \left[8C(\gamma, s) e^{\nu\gamma^2} \left(\|\omega_0\|_{H^s(\Omega)} + 2\|v_0\|_{H^{s+1}(\Omega)} \right) \right]^{-1}, \left[2C(\gamma, s) e^{\nu\gamma^2} \right]^{-1}, \frac{1}{2M}, 1 \right\}. \tag{66}$$

Then, we have

$$\left\| U^{k+1} - U^k \right\|_X \leq \frac{1}{2} \left\| U^k - U^{k-1} \right\|_X.$$

This proves that the sequence U^k converges to a fixed point of the map $\Phi : X \to X$, and the limiting fixed point $U = (v, \omega)$ is a solution of the 3D model with partial viscosity. Moreover, by passing the limit in (59)-(60), we obtain the following *a priori* estimate for the solution v and ω:

$$\|v\|_{H^{s+1}(\Omega)} \le 2\|v_0\|_{H^{s+1}(\Omega)}, \tag{67}$$

$$\|\omega\|_{H^s(\Omega)} \le C(\gamma, s)e^{\nu\gamma^2 T}\left(\|\omega_0\|_{H^s(\Omega)} + 2T\|v_0\|_{H^{s+1}(\Omega)}\right), \tag{68}$$

for $0 \le t \le T$ with T defined in (66).

It remains to show that the smooth solution of the 3D model with partial viscosity is unique. Let (v_1, ω_1) and (v_2, ω_2) be two smooth solutions of the 3D model with the same initial data and satisfying $\|v_i\|_{H^{s+1}(\Omega)} \le M$ and $\|\omega_i\|_{H^s(\Omega)} \le M$ for $i = 1, 2$ and $0 \le t \le T$, where M is a positive constant depending on the initial data, γ, s, and T. Since $s > 3/2$, the Sobolev embedding theorem [1] implies that

$$\|v_i\|_{L^\infty(\Omega)} \le \|v_i\|_{H^{s+1}(\Omega)} \le M, \quad i = 1, 2, \tag{69}$$

$$\|\mathcal{K}(\omega_i)_z\|_{L^\infty(\Omega)} \le \|\mathcal{K}(\omega_i)_z\|_{H^s(\Omega)} \le C_s\|\omega_i\|_{H^s(\Omega)} \le C_s M, \quad i = 1, 2. \tag{70}$$

Let $v = v_1 - v_2$ and $\omega = \omega_1 - \omega_2$. Then (v, ω) satisfies

$$\left\{\begin{array}{lcl} v_t &=& 4v\mathcal{K}(\omega_1)_z + 4v_2\mathcal{K}(\omega)_z \\ \omega_t &=& v_z + \nu\Delta\omega \end{array}\right., \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, \infty), \tag{71}$$

with $\omega|_{\partial\Omega\setminus\Gamma} = 0, \quad (\omega_z + \gamma\omega)|_\Gamma = 0$, and $\omega|_{t=0} = 0$, $v|_{t=0} = 0$. By using (69)-(70), and proceeding as the uniqueness estimate for the heat equation in (14), we can derive the following estimate for v and ω:

$$\frac{d}{dt}\|v\|^2_{L^2(\Omega)} \le C_1(\|v\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)}), \tag{72}$$

$$\frac{d}{dt}\|\omega\|^2_{L^2(\Omega)} \le C_3(\|v\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)}), \tag{73}$$

where C_i $(i = 1, 2, 3)$ are positive constants depending on M, ν, γ, C_s. In obtaining the estimate for (73), we have performed integration by parts in the estimate of the v_z-term in the ω-equation and absorbing the contribution from ω_z by the diffusion term. There is no contribution from the boundary term since $v|_{z=0} = 0$. We have also used the property $\|\mathcal{K}(\omega)_z\|_{L^2(\Omega)} \le C_s\|\omega\|_{L^2(\Omega)}$, which can be proved directly by following the argument in the Appendix of [5]. Since $v_0 = 0$ and $\omega_0 = 0$, the Gronwall inequality implies that $\|v\|_{L^2(\Omega)} = \|\omega\|_{L^2(\Omega)} = 0$ for $0 \le t \le T$. Furthermore, since $v \in H^{s+1}$ and $\omega \in H^s$ with $s > 3/2$, v and ω are continuous. Thus we must have $v = \omega = 0$ for $0 \le t \le T$. This proves the uniqueness of the smooth solution for the 3D model. □

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