On hypergraph cliques with chromatic number 3*

D.D. Cherkashin

This work is devoted to a problem in extremal hypergraph theory, which goes back to P. Erdős and L. Lovász (see [1]). Before giving an exact statement of the problem, we recall some definitions and introduce some notation.

Let $H = (V, E)$ be a hypergraph without multiple edges. We call it *n-uniform*, if any of its edges has cardinality n: for every $e \in E$, we have $|e| = n$. By the *chromatic number* of a hypergraph $H = (V, E)$ we mean the minimum number $\chi(H)$ of colors needed to paint all the vertices in V so that any edge $e \in E$ contains at least two vertices of some different colors. Finally, a hypergraph is said to form a *clique*, if its edges are pairwise intersecting.

In 1973 Erdős and Lovász noticed that if an n-uniform hypergraph $H = (V, E)$ forms a clique, then $\chi(H) \in \{2, 3\}$. They also observed that in the case of $\chi(H) = 3$, one certainly has $|E| \leqslant n^n$ (see [1]). Thus, the following definition has been motivated:

$$M(n) = \max\{|E| : \exists \text{ an } n-\text{uniform clique } H = (V, E) \text{ with } \chi(H) = 3\}.$$

Obviously such definition has no sense in the case of $\chi(H) = 2$.

Theorem 1 (P. Erdős, L. Lovász, [1]). *The inequalities hold*

$$n!\left(\frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) \leqslant M(n) \leqslant n^n.$$

Almost nothing better has been done during the last 35 years. In the book [2] the estimate $M(n) \leqslant \left(1 - \frac{1}{e}\right) n^n$ is mentioned as "to appear". However, we have not succeeded in finding the corresponding paper.

At the same time, another quantity $r(n)$ was introduced in [3]:

$$r(n) = \max\{|E| : \exists \text{ an } n-\text{uniform clique } H = (V, E) \text{ s.t. } \tau(H) = n\},$$

where $\tau(H)$ is the *covering number* of H, i.e.,

$$\tau(H) = \min\{|f| : f \subset V, \forall e \in E \ \ f \cap e \neq \emptyset\}.$$

Clearly, for any n-uniform clique H, we have $\tau(H) \leqslant n$ (since every edge forms a cover), and if $\chi(H) = 3$, then $\tau(H) = n$. Thus, $M(n) \leqslant r(n)$. Lovász noticed that for $r(n)$ the same estimates as in Theorem 1 apply and conjectured that the lower estimate is best possible. In 1996 P. Frankl, K. Ota, and N. Tokushige (see [4]) disproved this conjecture and showed that $r(n) \geqslant \left(\frac{n}{2}\right)^{n-1}$.

We discovered a new upper bound for the initial value $M(n)$.

Theorem 2. *There exists a constant $c > 0$ such that*

$$M(n) \leqslant c n^{n-\frac{1}{2}} \ln n.$$

We shall prove Theorem 2 in the next section.

*This work was supported by the grant of RFBR N 09-01-00294.

1 Proof of Theorem 2

We shall proceed by citing or proving successive propositions that will eventually lead us to the proof of the theorem.

Proposition 1 (P. Erdős, L. Lovász, [1]). *Let $H = (V, E)$ be an n-uniform clique with $\chi(H) = 3$. Let k be an arbitrary integer such that $1 \leqslant k \leqslant n$. Take any set $W \subseteq V$ of cardinality k. Let $E(W)$ denote the set of all edges $e \in E$ such that $W \subseteq e$. Then $|E(W)| \leqslant n^{n-k}$.*

Note that in particular, the degree $\deg v$ of any vertex $v \in V$ does not exceed n^{n-1} (here $k = 1$). This fact entails immediately the estimate $M(n) \leqslant n^n$. Although we suppose to prove a much better bound, we shall frequently use Proposition 1 during the proof.

To any n-uniform hypergraph $H = (V, E)$ we assign the set

$$B(H) = \left\{ v \in V : \deg v > \frac{|E|}{n^2} \right\}.$$

Proposition 2. *Let $H = (V, E)$ be an n-uniform clique with $\chi(H) = 3$. Then the two following assertions hold:*

1. *$|B(H)| < n^3$;*
2. *any edge $e \in E$ intersects the set $B(H)$.*

Proof. We start by proving the first assertion. Fix an $H = (V, E)$. Let $B = B(H)$. We know that $\sum_{v \in V} \deg v = n|E|$. Furthermore,

$$\sum_{v \in V} \deg v \geqslant \sum_{v \in B} \deg v > \frac{|B||E|}{n^2}.$$

Thus, $\frac{|B||E|}{n^2} < n|E|$, which means that actually $|B| < n^3$.

To prove the second assertion fix an arbitrary edge e. Since H is a clique, any $f \in E$ intersects e. Therefore, $\sum_{v \in e} \deg v \geqslant |E|$. By pigeon-hole principle, there is a vertex $v \in e$ with $\deg v \geqslant \frac{|E|}{n} \geqslant \frac{|E|}{n^2}$. So $v \in B$, and the proof is complete.

Proposition 3. *Let $H = (V, E)$ be an n-uniform clique with $\chi(H) = 3$. Let $t \in \{1, \ldots, n\}$ and suppose there is an edge $e \in E$ that intersects the set $B = B(H)$ by at most t vertices. Then there is a vertex $v \in V$ with $\deg v \geqslant \frac{|E|}{t+1}$.*

Proof. Fix a hypergraph $H = (V, E)$ and an $e \in E$ with $|e \cap B| \leqslant t$. Put $f = e \cap B$ and $a = |f| \leqslant t$. We know that for any $v \in f$, one has $\deg v > \frac{|E|}{n^2}$. We also know that for any $v \in (e \setminus f)$, one has $\deg v \leqslant \frac{|E|}{n^2}$. Finally, we know that H is a clique. Consequently,

$$\sum_{v \in f} \deg v = \sum_{v \in e} \deg v - \sum_{v \in (e \setminus f)} \deg v \geqslant |E| - (n - a)\frac{|E|}{n^2}.$$

By pigeon-hole principle, there is a vertex $v \in f$ with

$$\deg v \geqslant \frac{|E| - (n - a)\frac{|E|}{n^2}}{a}.$$

The right-hand side of the above inequality decreases in $a \leqslant t$, so that anyway

$$\deg v \geqslant \frac{|E| - (n - t)\frac{|E|}{n^2}}{t} = |E| \cdot \frac{n^2 - n + t}{n^2 t} \geqslant \frac{|E|}{t + 1},$$

where the last inequality is true, since $t \in \{1, \ldots, n\}$. Proposition 3 is proved.

Proposition 4. *Let $H = (V, E)$ be an n-uniform clique with $\chi(H) = 3$. Let $t \in \{1, \ldots, n\}$. Then either $|E| \leqslant tn^{n-1}$, or for any $e \in E$, we have $|e \cap B(H)| \geqslant t$.*

Proof. Fix an $H=(V,E)$ with $B(H)=B$. Assume that $|E|>tn^{n-1}$ and there exists an $e\in E$ such that $|e\cap B|\leqslant t-1$. By Proposition 3 we can find a vertex v with $\deg v\geqslant \frac{|E|}{t}>n^{n-1}$, which is in conflict with Proposition 1. Thus, our assumption is false, and the proof is complete.

Proposition 5. *Let $H=(V,E)$ be an n-uniform clique with $\chi(H)=3$ and $|E|>n^{n-\frac{1}{2}}$. Suppose that $n\geqslant 100$. Then there exist edges $e,f\in E$ such that $[\sqrt{n}]\leqslant|e\cap f|\leqslant n-[\sqrt{n}]$.*

Proof. Fix an $H=(V,E)$ with $B(H)=B$. Put $t=[\sqrt{n}]$. Since $tn^{n-1}\leqslant n^{n-\frac{1}{2}}<|E|$, Proposition 4 tells us that for any $e\in E$, we have $|e\cap B|\geqslant t$.

Consider the family $\mathcal{B}_t=C_B^t$ consisting of all the t-element subsets of the set B. By the first assertion of Proposition 2, we have

$$|\mathcal{B}_t|=C_{|B|}^t\leqslant|B|^t<n^{3t}.$$

Also we know that any $e\in E$ must contain a set $T\in\mathcal{B}_t$, since $|e\cap B|\geqslant t$.

At the same time, $|E|>n^{n-\frac{1}{2}}>n^{5t}$ as $n\geqslant 100$. Thus, taking into account the notation $E(W)$ from the statement of Proposition 1, we see that there exists a set $T\in\mathcal{B}_t$ such that $|E(T)|>n^{5t}/n^{3t}=n^{2t}$.

Clearly for any $e,f\in E(T)$, we have $T\subseteq(e\cap f)$, so that $|e\cap f|\geqslant t=[\sqrt{n}]$. If there exist $e,f\in E(T)$ with $|e\cap f|\leqslant n-[\sqrt{n}]=n-t$, then the proposition is proved. Otherwise, every two edges from $E(T)$ intersect by at least $n-t+1$ vertices.

Take an arbitrary $A\in E(T)$. Put $s=n-t+1$ and consider the family $\mathcal{A}_s=C_A^s$ consisting of all the s-element subsets of the set A. We know that simultaneously **a)** $|E(T)|>n^{2t}$; **b)** any $e\in E(T)$ contains a set $S\in\mathcal{A}_s$ (since $|e\cap A|\geqslant s$); **c)** $|\mathcal{A}_s|=C_{|A|}^s=C_n^s=C_n^{t-1}<n^t$. Therefore, there is a set $S\in\mathcal{A}_s$ such that $|E(S)|>n^t$. Since $|S|=s$, by Proposition 1, we have $|E(S)|\leqslant n^{n-s}=n^{t-1}$, which is a contradiction. Proposition 5 is proved.

Remark 1. Note that the proof of Proposition 5 can be easily extended to support the following assertion: *Let $H=(V,E)$ be an n-uniform clique with $\chi(H)=3$. Let an $F\subseteq E$ be such that $|F|>n^{n-\frac{1}{2}}$. Suppose that $n\geqslant 100$. Then there exist edges $f_1,f_2\in F$ such that $[\sqrt{n}]\leqslant|f_1\cap f_2|\leqslant n-[\sqrt{n}]$.* Note also that a hypergraph $H'=(V,F)$ does not necessarily have chromatic number 3. It can be bipartite as well.

Proposition 6. *Let $n\in\mathbb{N}$, $t\in\{1,\dots,n\}$,*

$$t'=\min\left\{t,4\sqrt{n}\ln n\right\},\quad N(t)=(t+1)\left(n-\frac{\sqrt{n}}{4}\right)^{t'-1}n^{n-t'}.$$

Then $N(t)=O\left(n^{n-\frac{1}{2}}\ln n\right)$.

Proof. First, assume that $t\leqslant 4\sqrt{n}\ln n$. Then

$$N(t)\leqslant(t+1)\cdot n^{t-1}\cdot n^{n-t}=(t+1)n^{n-1}=O\left(n^{n-\frac{1}{2}}\ln n\right),$$

and we are done. Now, assume that $t>4\sqrt{n}\ln n$. In this case,

$$N(t)\leqslant(n+1)\left(n-\frac{\sqrt{n}}{4}\right)^{t'-1}n^{n-t'}=(n+1)\cdot n^{n-1}\left(1-\frac{1}{4\sqrt{n}}\right)^{4\sqrt{n}\ln n-1}=O\left(n^{n-1}\right),$$

and we are done again. Proposition 6 is proved.

Remark 2. Note that we may write, say, $N(t)\leqslant 10n^{n-\frac{1}{2}}\ln n$ for $n\geqslant n_0$ and all t.

Completion of the proof of the theorem. Fix an n-uniform clique $H=(V,E)$ with $\chi(H)=3$ and $n\geqslant\max\{n_0,10000\}$. We shall prove that $|E|\leqslant 10n^{n-\frac{1}{2}}\ln n$. This will be enough to complete the proof of Theorem 2.

Let

$$T=\max\{t:\ \forall\, e\in E\ \ |e\cap B|\geqslant t\}.$$

By the second assertion of Proposition 2, $T \in \{1, \ldots, n\}$.

Define T' in the same way as t' was defined by t in Proposition 6. Since $n \geqslant 10000$, we have $T' < n$, and thus $T' \in \{1, \ldots, n-1\}$. Also, since $n \geqslant n_0$, we have $N(T) \leqslant 10n^{n-\frac{1}{2}} \ln n$ (see Remark 2).

Assume that $|E| > 10n^{n-\frac{1}{2}} \ln n$. So we automatically assume that $|E| > N(T)$. By the definition of the value T, there exists an edge $e \in E$ that intersects B by at most T vertices. Hence, by Proposition 3 there is a vertex $v \in V$ with

$$\deg\, v \geqslant \frac{|E|}{T+1} > \frac{N(T)}{T+1} = \left(n - \frac{\sqrt{n}}{4}\right)^{T'-1} n^{n-T'}.$$

Put $I = \{v\}$, $i = 1$. Then

$$|E(I)| = \deg\, v > \left(n - \frac{\sqrt{n}}{4}\right)^{T'-1} n^{n-T'} = \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i} n^{n-T'}. \tag{1}$$

If $T' = 1$, then inequality (1) contradicts Proposition 1. Therefore, assume that $T' \in \{2, \ldots, n-1\}$.

Let inequality (1) serve as the base for an inductive procedure with $\leqslant T'$ steps. So assume that we have already found a set $I \subset V$ with $|I| = i \in \{1, \ldots, T'-1\}$ (do not forget that $T' \geqslant 2$) and

$$|E(I)| > \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i} n^{n-T'}.$$

We shall prove that either we can take an $a \in (V \setminus I)$ such that

$$|E(I \cup \{a\})| > \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-1} n^{n-T'}, \tag{2}$$

or we can take $a, b \in (V \setminus I)$ such that

$$|E(I \cup \{a, b\})| > \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-2} n^{n-T'}. \tag{3}$$

(Here for all i, $T' - i - 2 \geqslant -1$ and $i + 2 \leqslant T' + 1 \leqslant n$, so that the choice of the parameters is correct.)

Indeed, to prove (2) or (3) let

$$E_I = \{e \in E : \ e \cap I \neq \emptyset\}.$$

By Proposition 1 $|E_I| \leqslant in^{n-1}$. Hence, $|E_I| < T'n^{n-1} \leqslant 4n^{n-\frac{1}{2}} \ln n$. Putting

$$E^I = \{e \in E : \ e \cap I = \emptyset\}$$

we immediately get the estimate

$$|E^I| = |E| - |E_I| > 10n^{n-\frac{1}{2}} \ln n - 4n^{n-\frac{1}{2}} \ln n = 6n^{n-\frac{1}{2}} \ln n > n^{n-\frac{1}{2}}.$$

Since $n > 100$, Remark 1 tells us that there exist $f_1, f_2 \in E^I$ with $[\sqrt{n}] \leqslant |f_1 \cap f_2| \leqslant n - [\sqrt{n}]$.

Since H is a clique, any edge e from $E(I)$ intersects both f_1 and f_2. So either e goes through a vertex $v \in (f_1 \cap f_2)$ or it contains a vertex $v_1 \in (f_1 \setminus f_2)$ and a vertex $v_2 \in (f_2 \setminus f_1)$. Formally, we may write down the equality

$$E(I) = \left(\bigcup_{v \in (f_1 \cap f_2)} E_v(I)\right) \bigcup \left(\bigcup_{v_1 \in (f_1 \setminus f_2)} \bigcup_{v_2 \in (f_2 \setminus f_1)} E_{v_1, v_2}(I)\right),$$

where

$$E_v(I) = \{e \in E(I) : \ v \in e\}, \quad E_{v_1, v_2}(I) = \{e \in E(I) : \ v_1, v_2 \in e\}.$$

Of course

$$|E(I)| \leqslant \sum_{v \in (f_1 \cap f_2)} |E_v(I)| + \sum_{v_1 \in (f_1 \setminus f_2)} \sum_{v_2 \in (f_2 \setminus f_1)} |E_{v_1, v_2}(I)|.$$

If a summand in the first sum is greater than $\left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-1} n^{n-T'}$, then (2) is shown: indeed, the corresponding v is contained in this many edges $e \in E(I)$ that already contain I. If a summand in the second sum is greater than $\left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-2} n^{n-T'}$, then (3) is shown in turn. So suppose that there are no such summands. In this case, putting $k = |f_1 \cap f_2|$ we have

$$|E(I)| \leqslant k \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-1} n^{n-T'} + (n-k)^2 \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-2} n^{n-T'}.$$

On the other hand, $|E(I)| > \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i} n^{n-T'}$. Having proved that

$$\left(n - \frac{\sqrt{n}}{4}\right)^{T'-i} n^{n-T'} > k \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-1} n^{n-T'} + (n-k)^2 \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-2} n^{n-T'}$$

for any $k \in [[\sqrt{n}], n - [\sqrt{n}]]$, we would get a contradiction which would complete the proof of (2) or (3).

The needed inequality is equivalent to

$$\left(n - \frac{\sqrt{n}}{4}\right)^2 > k \left(n - \frac{\sqrt{n}}{4}\right) + (n-k)^2,$$

which can be proved by standard analytic calculations.

Thus, (2) or (3) take place. So after $\leqslant T'$ steps of the inductive procedure, we get either a set I of cardinality T' such that $|E(I)| > n^{n-T'}$ or a set I of cardinality $T'+1$ such that $|E(I)| > \left(n - \frac{\sqrt{n}}{4}\right)^{-1} n^{n-T'} > n^{n-T'-1}$. Both estimates are in conflict with Proposition 1. Consequently, our initial assumption $|E| > 10 n^{n-\frac{1}{2}} \ln n$ is false, and Theorem 2 is proved.

References

[1] P. Erdős, L. Lovász, *Problems and results on 3-chromatic hypergraphs and some related questions*, Infinite and Finite Sets, Colloquia Mathematica Societatis Janos Bolyai, North Holland, **10** (1975), 609 - 627.

[2] T. Jensen, B. Toft, *Graph coloring problems*, New York: Wiley Interscience, 1995.

[3] L. Lovász, *On minimax theorems of combinatorics*, Math. Lapok **26** (1975), 209 - 264 (in Hungarian).

[4] P. Frankl, K. Ota, N. Tokushige, *Covers in uniform intersecting families and a counterexample to a conjecture of Lovász*, Journal of Combin. Th., Ser. A **74** (1996), 33 - 42.