Modulation Instability of Ultrashort Pulses in Quadratic Nonlinear Media beyond the Slowly Varying Envelope Approximation

Amarendra K. Sarma* and Parvendra Kumar
Department of Physics, Indian Institute of Technology Guwahati, Guwahati-781039, Assam, India.
*Electronic address: aksarma@iitg.ernet.in

Abstract: We report a modulational instability (MI) analysis of a mathematical model appropriate for ultrashort pulses in cascaded quadratic-cubic nonlinear media beyond the so-called slowly varying envelope approximation. Theoretically predicted MI properties are found to be in good agreement with numerical simulation. The study shows the possibility of controlling the generation of MI and formation of solitons in a cascaded quadratic-cubic media in the few cycle regimes. We also find that stable propagation of soliton-like few-cycle pulses in the medium is subject to the fulfilment of the modulation instability criteria.

1. Introduction

Recently nonlinear optics research beyond the so called slowly varying envelope approximation (SVEA) have received tremendous boost due to various reasons, primarily for richness in physics and possible applications in many diverse areas such as, ultrafast spectroscopy, metrology, medical diagnostics and imaging, optical communications, manipulation of chemical reactions and bond formation, material processing etc. [1-2]. Particularly, the availability of sources of light in the near-single optical cycle has opened new possibilities for physicists and scientist to explore and doubt many of the fundamental concepts and assumptions [3-4].The validity of SVEA is already questioned by many authors in this new domain of optical science [5-10].Many authors have attempted to modify the SVEA so that it might be extended to the few cycle regimes. The first widely accepted model in this regard has been developed by Brabec and Krausz [5]. Some other authors have offered non-SVEA models also [11-12]. However, the model equation proposed by Brabec and Krausz have been used most extensively and successfully in various contexts [13-16]. Recently, following the model proposed by Brabec and Krausz, Moses and Wise have derived a coupled propagation equations for ultrashort pulses in a degenerate three-wave mixing process in quadratic ($\chi^{(2)}$) media [17]. In passing, it is worthwhile to mention that owing to the efficient manipulation of spectral and temporal properties of femtosecond pulses through cascaded processes in quadratic materials, both theoretical and experimental research is getting tremendous boost in recent years [18-21]. Moses and Wise went on to present, using cascaded quadratic nonlinearity, theoretical and experimental evidence of a new quadratic effect, namely the controllable self-steepening (SS) effect. The controllability of the SS effect is very useful in nonlinear propagation of ultrashort pulses as it may be used to cancel the propagation effects of group velocity mismatch. It may be noted that, traditionally, the intensity dependent refraction (IDR) effects in quadratic media are not expected in quadratically nonlinear media owing to the phase mismatch of the fundamental harmonic with the higher ones within the SVEA [22]. In this work, we have studied the modulation instability (MI) of the single-field equation for the fundamental field (FF) derived by Moses and Wise [17]. Our study is mainly motivated by the fact that IDR effect is closely related to MI, particularly to the existence of optical solitons in a nonlinear media. It is well known that modulation instability is a fundamental and ubiquitous process that appears in most nonlinear systems in nature [23-27]. It occurs as a result of interplay between the nonlinearity and dispersion in time domain or diffraction in spatial domain. In this work we are specifically interested to see the role of group velocity mismatch (GVM) between the fundamental (FF) and second-harmonic (SH) field on MI as well as the role of self-steepening (SS) parameter. Though our study indicates the possibility of getting MI even in the so-called normal dispersion regime, in this work we confine our attention to the anomalous dispersion regime only.

2. Theoretical model and MI analysis

The Moses-Wise model for ultrashort pulse propagation in a cascaded-quadratic media, under appropriate condition, can be written as follows [17]:

$$i\frac{\partial A}{\partial z}-\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2}+\left(\gamma+\frac{\Gamma^2}{\Delta k}\right)|A|^2 A+i\left(\frac{\Gamma^2}{\Delta k\,\omega_0}+\frac{\gamma}{\omega_0}\right)A^2\frac{\partial A^*}{\partial T}=0 \tag{1}$$

Here A is the complex envelope of the fundamental field travelling along z, β_2 is the group velocity dispersion(GVD) parameter, ω_0 is the carrier wave frequency, γ is the cubic nonlinear co-efficient, $\Delta k = 2k_1 - k_2$ is the wave vector mismatch between the FF and SH fields and $\Gamma^2 = 32\pi^2\omega_0^4\chi^{(2)}_{2\omega_0-\omega_0}\chi^{(2)}_{\omega_0+\omega_0}/k_1k_2c^4$. k_1 and k_2 are respectively the wave vectors associated with FF and SH fields. In this work we assume that, $\gamma+\Gamma^2/\Delta k \neq 0$. We rewrite Eq. (1) in the normalized units [23] as follows:

$$i\frac{\partial u}{\partial \xi}-\frac{\alpha}{2}\frac{\partial^2 u}{\partial \tau^2}+\beta|u|^2 u+i\beta s u^2\frac{\partial u^*}{\partial \tau}=0 \tag{2}$$

where u is the normalized amplitude, $\alpha=\operatorname{sgn}(\beta_2)$ and

$$\xi = z/L_D, \tau = T/T_0, L_D = T_0^2/|\beta_2|, A=\sqrt{P_0}N u, N=\sqrt{\gamma P_0 L_D}, \beta = 1+\left(\Gamma^2/\Delta k\,\gamma\right), s = 1/\omega_0 T_0 \tag{3}$$

in which ξ and τ are the normalized propagation distance and time respectively, P_0 is the peak power of the incident pulse, L_D is the dispersion length, N is the so called soliton order and s is the self-steepening (SS) parameter. On the basis of Eq. (2) we would now investigate the MI of few cycle pulses. Eq. (2) has a steady state solution given by $u = u_0\exp\left[i u_0^2\xi\right]$, where u_0 is the constant amplitude of the incident plane wave. We now introduce perturbation $a(\xi,\tau)$ together with the steady state solution to Eq. (2) and linearize in $a(\xi,\tau)$ to obtain:

$$i\frac{\partial a}{\partial \xi}+\beta u_0^2\left(a+a^*\right)-\frac{\alpha}{2}\frac{\partial^2 a}{\partial \tau^2}+i\beta s u_0^2\frac{\partial a^*}{\partial \tau}=0 \tag{4}$$

Separating the perturbation to real and imaginary parts, according to $a = a_1 + i a_2$, and assuming $a_1, a_2 \propto \exp\left[i\left(K\xi-\Omega\tau\right)\right]$, where K and Ω are the wave number and the frequency of perturbation respectively, from Eq. (4) we obtain the following dispersion relation

$$K=\left[\frac{\Omega^4}{4}+\left(1+\frac{\Gamma^4}{\Delta k^2\gamma^2}\right)s^2u_0^4\Omega^2+\frac{\Gamma^2}{\Delta k\,\gamma}\left(2s^2u_0^4\Omega^2+\alpha\Omega^2u_0^2\right)+\alpha\Omega^2u_0^2\right]^{\frac{1}{2}} \tag{5}$$

From Eq. (5), we observe that the modulation instability exists only if the quantity inside the bracket is <0. It may be possible to have MI in the normal dispersion regime, for which $\alpha = +1$, if $\Delta k < 0$ and other parameters are chosen judiciously, but we find that task to be a difficult one. However in the anomalous dispersion regime, for which $\alpha = -1$, the occurrence of MI is possible under appropriate conditions and judicious choice of parameters are easy. Now onwards we take $\alpha = -1$. The expression for the so-called gain spectrum $g(\Omega)$ could be put in the following form:

$$g(\Omega) = 2\,\mathrm{Im}(K) = 2\left[\Omega_B^2 - \Omega_A^2\right]^{\frac{1}{2}}, \tag{6}$$

where,

$$\Omega_B^2 = \Omega^2 u_0^2 - \eta\left(2s^2 u_0^4 - u_0^2\right)\Omega^2 ; \Omega_A^2 = \left(1+\eta^2\right)s^2 u_0^4 \Omega^2 + \left(\Omega^4/4\right) \quad \text{with } \eta = \Gamma^2/\gamma\Delta k \tag{7}$$

We note that for the occurrence of MI, one must have $\Omega_B^2 > \Omega_A^2$. The maximum of the gain occurs at two frequencies given by:

$$\Omega_{\max} = \pm\sqrt{2u_0^2 - 2\eta\left(2su_0^4 - u_0^2\right) - 2\left(1+\eta^2\right)s^2 u_0^4} \tag{8}$$

Now, we would try to see the role of various controllable parameters such as the self-steepening (SS) s and wave-vector mismatch Δk on MI. Fig.1 depicts the modulation instability gain spectrum $g(\Omega)$ vs. Ω for different values of η for $s = .01$ and $u_0 = 1$.

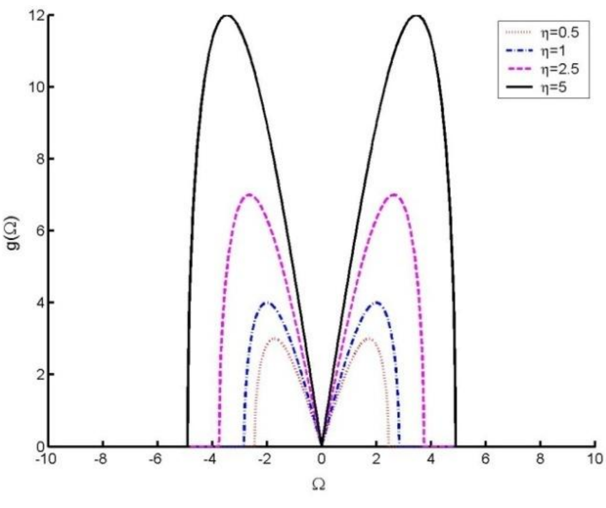

Fig. 1 (Color online) Modulational instability gain as a function of normalized frequency for four different values of η with u_0 =1 and $s = 0.1$.

It can be clearly seen that the gain spectrum is symmetric with respect to $\Omega = 0$. We observe from Fig. 1 that, for the given input power and a fixed self-steepening parameter, the modulation instability gain increases with increase in η. Physically speaking, MI gain increases with a decrease in the wave-vector mismatch Δk, as the parameter η is directly related to it through Eq. (7). If a probe wave at a frequency $\omega_0 + \Omega$ were to propagate with the CW beam at ω_0, it would experience a net power gain given by Eq.(6) as long as

$\text{Im}(K) > 0$.Eventually, due to MI gain, the CW beam would break up spontaneously into a periodic pulse train known as solitons. These soliton-like pulses exist whenever the conditions $\Omega_B^2 > \Omega_A^2$ and $\Delta k \neq 0$ are satisfied. The appearance of the sidebands located around $\Omega = 0$ is the clear evidence of modulation instability.

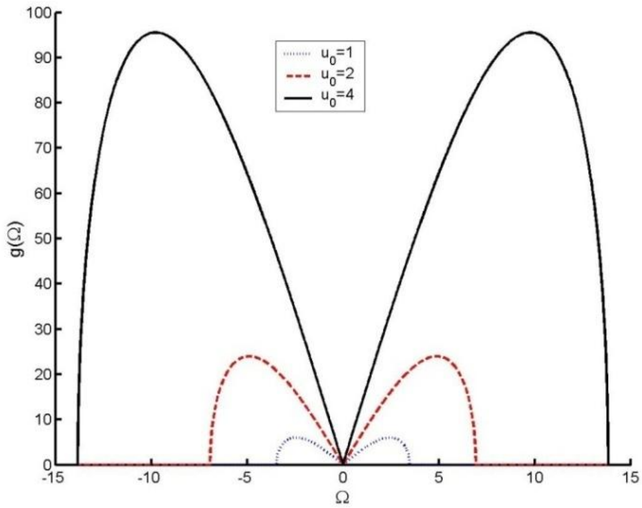

Fig. 2 (Color online) Modulation instability gain as a function of normalized frequency for three different values of u_0 with $\eta = 2$ and $s = 0.1$.

Fig.2 shows, quite expectedly, that with increase with the amplitude, the MI gain also increases. On the other hand we find that MI gain decreases with increase in the SS parameter, s, as could be observed from Fig.3. In fact our calculations show that MI gain vanishes if s is increased beyond $s = 0.55$ for the given parameters. As the SS parameter is inversely related to both the carrier-wave frequency, ω_0, and pulse width, T_0, we may

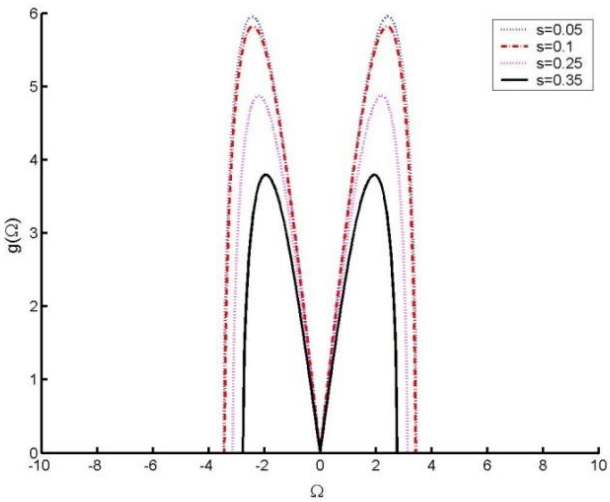

Fig. 3 (Color online) Modulation instability gain as a function of normalized frequency for four different values of s with $\eta = 2$ and $u_0 = 1$.

conclude that there is a threshold value of the pulse-width for a given operating frequency at which the MI could be invoked, and thereby generating solitonic pulses, in a cascaded quadratic media like the one considered in this work.

3. Numerical Simulation

To get more physical insight of the obtained properties of MI and dynamics of a CW beam under the MI gain, we numerically solve Eq. (2) using the so-called split-step Fourier method [23] in the anomalous dispersion regime. The incident field launched at $\xi = 0$ into the nonlinear medium is taken to be: $u(0,\tau) = u_0\left(1 + a_0\cos(\Omega_m \tau)\right)$, where a_0 is the normalized modulation amplitude and Ω_m is the normalized angular frequency of a weak sinusoidal modulation imposed on the CW beam. We choose $u_0 = 1, a_0 = 0.05$ and $\Omega_m = 1$ for our

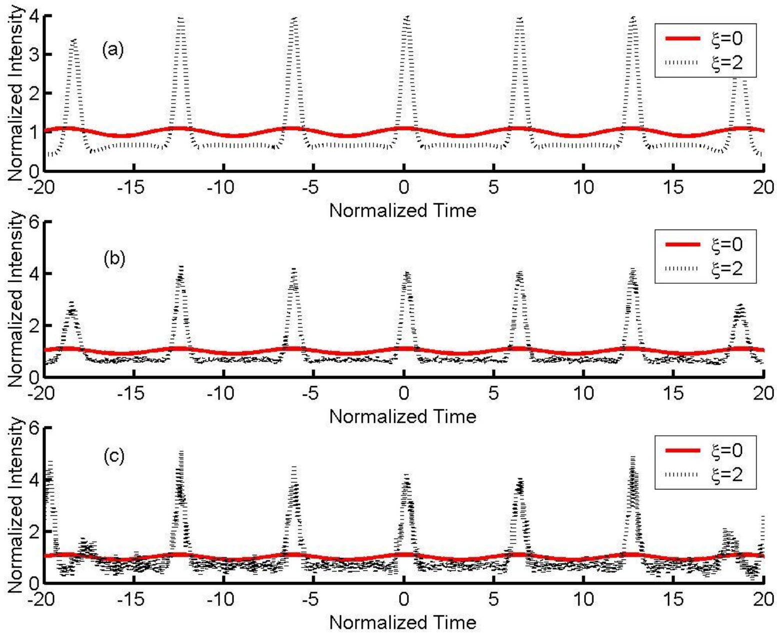

Fig. 4 (Color online) Temporal distribution of the normalized field distribution at two different normalized propagation distances with $\eta = 1$ for (a) $s = 0$ (b) $s = 0.05$ and (c) $s = 0.1$

numerical simulation. Fig. 4 depicts the temporal distribution of the field intensity at distances ξ =0 and 2 for various values of SS parameter with a fixed value of η .We observe that the co-sinoidally modulated plane wave evolves into a train of pulses with much higher amplitude than the initial modulation. Pulses got distorted with increasing value of the SS-parameter. In fact we find that arbitrary increase of s results in vanishing of MI, confirming our analytical results. In order to get an idea how the generation of MI is affected with increase of η, in Fig. 5 we plot the temporal evolution of the normalized intensity for three different values of η taking s =0. It can be clearly seen that as η increases the co-sinoidally modulated plane wave evolves into a train of shorter pulses. The breaking of the plane wave

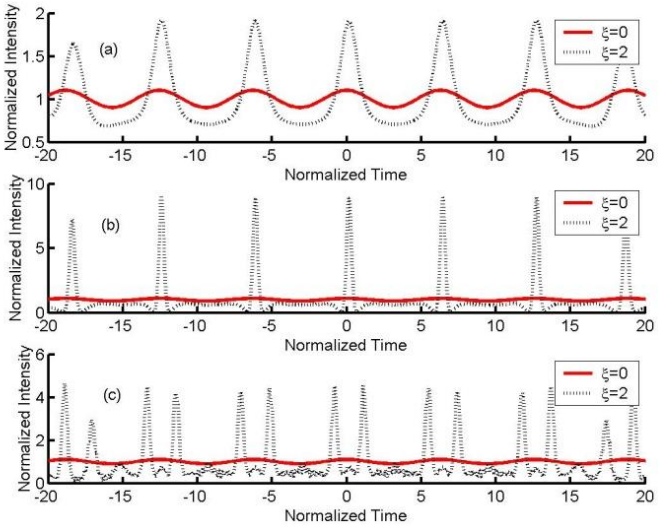

Fig. 5 (Color online) Temporal distribution of the normalized field distribution at two different normalized propagation distances for (a) $\eta = 0.5$ (b) $\eta = 1.5$ and (c) $\eta = 3.5$

into much shorter pulses get enhanced with increase in η, as could be expected owing to the increase of MI gain, predicted by our analytical calculations. Finally, we solve Eq. (2) numerically to test the stability of soliton propagation in a cascaded quadratic nonlinear

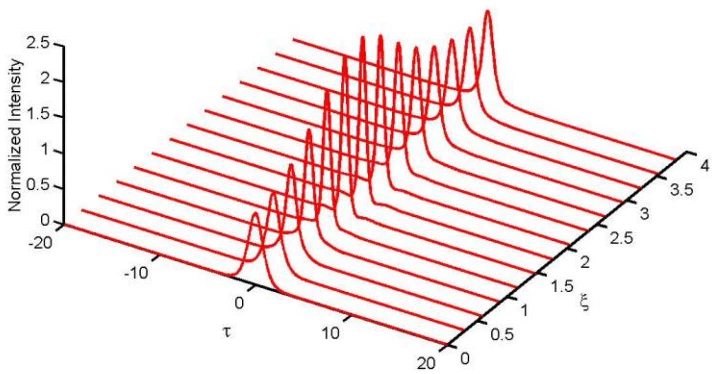

Fig. 6 (Color online) Spatio-temporal evolution of soliton

media like the one considered in this work. The input pulse is taken to be of the form: $u(0,\tau) = Sech(\tau)$. We choose s =0.01and η =1. The spatio-temporal evolution of the soliton pulse is depicted in Fig. 6, from which clear stable propagation may be observed.

4. Conclusion

To conclude, we have studied the modulation instability of the Moses-Wise model for ultrashort pulse propagation in a cascaded-quadratic media. A nonlinear dispersion relation is worked out using standard methods. We find that subject to the fulfilment of the MI criteria and judicious choice of the parameters, MI could be generated in a cascaded quadratic-cubic media in the anomalous dispersion regime. Numerical simulation confirms our theoretical predictions. Stable soliton propagation is also observed with appropriate choice of parameters. We hope that this work would throw some light or stimulate research work related to the control of MI generation and soliton formation in cascaded quadratic media.

References

[1] M. Wegener, *Extreme Nonlinear Optics* (Springer-Verlag, Berlin, 2005).
[2] T.Brabec, *Strong Field Laser Physics* (Springer, Berlin, 2008).
[3] E. Goulielmakis *et al.*, Science **305**, (2004)1267.
[4] T. Brabec and F. Krausz, Rev. Mod. Phys. **72**, (2000) 545.
[5] T. Brabec and F. Krausz, Phys.Rev. Lett. **78**, (1997) 3282.
[6] A.L.Gaeta, Phys.Rev. Lett. **84**, (2000) 3582.
[7] H. Leblond and D. Mihalache, Phys. Rev. A **79**, (2009) 063835.
[8] J. E. Rothenberg, Opt. Lett. **17**, (1992) 1340.
[9] K. E. Oughstun and H. Xiao, Phys.Rev. Lett. **78**, (1997) 642.
[10] J. K. Ranka and A. L. Gaeta, Opt. Lett. **23**, (1998) 534.
[11] V. G. Bespalov *et al*., Phys. Rev. A **66**, (2002) 013811.
[12] T. Tsurumi, J. Phys. Soc. Japan 75, (2006) 024002.
[13] D. Homoelle and A. L. Gaeta, Opt. Lett. **25**, (2000) 761.
[14] S. A. Trushi *et al*., Opt. Lett. 32, (2007) 2432.
[15] G. Fibich and A. L. Gaeta, Opt. Lett. **25**, (2000) 335.
[16] M. Nurhuda *et al*., Phys. Rev. A **66**, (2002) 023811.
[17] J. Moses and F. W. Wise, Phys.Rev. Lett. **97**, (2006) 073903.
[18] M. Marangoni *et al*., App. Phys. Lett. **93**, (2008) 021107.
[19]X. Liu *et al*., Opt. Lett. **24**, (1999) 1777.
[20]S. Ashihara *et al*., J. Opt. Sc. Am. B. **19**, (2002) 2505.
[21]M. Bache *et al*., Phys. Rev. A **82**, (2010) 063806.
[22] A.G. Kalocsai and J. W. Haus, Phys. Rev. A **49**, (1994)574.
[23] G. P. Agrawal, *Nonlinear Fiber Optics*, 4th Ed. (Academic Press, San Diego, 2007).
[24] A. K. Sarma and M. Saha, J. Opt. Sc. Am. B. **28**, (2011) 944.
[25] M. J. Potasek, Opt. Lett. **12**, (1987) 921.
[26] P. K. Shukla and J. J. Rasmussen, Opt. Lett. **11**, (1986) 171.
[27] A. K. Sarma, Eur. Phys. Lett. **92**, (2010) 24004.