Spectral Singularities of a General Point Interaction

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Abstract

We study the problem of locating spectral singularities of a general complex point interaction with a support at a single point. We also determine the bound states, examine the special cases where the point interaction is $\mathcal{P}$-, $\mathcal{T}$-, and $\mathcal{PT}$-symmetric, and explore the issue of the coalescence of spectral singularities and bound states.

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1 Introduction

In elementary courses on quantum mechanics we learn that the condition that observables are Hermitian operators ensures the reality of their spectrum. This is an indispensable requirement for the applicability of the quantum measurement theory, for the points of the spectrum correspond to readings of a measuring device. Surprisingly the obvious fact that the reality of the spectrum of an operator does not necessarily mean that it is Hermitian did not play much of a role in our understanding of quantum mechanics until the recent discovery of a large class of non-Hermitian $\mathcal{PT}$-symmetric Schrödinger operators with a real spectrum [1]. This led to a great deal of excitement among some theoretical physicists. At first it looked as if we could use this kind of non-Hermitian operators to obtain a quantum theory that differed (if not generalized or even replaced) the standard quantum mechanics [2, 3]. But soon it became clear that the former theory is an equivalent representation of the latter [4, 5]. The key was to note that not only the spectrum but the expectation values, that represented all the physical quantities, must be real. It turns out that the reality of expectation values is a stronger condition than the reality of the spectrum. In fact, it implies that the operator must be Hermitian (self-adjoint) with respect to

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the inner product used to compute the expectation values [4]. This in turn brings up the possibility of the use of non-standard inner products in quantum mechanics [7, 2, 8, 3, 4]. In short, we can use non-Hermitian operators as observables of a (unitary) quantum system provided that we Hermitize them by defining the physical Hilbert space appropriately. In trying to implement this procedure to a delta-function potential with a complex coupling constant [9], it was noticed that the standard computational techniques [4] for determining the inner product (or the corresponding metric operator) of the physical Hilbert space failed when the coupling constant was purely imaginary. This marked the occurrence of an intriguing mathematical phenomenon known as a spectral singularity [10]. In Ref. [11] we explored this phenomenon for a point interaction consisting of two delta-function potentials with complex coupling constants. In Refs. [12, 13] we provided the physical meaning of spectral singularities and outlined their possible realizations and applications in optics. This followed by further study of the physical implications of spectral singularities [14]. The purpose of the present paper is to examine spectral singularities of a general point interaction with support at a single point.

First we give the definition of the point interaction in question.

Let ψ be a solution of the time-independent Schrödinger equation

$$-\psi''(x) = k^2\psi(x), \qquad \text{for} \qquad x \in \mathbb{R}\setminus\{0\}, \tag{1}$$

ψ_- and ψ_+ be respectively the restrictions of ψ to the sets of negative and positive real numbers, i.e.,

$$\psi_\pm(x) := \psi(x) \qquad \text{for} \qquad \pm x > 0. \tag{2}$$

We also set $\psi_\pm(0) := \lim_{\epsilon\to 0^\pm}\psi_\pm(\epsilon)$ and introduce the two-component wave function:

$$\Psi_\pm(x) := \left(\begin{array}{c} \psi_\pm(x) \\ \psi'_\pm(x) \end{array}\right) \qquad \text{for} \qquad \pm x \geq 0. \tag{3}$$

Then we can define the point interaction of interest by imposing the matching condition:

$$\Psi_+(0) = \mathbf{B}\Psi_-(0), \qquad \mathbf{B} := \left(\begin{array}{cc} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{array}\right), \qquad \mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d} \in \mathbb{C}. \tag{4}$$

Depending on the choice of the coupling constants $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}$, this interaction may display $\mathcal{P}$, $\mathcal{T}$, or $\mathcal{PT}$-symmetry. The effects of $\mathcal{PT}$-symmetry on the spectrum of this class of point interactions have been studied in [15]. Here we consider the problem of locating spectral singularities of these interactions. We will also study the corresponding bound states (eigenvalues with square-integrable eigenfunctions) using a different approach than the one pursued in [15].

The problem of finding spectral singularities of the above point interaction turns out not to be as trivial as that of a complex delta function potential [5] and not as complicated as that of a pair of complex delta function potentials [11]. It is nevertheless exactly solvable and provides a useful toy model to examine the nature of spectral singularities and the effects of symmetries on them. For example it allows for the examination of the possibility of the coalescence of two spectral singularities.

2 Spectral Singularities and Bound States

As discussed in [11], both the spectral singularities and bound states correspond to zeros of the M_{22} entry of the transfer matrix $\mathbf{M}$ of the system. For the point interaction given by (4),

$$\psi_\pm(x) = A_\pm e^{ikx} + B_\pm e^{-ikx}, \tag{5}$$

and $\mathbf{M}$ is the 2×2 matrix satisfying

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \mathbf{M} \begin{pmatrix} A_- \\ B_- \end{pmatrix}. \tag{6}$$

Combining (3) – (6), we find

$$\mathbf{M} = \mathbf{N}^{-1}\mathbf{B}\,\mathbf{N} = \frac{-i}{2k}\begin{pmatrix} -\mathfrak{b}k^2 + i(\mathfrak{a}+\mathfrak{d})k + \mathfrak{c} & \mathfrak{b}k^2 + i(\mathfrak{a}-\mathfrak{d})k + \mathfrak{c} \\ -\mathfrak{b}k^2 + i(\mathfrak{a}-\mathfrak{d})k - \mathfrak{c} & \mathfrak{b}k^2 + i(\mathfrak{a}+\mathfrak{d})k - \mathfrak{c} \end{pmatrix}, \tag{7}$$

where

$$\mathbf{N} := \begin{pmatrix} 1 & 1 \\ ik & -ik \end{pmatrix}.$$

According to (7), spectral singularities correspond to real k values for which

$$\mathfrak{b}k^2 + i(\mathfrak{a}+\mathfrak{d})k - \mathfrak{c} = 0, \tag{8}$$

and bound states are given by the complex solutions of this equation that have a positive imaginary part, [11].

Recall that the transfer matrix for a piecewise continuous scattering potentials has unit determinant [11]. For the point interaction we consider,

$$\det\mathbf{M} = \det\mathbf{B} = \mathfrak{a}\mathfrak{d} - \mathfrak{b}\mathfrak{c}. \tag{9}$$

Therefore, for the point interactions that are obtained as "limits" of a sequence of piecewise continuous functions, $\mathfrak{a}\mathfrak{d} - \mathfrak{b}\mathfrak{c} = 1$. We will call the point interactions violating this condition *anomalous point interactions.*

In order to examine the solutions of (8) we consider the following cases.

Case I) $\mathfrak{b}\neq 0$: In this case (8) gives

$$k=-i(\mu\pm\sqrt{\mu^2-\nu}), \tag{10}$$

where

$$\mu:=\frac{\mathfrak{a}+\mathfrak{d}}{2\mathfrak{b}}, \qquad \nu:=\frac{\mathfrak{c}}{\mathfrak{b}}. \tag{11}$$

Therefore a spectral singularity appears whenever

$$\mathrm{Re}(\mu\pm\sqrt{\mu^2-\nu})=0, \tag{12}$$

and a bound state, with a possibly complex energy, $k^2=-(\mu\pm\sqrt{\mu^2-\nu}\,)^2$, exists if

$$\mathrm{Re}(\mu\pm\sqrt{\mu^2-\nu})<0. \tag{13}$$

The following are some notable special cases:

I.a) $\mathfrak{d}=-\mathfrak{a}$ (i.e., tr $\mathbf{B}=0$): Then $\mu=0$, $k=\pm\sqrt{\nu}$ and conditions (12) and (13) become $\nu\in\mathbb{R}^+$ and $\mathrm{Im}(\pm\sqrt{\nu})<0$, respectively.

I.b) $\mathfrak{c}=0$: Then $\nu=0$, $k=-2i\mu$, we have a spectral singularity if μ is imaginary, and a bound state if $\mathrm{Re}(\mu)<0$.

I.c) $\mu^2=\nu$: Then the left-hand side of (8) has a double root, namely $k=-i\mu$. This corresponds to a spectral singularity, if $\mathrm{Re}(\mu)=0$ and $\nu\in\mathbb{R}^-$. It gives a bound state of energy $k^2=-\mu^2=-\nu$, if $\mathrm{Re}(\mu)<0$.

Case II) $\mathfrak{b}=0$: In this case we consider the following two possibilities.

II.a) $\mathfrak{a}+\mathfrak{d}=$ tr $\mathbf{B}\neq 0$: Then $k=-i\mathfrak{c}/(\mathfrak{a}+\mathfrak{d})$, and we have a spectral singularity if $\mathrm{Re}(\mathfrak{c}/(\mathfrak{a}+\mathfrak{d}))=0$ and a bound states if $\mathrm{Re}(\mathfrak{c}/(\mathfrak{a}+\mathfrak{d}))<0$. A concrete example is the delta-function potential with a possibly complex coupling constant $\mathfrak{z}$ which corresponds to the choice: $\mathfrak{a}=\mathfrak{d}=1$, $\mathfrak{b}=0$, and $\mathfrak{c}=\mathfrak{z}$. These imply $\mathfrak{c}/(\mathfrak{a}+\mathfrak{d})=\mathfrak{z}/2$. Therefore, the system has a spectral singularity if $\mathfrak{z}$ is purely imaginary and a bound state if $\mathrm{Re}(\mathfrak{z})<0$, as noted in [9, 5].

II.b) $\mathfrak{a}+\mathfrak{d}=$ tr $\mathbf{B}=0$: Then the condition of the existence of a spectral singularity or a bound state, namely $M_{22}=0$, implies that $\mathfrak{c}=0$. In this case $\mathbf{B}=\mathfrak{a}\,\sigma_{\mathbf{3}}$ and $\mathbf{M}=\mathfrak{a}\,\sigma_{\mathbf{1}}$, where

$$\sigma_{\mathbf{1}}=\left(\begin{array}{cc}0&1\\1&0\end{array}\right),\qquad \sigma_{\mathbf{3}}=\left(\begin{array}{cc}1&0\\0&-1\end{array}\right).$$

In particular, $\mathbf{M}$ is independent of k, M_{22} vanishes identically, and the interaction is anomalous for $\mathfrak{a}\neq\pm i$.

3 Coalescing Spectral Singularities and Bound States

Consider case I of the preceding section, i.e., $\mathfrak{b} \neq 0$. Suppose that $\mu_r := \text{Re}(\mu) \leq 0$ and $\nu = (1 + \frac{\epsilon}{4})\mu^2$ where $\epsilon \in [-1, 1]$. Then (10) takes the form

$$k = -i \left(1 \pm \frac{\sqrt{-\epsilon}}{2}\right) \mu, \tag{14}$$

and we find

$$\text{Re}(k) = \begin{cases} \left(1 \pm \frac{\sqrt{|\epsilon|}}{2}\right) \mu_i & \text{for} \quad -1 \leq \epsilon < 0, \\ \mu_i & \text{for} \qquad \epsilon = 0, \\ \mu_i \pm \frac{\sqrt{|\epsilon|}\ \mu_r}{2} & \text{for} \quad 0 < \epsilon \leq 1, \end{cases} \tag{15}$$

$$\text{Im}(k) = \begin{cases} -\left(1 \pm \frac{\sqrt{|\epsilon|}}{2}\right) \mu_r & \text{for} \quad -1 \leq \epsilon < 0, \\ -\mu_r & \text{for} \qquad \epsilon = 0, \\ -\mu_r \pm \frac{\sqrt{|\epsilon|}\ \mu_i}{2} & \text{for} \quad 0 < \epsilon \leq 1, \end{cases} \tag{16}$$

where $\mu_i := \text{Im}(\mu)$.

According to (16), because $\mu_r \leq 0$ the system has spectral singularities or bound states. Specifically, if $\mu_r < 0$, then $\text{Im}(k) > 0$ and for $\epsilon \in [-1, 0)$ there are a pair of bound states that coalesce into a single bound state at $\epsilon = 0$ with energy $-\mu^2$. This marks an exceptional point [16]. For $\epsilon \in (0, 1]$, the system acquires a pair of bound states, provided that $|\mu_i\sqrt{\epsilon}| < -2\mu_r$. If $|\mu_i\sqrt{\epsilon}| = -2\mu_r$ then there will be a bound state and a spectral singularity. If $|\mu_i\sqrt{\epsilon}| > -2\mu_r$, then there will be a single bound state as well as a solution of the Schrödinger equation that grows exponentially as $x \to \pm\infty$.

Figure 1 shows the plots of $\text{Re}(k)$ and $\text{Im}(k)$ for the case that $\mu = -1 + 4i$. As ϵ changes from -1 to 1, the two bound states coalesce at $\epsilon = 0$ and then split into another pair of bound states that survive as ϵ ranges between 0 and $\frac{1}{4}$. For $\epsilon = \frac{1}{4}$ one of these turns into a spectral singularity, and for $\epsilon \in (\frac{1}{4}, 1)$ it turns into a non-normalizable solution of the Schrödinger equation. The latter corresponds to the part of the graph of $\text{Im}(k)$ that appears below the ϵ-axis.

A similar scenario holds for the case that $\mu_r = 0$ and $\mu_i \neq 0$. Then for $\epsilon \in [-1, 0)$, the system has a pair of spectral singularities that coalesce at $\epsilon = 0$. For $\epsilon \in (0, 1]$ the resulting (second order) spectral singularity turns into a bound state with $k = (1 + i\frac{\sqrt{\epsilon}}{2})\mu_i$. There also appears a solution of the Schrödinger equation that grows exponentially as $x \to \pm\infty$. Figure 2 shows this behavior for $\mu = 2i$.

At $\epsilon = 0$ both the coalescing bound states and spectral singularities correspond to a repeated root of M_{22}, equivalently a second order pole of (the singular eigenvalue of) the S-matrix [12].

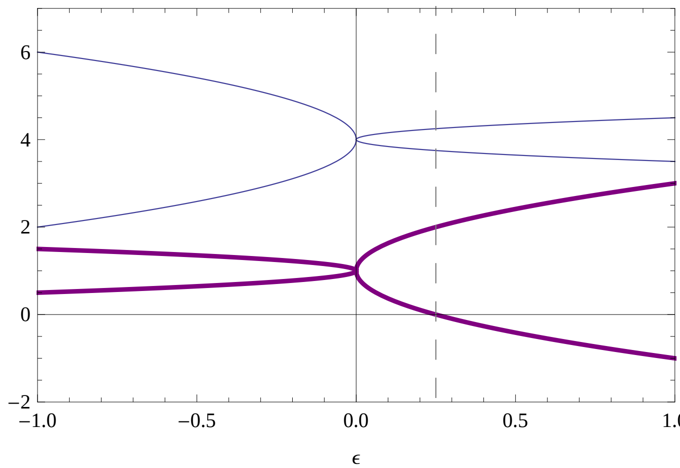

Figure 1: (Color online) Graphs of $\mathrm{Re}(k)$ (the thin blue curve) and $\mathrm{Im}(k)$ (the thick purple curve) as a function of $\epsilon \in [-1, 1]$ for $\mu = -1 + 4i$, $\nu = (1 + \frac{\epsilon}{4})\mu^2$. The dashed (grey) line is the graph of $\epsilon = \frac{1}{4}$. As ϵ increases starting from $\epsilon = -1$, the initial pair of bound states coalesce at $\epsilon = 0$ and then split into another pair of bound states. One of these only survives until ϵ reaches the critical value $1/4$ at which it turns into a spectral singularity and then disappears from the spectrum.

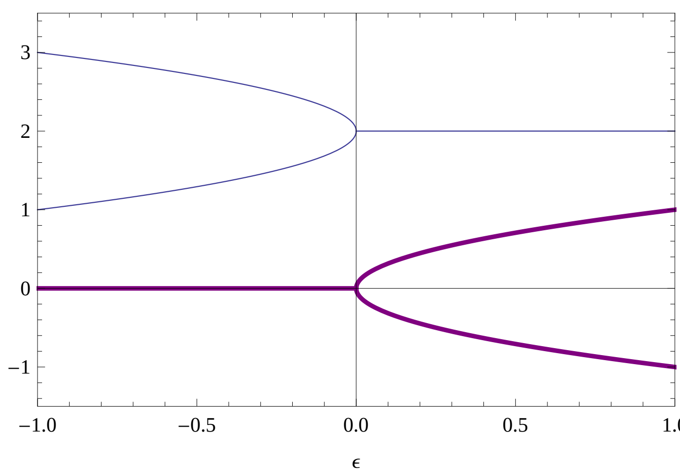

Figure 2: (Color online) Graphs of $\mathrm{Re}(k)$ (the thin blue curve) and $\mathrm{Im}(k)$ (the thick purple curve) as a function of $\epsilon \in [-1, 1]$ for $\mu = 2i$ and $\nu = -(4 + \epsilon)$. As ϵ increases starting from $\epsilon = -1$, the initial pair of spectral singularities coalesce at $\epsilon = 0$. For $\epsilon > 0$ the system has a bound state and no spectral singularities.

4 $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}$-Symmetries

In this section we examine the consequences of imposing $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}$-symmetries on the point interaction (4) and their spectral singularities and bound states.

4.1 $\mathcal{P}$-Symmetry

Let $\mathcal{P}$ be the parity (reflection) operator acting in the space of all differentiable complex-valued functions $\psi : \mathbb{R} \to \mathbb{C}$. Then for all $x \in \mathbb{R}$ we have $(\mathcal{P}\psi)(x) := \psi(-x)$ and $(\mathcal{P}\psi')(x) = -\psi'(-x)$. Therefore in terms of the two-component wave functions $\Psi := \begin{pmatrix} \psi \\ \psi' \end{pmatrix}$, we have

$$(\mathcal{P}\Psi)(x) = \sigma_{\mathbf{3}}\Psi(-x). \tag{17}$$

We say that the point interaction (4) is $\mathcal{P}$-invariant (has $\mathcal{P}$-symmetry), if

$$(\mathcal{P}\Psi_+)(0) = \mathbf{B}\,(\mathcal{P}\Psi_-)(0). \tag{18}$$

We can use this relation and (17) to obtain the following simple expression for $\mathcal{P}$-invariance of the point interaction (4).

$$\mathbf{B}\,\sigma_{\mathbf{3}}\mathbf{B} = \sigma_{\mathbf{3}}. \tag{19}$$

In terms of the entries of $\mathbf{B}$, this is equivalent to

$$\mathfrak{a}^2 - \mathfrak{b}\mathfrak{c} = 1, \qquad \mathfrak{d} = \pm\mathfrak{a}, \qquad \mathfrak{b}(\mathfrak{a} - \mathfrak{d}) = \mathfrak{c}(\mathfrak{a} - \mathfrak{d}) = 0. \tag{20}$$

Next, consider the problem of spectral singularities and bound states for $\mathcal{P}$-invariant point interactions.

Case I) $\mathfrak{b} \neq 0$: Then $\mathfrak{d} = \mathfrak{a}$, $\mathfrak{c} = (\mathfrak{a}^2 - 1)/\mathfrak{b}$, $\mu = \mathfrak{a}/\mathfrak{b}$, $\nu = (\mathfrak{a}^2 - 1)/\mathfrak{b}^2$, $\mu^2 - \nu = 1/\mathfrak{b}^2$, and $k = -i(\mathfrak{a} \pm 1)/\mathfrak{b}$. Therefore, a spectral singularity exists if $\mathrm{Re}((\mathfrak{a} \pm 1)/\mathfrak{b}) = 0$, and a bound states arises if $\mathrm{Re}((\mathfrak{a} \pm 1)/\mathfrak{b}) < 0$.

Case II.a) $\mathfrak{b} = 0$ and $\mathfrak{a} + \mathfrak{d} \neq 0$: Then $\mathfrak{a} = \mathfrak{d} = \pm 1$, $k = \mp i\mathfrak{c}/2$, a spectral singularity appears if $\mathrm{Re}(\mathfrak{c}) = 0$, and a bound state exists if $\mathrm{Re}(\mp\mathfrak{c}) > 0$.

Case II.b) $\mathfrak{b} = 0$ and $\mathfrak{a} + \mathfrak{d} = 0$: Then $\mathfrak{d} = -\mathfrak{a} = \mp 1$, the interaction is anomalous, and spectral singularities and bound states exist for $k \in \mathbb{R}^+$ and $\mathrm{Im}(k) > 0$, respectively.

4.2 $\mathcal{T}$-Symmetry

We identify the time-reversal operator $\mathcal{T}$ as the operator that acts on complex-valued functions $\psi : \mathbb{R} \to \mathbb{C}$ according to $(\mathcal{T}\psi)(x) := \psi(x)^*$. The point interaction (4) is time-reversal invariant (has $\mathcal{T}$-symmetry), if

$$(\mathcal{T}\Psi_+)(0) = \mathbf{B}\,(\mathcal{T}\Psi_-)(0), \tag{21}$$

where the action of $\mathcal{T}$ on a two-component wave function Ψ is defined componentwise. It is easy to see that this relation is equivalent to the requirement that $\mathbf{B}$ is a real matrix, i.e., $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ must be real. In this case, we can summarize the conditions for the existence of spectral singularities and bound states as follows.

Case I) $\mathfrak{b} \neq 0$: Then a spectral singularity exists provided that $\mu = 0$ and $\nu \in \mathbb{R}^+$. In terms of the entries of $\mathbf{B}$, these relations take the form $\mathfrak{d} = -\mathfrak{a}$ and $\mathfrak{c}/\mathfrak{b} \in \mathbb{R}^+$, respectively. The k value associated with this spectral singularity is $k = \sqrt{\nu} = \sqrt{\mathfrak{c}/\mathfrak{b}}$. Similarly a bound state with $k = \sqrt{\nu - \mu^2} - i\mu$ exists, if $\mu < 0$ and $\nu - \mu^2 > 0$. The latter conditions can also be expressed as $(\mathfrak{a} + \mathfrak{d})/\mathfrak{b} < 0$ and $4\mathfrak{b}\mathfrak{c} - (\mathfrak{a} + \mathfrak{d})^2 > 0$, respectively.

Case II.a) $\mathfrak{b} = 0$ and $\mathfrak{a} + \mathfrak{d} \neq 0$: Then no spectral singularities exists, and a bound state with $k = -i\mathfrak{c}/(\mathfrak{a} + \mathfrak{d})$ is present provided that $\mathfrak{c}/(\mathfrak{a} + \mathfrak{d}) < 0$.

Case II.b) $\mathfrak{b} = 0$ and $\mathfrak{a} + \mathfrak{d} = 0$: Then $M_{22} = 0$ implies $\mathfrak{c} = 0$. As a result, the interaction is anomalous, and spectral singularities and bound states exist for $k \in \mathbb{R}^+$ and $\mathrm{Im}(k) > 0$, respectively.

4.3 $\mathcal{PT}$-Symmetry

We say that the point interaction (4) is $\mathcal{PT}$-invariant or $\mathcal{PT}$-symmetric if

$$(\mathcal{PT}\Psi_+)(0) = \mathbf{B}\,(\mathcal{PT}\Psi_-)(0), \tag{22}$$

This is equivalent to

$$\mathbf{B}^*\boldsymbol{\sigma_3}\mathbf{B} = \boldsymbol{\sigma_3}. \tag{23}$$

Expressing this relation in terms of the entries of $\mathbf{B}$ and solving the resulting equations yield

$$\mathfrak{a} = \sqrt{1 + \epsilon_1 bc}\, e^{i\alpha}, \qquad \mathfrak{b} = \epsilon_1\epsilon_2 b\, e^{i(\alpha+\delta)/2}, \qquad \mathfrak{c} = \epsilon_2 c\, e^{i(\alpha+\delta)/2}, \qquad \mathfrak{d} = \sqrt{1 + \epsilon_1 bc}\, e^{i\delta}, \tag{24}$$

where $\epsilon_2 = \pm 1$, $b, c \in [0, \infty)$, $\alpha, \delta \in [0, 2\pi)$, and

$$\epsilon_1 = \begin{cases} +1 & \text{if} \quad bc \geq 1, \\ \pm 1 & \text{if} \quad bc < 1. \end{cases} \tag{25}$$

Therefore, the $\mathcal{PT}$-symmetric point interactions are determined by

$$\mathbf{B} = e^{i(\alpha+\delta)/2}\left(\begin{array}{cc} \sqrt{1+\epsilon_1 bc}\; e^{i(\alpha-\delta)/2} & \epsilon_1\epsilon_2 b \\ \epsilon_2 c & \sqrt{1+\epsilon_1 bc}\; e^{-i(\alpha-\delta)/2} \end{array}\right). \tag{26}$$

In particular, $\det\mathbf{B} = e^{i(\alpha+\delta)}$, and the interaction is anomalous unless $\alpha+\delta$ is an integer multiple of 2π.

The conditions for the existence of spectral singularities and bound states are as follows.

Case I) $\mathfrak{b}\neq 0$: A straightforward consequence of (24) is that both the parameters μ and ν take real values. More specifically, we have

$$\mu = \frac{\sqrt{1+\epsilon_1 bc}\;\cos(\frac{\alpha-\delta}{2})}{\epsilon_1\epsilon_2 b}, \qquad \nu = \frac{\epsilon_1 c}{b}, \qquad \mu^2-\nu = \frac{(1+\epsilon_1 bc)\cos^2(\frac{\alpha-\delta}{2}) - c^2}{b^2}. \tag{27}$$

Now, if we impose the condition of the presence of spectral singularities (12) we find $\mu = 0$ and $\nu > 0$. In view of (27), these imply that $\epsilon_1 = +1$ and $\delta = \alpha + (2\ell+1)\pi$, where ℓ is an arbitrary integer. Therefore, the $\mathcal{PT}$-symmetric point interactions that have a spectral singularity (with $k = \sqrt{\nu}$) are given by

$$\mathbf{B} = e^{i\alpha}\left(\begin{array}{cc} \sqrt{1+bc} & i\epsilon b \\ i\epsilon c & -\sqrt{1+bc} \end{array}\right), \tag{28}$$

where $\epsilon = (-1)^\ell\epsilon_2 = \pm 1$. Similarly, we can check that the system has a bound state provided that $\mu < 0$. This is equivalent to $\epsilon_1\epsilon_2\cos[(\alpha-\delta)/2] < 0$. In this case there are three possibilities:

1) $\mu^2-\nu > 0$: Then there is a pair of bound states with real and negative energies $k^2 = -(2\mu^2-\nu\pm\mu\sqrt{\mu^2-\nu})$.

2) $\mu^2-\nu = 0$: This corresponds to an exceptional point [16], where there is single bound state with a real and negative energy $k^2 = -\mu^2$.

3) $\mu^2-\nu < 0$: Then there is a pair of bound states with complex-conjugate energies $k^2 = \nu - 2\mu^2 \pm i\mu\sqrt{\nu-\mu^2}$.

Case II.a) $\mathfrak{b} = 0$ and $\mathfrak{a}+\mathfrak{d}\neq 0$: In this case there are no spectral singularities, but a bound state exists provided that $\epsilon_2\cos[(\alpha-\delta)/2] < 0$. It has a real and negative energy given by $k^2 = -\frac{1}{4}c^2\sec^2[(\alpha-\delta)/2]$.

Case II.b) $\mathfrak{b} = 0$ and $\mathfrak{a}+\mathfrak{d} = 0$: Then the condition $M_{22} = 0$ implies $c = 0$,

$$B = \left(\begin{array}{cc} e^{i\alpha} & 0 \\ 0 & -e^{i\alpha} \end{array}\right),$$

the interaction is anomalous unless $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, a spectral singularity arises for all $k\in\mathbb{R}$, and there is a bound state for all $k\in\mathbb{C}$ with $\mathrm{Im}(k) > 0$.

5 Concluding Remarks

In this article we present a complete solution for the problem of spectral singularities of a general point interaction with a support at a single point. We also examine the consequences of the presence of $\mathcal{P}$-, $\mathcal{T}$-, and $\mathcal{PT}$-symmetries. Unlike the case of complex delta-function potential and double-delta function potential, for a generic point interaction that we consider, the function whose zeros give the spectral singularities and bound states is a quadratic polynomial in k. This in turn implies the possibility of having a spectral singularity or a bound state that is related to a second order zero of this polynomial. For the case of a bound state this corresponds to a degeneracy or exceptional point. The latter leads to a well-known type of geometric phases [16]. The analogy with coalescing spectral singularities calls for a thorough examination of the geometric phase problem for systems supporting second and higher order spectral singularities.

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