How to be causal

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I explain a simple definition of causality in widespread use, and indicate how it links to the Kramers Kronig relations. The specification of causality in terms of temporal differential eqations then shows us the way to write down dynamical models so that their causal nature *in the sense used here* should be obvious to all. In particular, I apply this reasoning to Maxwell's equations, which is an instructive example since their casual properties are sometimes debated.

I. INTRODUCTION

Causality is a basic concept in physics - so basic, in fact, that it is hard to conceive of a useful model in which effects do not have causes. Indeed, the whole point of a physical model could be said to describe the process of cause and effect in some particular situation. But what do we generally mean by word like "causality", and phrases such as "cause and effect"? Usually, we mean that the cause of any event must not be later than any of its effects. But even such simple-sounding statements are rarely as uncomplicated as they seem: when trying to clarify the details and built-in assumptions, it is easily possible to get into philosophical discussions [1, 2], issues regarding statistical inference and induction [3], or particular physical arguments [4]. Here I instead follow the physics tradition characterized by Mermin as "shut up and calculate" [5]. But *what* should we calculate, and *how*?

A good starting point uses the fact that I can test whether some effect R occurs after its cause Q by means of a mathematical step function $h(t)$ which has $h(t) = 0$ for $t < 0$, and $h(t) = 1$ for $t \geq 0$. The definition of causality applied by this temporal step function is the same as that enforced by the famous Kramers Kronig relations [6, 7] that apply to spectral quantities. Therefore I call the step-like causality discussed here "KK causality", to distinguish it from any alternative definitions.

This step function is fine for analysing causal relationships present in existing data or mathematical functions, but it does not specify causal relationships. To do this we need to use temporal differential equations, which can then of themselves generate the step function criteria. Note that common expressions such as $F = ma$ do not express a causal relationship in the sense used here. Although an equation such as $F = ma$ does tie together the force F and acceleration a, there is no means of telling whether F is supposed to cause a, or a cause F, or even if the equation is instead intended as a constraint of some kind.

Differential equations containing temporal derivatives are useful because they are *open-ended specifications for the future behaviour*. They require only a knowledge of initial conditions and the on-going behaviour of the environment to solve. This is why they can be regarded as being causal, while (e.g.) $F = ma$ is not. After discussing the role of differential equations in section II, I consider causality in the spectral domain in section III, present a simple example in section IV, and consider Maxwell's Equations in section V. After a discussion in section VI, I summarize in section VII.

The presentation introduces at an undergraduate level the basic ideas and constraints arising from considerations of causality, with specific reference to the construction of causal models, as opposed to the philosophical, statistical, or more abstract technical aspects of casuality. Further, I address only purely temporal causality, and not spacetime causality which also allows for the effect of the finite speed of light.

II. CAUSAL DIFFERENTIAL EQUATIONS

Let us first write down a simple model, where some system R responds to its environment Q. Here R can be any quantity – e.g. a position or velocity, a level of excitation of some system, and even – if a position $\mathbf{r}$ is also specified – a probability distribution or wave function. Likewise Q might be anything, depending on some pre-set behaviour, the behaviour of R or other systems, or (e.g.) spatial derivatives of fields, potentials, distributions, and so on. Whatever the specific meaning of R (or indeed of Q), if keeping things simple we would most likely start by writing something like

$$\partial_t R(t) = Q(t), \tag{1}$$

where ∂_t is just the time derivative d/dt. To determine how causal this model equation is, consider the case where the environment generates a cause in the form of a brief delta-function impulse, i.e. $Q(t) = Q_0 \delta(t - t_0)$. Reassuringly, if I integrate eqn. (1), then $R(t)$ will gain a step at t_0 – i.e. the effect of the impulsive Q is to cause R to increase discontinuously by Q_0 at t_0; as depicted on fig. 1. Thus we see how the step function $h(t)$ arises quite simply from the most basic temporal differential equation.

I describe this situation, where an impulse gives rise to a stepped response (as opposed to e.g. a ramp or

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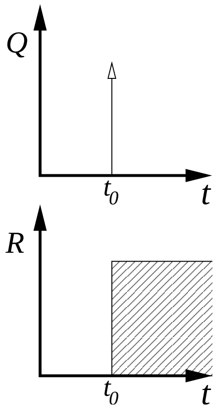

FIG. 1: An impulse Q at a time t_0 causes an effect (a step-change in value) on R as a result of the causal model specified in eqn. (1).

some smoother response) as "barely causal". That is, it is causal by the terms of our definition, but it is (in a sense) on the edge of allowed causal behaviour: a finite-valued effect is simultaneous with the cause – albeit only an infinitesimal slice. Nevertheless, having part of the effect being simultaneous with the cause is not usually regarded as indicating a problem with the casuality of the model. Other model equations, such as those for a simple damped pendulumn, or the Drude model or Lorentz oscillator used in electromagnetism [8], are more complicated and contain higher-order time derivatives – and can therefore be regarded as being more comfortably causal.

Simple examples from kinematics can illustrate the meaning of "causal" as used here. If we were to write down $\partial_t x = v$, then we could make the statement that "velocity v causes a change in position x"; likewise $\partial_t v = a$ means that "acceleration a causes a change in velocity v"; and $\partial_t^2 x = a$ means "acceleration a causes changes in position x".

More general differential equations can be written down as weighted sums of different orders of time derivatives e.g.

$$\sum_{n=0}^{N} T_n \partial_t^n R(t) = \sum_{m=0}^{N-1} a_m \partial_t^m Q(t), \tag{2}$$

for parameters T_n, a_m, and a defined maximum order of derivatives N. These will remain KK causal as long as the derivatives on the RHS are always of lower order than those on the LHS [9].

We might consider recasting the differential equations used here in an integral form; e.g. for an evolution starting at a time t_i, eqn. (1) becomes

$$R(t) = \int_{t_i}^{t} Q(t')dt', \tag{3}$$

although in most cases this is not as easy as writing down the differential equation. Also, as discussed next, differential equations make it easier to consider the spectral properties. And on a more intuitive note, writing down a differential equation does not imply you have solved it – it is a notation more compatible with the concept of an unknown future outcome, dependent on as-yet unknown future causes.

III. CAUSALITY AND SPECTRA

Often, the more complicated a model response is, the more likely it is that its spectral response will be analyzed in the frequency domain. That is, we take a known function of time $S(t)$ and use the well known Fourier transform (FT) [10, 11] to convert it into a spectrum $\tilde{S}(\omega)$. Although it is quite common to write down (or use) two complementary forms, i.e. the sin and cosine Fourier transforms, it is most convenient to combine them using $e^{-\imath\omega t} = \cos(\omega t) + \imath \sin(\omega t)$. This gives us the complex valued FT

$$\tilde{S}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S(t) e^{-\imath\omega t} dt. \tag{4}$$

Note that even for real-valued $S(t)$, the spectrum $\tilde{S}(\omega)$ can be complex valued. Since if $S(t)$ is consistent with casuality, then its spectrum $\tilde{S}(\omega)$ must also be so, and this insistence that measured spectral data must be consistent with causality can be of considerable use [12]. So useful, in fact, that even quite long articles on causality and dispersion [13] can get away without any discussion of time-domain dynamics at all!

Let us therefore take our simple eqn. (1) and Fourier transform it from the time domain into the frequency domain. Since $\partial_t A(t)$ transforms to $-\imath\omega\tilde{A}(\omega)$, we get

$$-\imath\omega\tilde{R}(\omega) = \tilde{Q}(\omega) \tag{5}$$

$$\implies \qquad \tilde{R}(\omega) = +\imath\frac{\tilde{Q}(\omega)}{\omega}. \tag{6}$$

If $Q(t)$ is a delta function impulse, then spectrally this gives $Q(\omega) = Q_0$, since the Fourier transform of a delta function is a constant, so that

$$\tilde{R}(\omega) = +\imath\frac{Q_0}{\omega}. \tag{7}$$

Since we already know that the solution for R contains a step at t_0, then we now also know (and can check) that the Fourier transform of a step function is proportional to $1/\omega$.

An important and useful was of checking and /or enforcing causality on spectral data or spectral models are the Kramers Kronig (KK) relations [6, 7, 12]. Derivations of KK relations tend to be complicated, but if we leave the technical details aside, their construction is based on two concepts:

1. The Hilbert transform [11, 14], is a transform based on the step function $h(t)$. This step $h(t)$ establishes

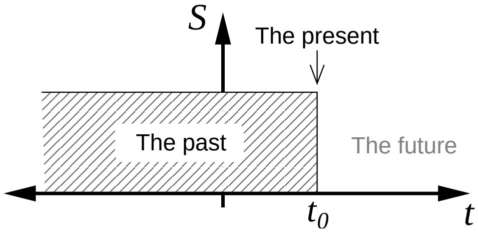

FIG. 2: A historical record $S(t)$ taken at a time t_0 can only contain data prior to that time.

or enforces the one sided nature of effects that are generated causally:

(a) A cause at $t = t_0$ can only have effects $R(t)$ that appear for times $t \geq t_0$, so that multiplying by $h(t-t_0)$ has no effect, i.e. $R(t) = R(t)h(t-t_0)$, as on fig. 1.

(b) A linear response function $u(t) = u(t)h(t)$, where $R(t) = \int_0^\infty u(t-t')Q(t)dt'$ only depends on past values of $Q(t)$.

(c) A change of sign to $h(-t)$ allows us to assert that any historical record $S(t)$ at a time t_0 must only contain data on the past ($t \leq t_0$), i.e. $S(t) = S(t)h(t_0 - t)$, as on fig. 2.

2. The Fourier transform is a transform based on the exponential $e^{-\imath\omega t}$ (see eqn. (4)). It is widely used to re-represent a time history $S(t)$, response $R(t)$, or response function $u(t)$ in a spectral form. This requires that the Fourier transforms of S, R, or u are well behaved enough to exist, which usually requires them to be normalizable and to vanishes fast enough as $\omega \rightarrow \infty$.

The Hilbert and Fourier transforms combine to turn time-domain "step" restrictions on the real-valued $X(t) \in \{R, u, S\}$ into spectral constraints on the complex $\tilde{X}(\omega)$. Following a well known theorem of Titchmarsh [12, 15], which enables us to state that for some causal (i.e. step-like) function $X(t)$ which depends only on the past (i.e. $t < 0$), the real and imaginary parts of its frequency spectrum are connected by

$$\tilde{X}(\omega) = \frac{\sigma}{\imath\pi} \mathcal{P} \int_{-\infty}^{+\infty} \frac{\tilde{X}(\omega')}{\omega' - \omega} d\omega'. \qquad (8)$$

Eqn. (8) is a compact (re)statement of the KK relations, with the factor $\imath$ in the prefactor serving to cross-link the real and imaginary parts of $\tilde{X}$; the parameter $\sigma = \pm 1$ allows us to swap the preferred direction for "the past"; with the operator $\mathcal{P}$ taking the principal part [16] of the integral[1]. This eqn. (8) thus informs us as to the spectral effect of temporal causality. However, it is not necessary to understand the mathematics it relies on – integral transforms and (complex) contour integration – in order to appreciate what it tells us.

What the KK relations tell us is that local properties are tied to global ones, as noted later on fig. 4. If $\tilde{X}(\omega)$ represents a response function, then the KK relation can be said to link the global dispersive properties ($\mathbb{R}\mathrm{e}[\tilde{X}(\omega')]$) to losses at a specific frequency ($\mathbb{I}\mathrm{m}[\tilde{X}(\omega)]$); as well as to link global loss properties ($\mathbb{I}\mathrm{m}[\tilde{X}(\omega')]$) to the response at a specific frequency ($\mathbb{R}\mathrm{e}[\tilde{X}(\omega)]$). More generally, any effect requiring complicated behaviour for the real part of the spectra (usually described as "dispersion") usually has an associated imaginary component, which for passive systems can usually be interpreted as loss. This is the origin of commonly made (and not always true) statements along the lines of "dispersion requires (or induces) loss" [17, 18].

IV. A DRIVEN, DAMPED OSCILLATOR

Let us now consider a causal response more complicated than the simple case shown in eqn. (1). An ideal example is that of a driven, damped oscillator such as a mass on a spring [19, 20], whose temporal differential equation matches that often used in electromagnetism to model the Lorentz response in a dielectric medium [8]. For the mass on a spring, with a spring constant k, we have a resonant frequency of $\omega_0 = \sqrt{k/m}$, and a friction (or "loss") parameter γ; likewise the Lorentz response also has a resonant frequency and loss. Given these parameters, the displacement of the pendulumn bob $x(t)$ (or dielectric polarization $\mathbf{P}$) could then be affected by the driving force per unit mass $F(t)/m$ (or electric field $\mathbf{E}$) according to equations of the form

$$\partial_t^2 x(t) + \gamma \partial_t x(t) + \omega_0^2 x(t) = F(t)/m, \qquad (9)$$

$$\partial_t^2 \mathbf{P}(t) + \gamma \partial_t \mathbf{P}(t) + \omega_0^2 \mathbf{P}(t) = \alpha \mathbf{E}(t). \qquad (10)$$

Here a delta function impulse in force $F(t) = p_0 \delta(t)$ does not induce an initial step change in position, but in velocity $\partial_t x(t) \simeq p_0/m$; with the likewise *initial* response of a linear (or ramp-like) change in position, with $x(t) \simeq t p_0/m$. In the same way, in an electromagnetic Lorentz dielectric medium, an impulsive $\mathbf{E}(t) = \mathbf{j}_0 \delta(t)$ gives rise to an initial step change in polarization current $\partial_t \mathbf{P}(t) \simeq \alpha \mathbf{j}_0$, and a concomittant ramp/linear change in polarization initially, i.e. $\mathbf{P}(t) \simeq \alpha t \mathbf{j}_0$. Fig. 3 shows some typical oscillating (under-damped) time responses to an impulsive driving force.

Both eqns. (9) and (10) are linear, so that the model can also be expressed in terms of a response function $u(t)$ – e.g. for the dielectric, we would have that $\mathbf{P}(t) = \int_0^\infty u(t')\mathbf{E}(t-t')dt'$. We can then Fourier transform this, and when the transform of $u(t)$ is denoted $\tilde{u}(\omega)$, we have that $\tilde{\mathbf{P}}(\omega) = \tilde{u}(\omega)\tilde{\mathbf{E}}(\omega)$. Since the Fourier transform of

[1] I.e., the value corresponding to that we would get for the integral if those points at which the integrand diverges were skipped.

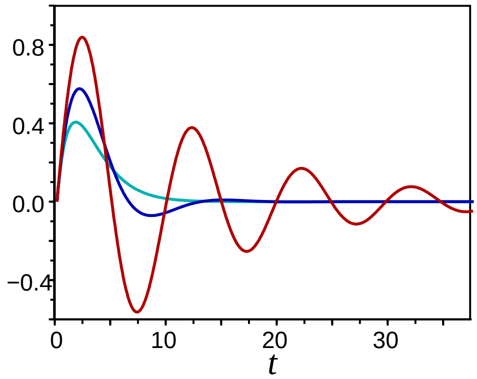

FIG. 3: Typical temporal responses (e.g. either $x(t)$ or $P(t)$) to an impulsive driving force. for a damped oscillator in the underdamped (oscilliatory) regime. The initial ramp-like response can be seen close to the vertical axis near $t = 0$.

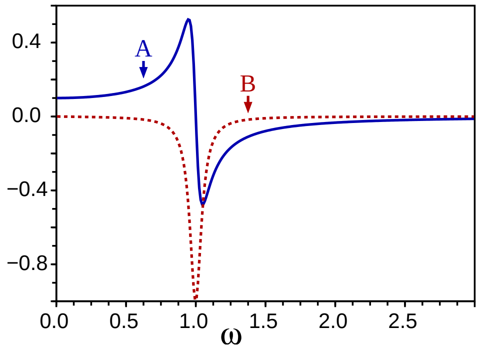

FIG. 4: A typical spectral response $\tilde{u}(\omega)$ for the damped oscillator model. The solid line shows the real part of the response, the dashed line the imaginary part. The real part of the response at any point (e.g. A at $\omega \simeq 0.6$) depends on an integral of the imaginary part over all frequencies – i.e. over the whole dashed line, $\omega' \in [0, \infty]$. Similarly, likewise the imaginary part of the response at any point (e.g. B at $\omega \simeq 1.4$) depends on an integral of the real part over all frequencies – i.e. over the whole solid line, $\omega' \in [0, \infty]$.

eqn. (10) is

$$\left[-\omega^2 - \imath\gamma\omega + \omega_0^2\right] \tilde{\mathbf{P}}(\omega) = \alpha \tilde{\mathbf{E}}(\omega), \tag{11}$$

the spectral response $\tilde{u}(\omega)$ is then easily obtained, being

$$\tilde{u}(\omega) = \frac{-\alpha}{\omega^2 - \omega_0^2 + \imath\gamma\omega}. \tag{12}$$

We can see from eqn. (12) that the real part of $\tilde{u}(\omega)$, which in electromagnetism relates to the refractive index n (squared) [21], has a frequency dependent variation with an explicit dependence on the loss parameter γ. Likewise, the loss-like part of the response, i.e. the imaginary part of $\tilde{u}(\omega)$, has an explicit dependence on frequency.

V. MAXWELL'S EQUATIONS

The curl Maxwell's equations control the behaviour of the electric and electric displacement fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{D}(\mathbf{r}, t)$, and the magnetic and magnetic induction fields $\mathbf{B}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$; and depend on a current density $\mathbf{J}(\mathbf{r}, t)$. They have single first order time derivatives as per our simple causal model of eqn. (1), and are usually written [8]

$$\nabla \times \mathbf{H} = +\partial_t \mathbf{D} + \mathbf{J}, \qquad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{13}$$

although here it is preferrable to write

$$\partial_t \mathbf{D} = \nabla \times \mathbf{H} - \mathbf{J}, \qquad \partial_t \mathbf{B} = -\nabla \times \mathbf{E}. \tag{14}$$

These otherwise independent pairs $\mathbf{E}, \mathbf{B}$ and $\mathbf{D}, \mathbf{H}$ [22] are connected together by the constitutive relations involving the dielectric polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ of the background medium, which are

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}, \tag{15}$$

and are subject to the constraint imposed by the divergence Maxwell's equations, which depend on the free electric charge density $\rho(\mathbf{r}, t)$ and the zero magnetic charge density, and are

$$\nabla \cdot \mathbf{D} = \rho/\epsilon_0, \qquad \nabla \cdot \mathbf{B} = 0. \tag{16}$$

Perhaps surprisingly, the causal nature of Maxwell's equations remains a subject of debate (see e.g. [23, 24]). Nevertheless, the curl Maxwell's equations must be causal in the "step" KK sense: eqns. (14) have the same form as our simple causal model in eqn. (1); and the presence of the spatial curl operator on the RHS's makes no difference to the temporal causality, only to the details of how $\mathbf{H}$ affects $\mathbf{D}$, or $\mathbf{E}$ affects $\mathbf{B}$. Take as a starting point the case in vacuum, where $\mathbf{P}$ and $\mathbf{M}$ are both zero, so that $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$. Then we can then rewrite eqns. (14) solely in terms of any pair of electric-like ($\mathbf{E}$ or $\mathbf{D}$) and magnetic-like ($\mathbf{H}$ or $\mathbf{B}$) fields; e.g.

$$\epsilon_0 \partial_t \mathbf{E} = \nabla \times \mathbf{H} - \mathbf{J}, \qquad \mu_0 \partial_t \mathbf{H} = -\nabla \times \mathbf{E}. \tag{17}$$

These vacuum Maxwell's equations are self-evidently KK causal.

More generally, however, the background medium for the electromagnetic fields can have non-trivial and dynamical responses to those fields encoded in $\mathbf{P}$ and $\mathbf{M}$. To avoid specifying particular response models for $\mathbf{P}$ and $\mathbf{M}$, which are many and varied, depending on the problem under consideration, I will use an abbreviated notation, i.e.,

$$\delta_t \mathbf{P} = \mathbf{f}(\cdot), \qquad \delta_t \mathbf{M} = \mathbf{f}(\cdot), \tag{18}$$

where these represent differential equations for $\mathbf{P}$ and $\mathbf{M}$ along the lines of eqn. (2). As an example, the Lorentz response in a dielectric, written out explicitly in eqn. (10), will now be represented by $\delta_t \mathbf{P} = \mathbf{f}(\mathbf{E})$.

A straightforward expression of Maxwell's equations that emphasizes their causal nature is achieved by insisting that any given field should not appear on both the LHS and RHS of the equations. Thus every field that has a dynamical response (i.e. is modelled by a temporal differential equation) can be updated simultaneously. There is no need to follow some specified sequence, although that can be useful, as in e.g. finite element simulations [25]. We can even do this even whilst incorporating magneto-electric material responses, where the electric field affects the magnetization, or the magnetic field affects the dielectric polarization. Maxwell's equation are KK causally written as

$$\partial_t \mathbf{D} = + \nabla \times \mathbf{H} - \mathbf{J}, quad \quad \delta_t \mathbf{M} = \mathbf{f}(\mathbf{H}, \mathbf{E}) \tag{19}$$

$$\text{and} \qquad \partial_t \mathbf{B} = - \nabla \times \mathbf{E}, \qquad \delta_t \mathbf{P} = \mathbf{f}(\mathbf{E}, \mathbf{H}). \tag{20}$$

Although interdependent, these equations remain explicitly KK causal in the sense that $\mathbf{H}$ and $\mathbf{E}$ are uniquely defined as causes, and $\mathbf{D}, \mathbf{P}$ and $\mathbf{B}, \mathbf{M}$ are affected by those causes (i.e. show "effects"). In the ordinary (non magneto-electric) case where $\delta_t \mathbf{M} = \mathbf{f}(\mathbf{H})$ and $\delta_t \mathbf{P} = \mathbf{f}(\mathbf{E})$, the two eqns. (19) and (20) are independent of one another. This further possible separation is the reason for associating the equation for $\mathbf{M}$ with that for $\mathbf{D}$ in eqn. (19), and associating that for $\mathbf{P}$ with $\mathbf{B}$ in eqn. (20).

Once the RHS's of eqns. (19) and (20) have been evaluated, the LHS's can be integrated directly in an explicitly KK causal manner – the "cause" fields $\mathbf{E}, \mathbf{H}$ have affected changes on $\mathbf{D}, \mathbf{P}$ and $\mathbf{B}, \mathbf{M}$. Then, the usual constitutive relations can be rearranged to connect the fields according to

$$\mathbf{E} = \epsilon_0^{-1} \left[\mathbf{D} - \mathbf{P} \right], \qquad \mathbf{H} = \mu_0^{-1} \left[\mathbf{B} - \mathbf{M} \right], \tag{21}$$

to allow us to directly update the the "cause" fields $\mathbf{E}$ and $\mathbf{H}$.

Note that if the evolution of $\mathbf{M}$ or $\mathbf{P}$ were (e.g.) to be written as dependent on $\mathbf{D}$ or $\mathbf{B}$, (so that $\delta_t \mathbf{M} = \mathbf{f}(\mathbf{B}, \mathbf{D})$), then there is no longer a perfect separation between "cause fields" and "effect fields". Nevertheless, such a rewriting will not violate causality, since the differential equations still have the correct form. Further, we cannot regard the displacement current $\partial_t \mathbf{D}$ (or indeed its magnetic counterpart $\partial_t \mathbf{B}$) as "causes" in the manner reviewed by Heras [26]; these changes in $\mathbf{D}$ and $\mathbf{B}$ are instead *effects*.

As a final note, we can easily replace an abstract current density by incorporating the motion of a particles of mass m_j and charge q_j at position $\mathbf{x}_j(t)$ with velocity $\mathbf{v}_j(t)$, by using additional causal equations

$$\partial_t \mathbf{v}_j(t) = \frac{q_j}{m_j} \left[\mathbf{E}(\mathbf{x}_j(t), t) + \mathbf{v}_j(t) \times \mathbf{B}(\mathbf{x}_j(t), t) \right] \tag{22}$$

$$\partial_t \mathbf{x}_j(t) = \mathbf{v}_j(t), \tag{23}$$

along with the connection between the electric current density and the particle motion, and the charge density which are now

$$\mathbf{J}(\mathbf{r}, t) = \sum_j q_j \delta(\mathbf{r} - \mathbf{x}_j(t)) \mathbf{v}_j(t) \tag{24}$$

$$\rho(\mathbf{r}, t) = \sum_j q_j \delta(\mathbf{r} - \mathbf{x}_j(t)). \tag{25}$$

VI. SIMULTANEITY, LOCALITY, AND SMOOTHNESS

It is worth briefly considering some issues related to the specific definition of causality used here, and whether it should be modified somehow. Here I discuss two points raised by the stance of Jefimenko [24], and another regarding the permitted smoothness of any casual response.

Local vs retarded causes: One of the benefits of the *local* causality advanced in this article is that it works knowing only the current state and current influences, and makes only minimal assumptions. In contrast, some (e.g. Jefimenko [24], also see Heras [26]) prefer to relate effects back to their original causes. Thus the electromagnetic fields would be directly obtained from the integral equations over the past behaviour of charges and currents (see eqns. (7,8) in [24]). From a practical perspective, this can raise difficulties: we often want to solve electromagnetic problems for free fields on the basis of some stated initial boundary conditions for the fields – where we do not know, nor want to calculate, whatever sources may have been required to generate them. KK causality, being local, neither knows or cares about this "deep" past; but casuality is still enforced and remains testable.

Simultaneity: We might take the position that having any part (however infinitesimal) of the effect simultaneous with the cause is unsatisfactory. Indeed, this is one of the reasons that Jefimenko [24] considers that Maxwell's equations eqn. (14) (and hence even our simple eqn. (1)) are causally ambiguous. This point can be answered in two ways.

Firstly, I have shown in this article that Maxwell's equations remain unambigously casual in the sense utilised by the Kramers Kronig relations; i.e. they are KK causal (even if they might not be "Jefimenko causal" as well).

Secondly, it is easy to examine the consequences of excluding effects simultaneous with the cause: mathematically we simply replace our step function H with by a new H' defined as $h'(t) = 0$ for $t \leq 0$ and $h'(t) = 1$ for $t > 0$. From a physicists perspective, this amounts to an infinitesimal time delay τ in the step w.r.t. the usual case; it then follows from eqn. (4) that this time delay gives rise to a phase shift $\phi = \omega\tau$ in the spectra. In the limit as $\tau \to 0^+$, we also have $\phi \to 0^+$. Since nothing mathematically singular or badly behaved happens in this limit it is hard to see how any non-negligible effects could persist[2].

Smoothness: We might feel that a step-like response is too abrupt, and that something smoother is more appropriate: e.g. for short times, the response should be ramp-like. This requires a minimal causal differential equation, which would be like a simplified version of the damped oscillator model in section IV, i.e.

$$\partial_t^2 R(t) = Q(t). \tag{26}$$

Further along this line of thought, we may desire the response to be smoother still – e.g. perhaps N times differentiable. For $N = 2$, i.e. where eqn. (26) was considered causal, we could no longer use $\partial_t x = v$ to say that "velocity causes change in position", but would instead need to use $\partial_t^2 x = a$ to say that "acceleration causes change in position".

I will not argue here for (or against) requiring this extra smoothness, save to mention the following fact: if we demand our causal responses to be smoother than a step function, we find that the curl Maxwell's equations – with their single time derivative – no longer count as causal. Perhaps, however, the basic vacuum behaviour of Maxwell's equations has nothing to do with a causal response between the fields themselves, but is rather the result of the spacetime metric. After all, the vacuum is not an electromagnetic ether! If this point of view could be sucessfully argued, then the additional smoothness requirements need to be demanded only from the medium responses $\mathbf{P}$ and $\mathbf{M}$, and the response of charges and currents to forces generated by the electromagnetic fields.

Relativity: For simplicity, in this article I have not addressed any relativistic considerations; the causality as implemented is purely temporal, and the role of the speed-of-light as a maximum signal velocity is not considered. However, this is not excluded by the formulation advanced here, although neither is it enforced. To ensure that spacetime causality holds as well, the causes need to be specified in accordance with relativistic principles, i.e. depending only on information in the past light cone, with the dynamical equations being written in a covariant way (as in e.g. Maxwell's equations).

VII. SUMMARY

KK causal models are constructed as temporal differential equations where the changes with time of the effect depends on the strength of some cause. The simplest possible model is written down as eqn. (1), containing a cause Q, something to be affected R, and a single time derivative applied to R. In such a case an impulsive cause leads to a step-like effect. A spectral analysis then gives rise to the Kramers Kronig relations, which can be used to apply constraints to measured spectral data; these link the real and imaginary parts of the spectral functions in a global-to-local manner.

For example, in kinematics, the KK causality definition used here along with the equation $\partial_t x = v$, allows us to make the statement that "velocity v causes a change in position x". In contrast, writing down e.g. $F = ma$ does not allow me to claim either that "ma causes F" or "F/m causes a". But I could instead write $\partial_t v = F/m$ and *then* make the KK causal statement "a given F/m causes a change in velocity v".

Finally, a careful writing of Maxwell's equations allows us to break down electrodynamic *solutions* into two independent steps: integration of explicitly causal temporal differential equations, and direct recalculation of the "next causes" using the equality enforced by the constitutive relations. Maxwells equations are unambiguously causal in the KK sense, whether or not any useful alternative definitions of causality exist.

Lastly, the definition of KK causality allows us to see the validity of Norton's argument [29] that it is not *a priori* necessary to *add* causality as an extra assumption to physical models. Instead, we need only take the model under discussion and determine whether or not it is compatible with KK causality by its construction – i.e. from its differential equation(s). We have seen that Maxwell's equations are compatible with causality, but even so, it is worth noting that even that fact cannot preserve causality for us if we insist (e.g. for practical reasons) on integrating them forward in space [30].

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