POINCARÉ INEQUALITIES AND RIGIDITY FOR ACTIONS ON BANACH SPACES

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Abstract. The aim of this paper is to extend the framework of the spectral method for proving property (T) to the class of reflexive Banach spaces and present conditions implying that every affine isometric action of a given group G on a reflexive Banach space X has a fixed point. This last property is a strong version of Kazhdan's property (T) and is equivalent to the fact that $H^1(G,\pi) = 0$ for every isometric representation π of G on X. The condition is expressed in terms of certain Poincaré constants and we provide examples of groups, which satisfy such conditions and for which $H^1(G,\pi)$ vanishes for every isometric representation π on an L_p space for some $p > 2$. We also obtain quantitative estimates for vanishing of 1-cohomology with coefficients in uniformly bounded representations on a Hilbert space. Our results also have several other applications. In particular, we give lower bounds on the conformal dimension of the boundary of a hyperbolic group in the Gromov density model.

1. Introduction

Kazhdan's property (T) is a powerful rigidity property of groups with numerous applications and several characterizations. In this article we focus on the following description of property (T): *a group G has property (T) if and only if every affine isometric action of G on the Hilbert space has a fixed point*. This characterization can be rephrased as the cohomological condition $H^1(G,\pi) = 0$ for every unitary representation π of G. A generalization of property (T) to other Banach spaces is then natural. We are interested in conditions implying that every affine isometric action of a given group on a reflexive Banach space has a fixed point. Such rigidity properties for actions on Banach spaces, as well as other generalizations of property (T), and their applications, were studied earlier in [2, 8, 12, 20].

One very successful method of proving property (T) is through spectral conditions on links of vertices of complexes acted upon by a group. Various variations of such conditions were studied in [3, 9, 10, 14, 16, 18, 30, 32, 33, 35, 36] in the context of Hilbert spaces and non-positively curved spaces. The main idea is as follows: given a group G acting on a 2-dimensional simplicial complex, one considers the link of a vertex. This link is a finite graph. If for every vertex, the first positive eigenvalue of the discrete Laplacian is strictly larger than $1/2$, then G has property (T).

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The main purpose of this work is to extend the framework of spectral method, and some of the rigidity results, beyond Hilbert spaces. Our main result provides such a framework for the class of reflexive Banach spaces. The difficulty in doing this lies in the fact, that in the Hilbert space case the spectral method relies heavily on duality, in particular self-duality of Hilbert spaces. When passing to other Banach spaces, dual spaces of certain Banach spaces and of their subspaces have to be identified, and this is often a difficult task. We show, that when the representation is isometric such computations are possible and we can use duality effectively.

We focus on certain link graphs constructed using generating sets of a group, as in [36]. For a finite, symmetric generating set S not containing the identity element the vertices of the link graph $\mathcal{L}(S)$ are the elements of S; generators s and t are connected by an edge if $s^{-1}t$ is a generator. We will also assume that the graph is equipped with a weight ω on the edges.

Let $\kappa_p(S,X)$ be the optimal constant in the p-Poincaré inequality for the link graph $\mathcal{L}(S)$ of G in the norm of X,

$$\sum_{s\in S} \|f(x) - Qf\|_X^p \deg_\omega(s) \leq \kappa_p^p \sum_{s\sim t} \|f(x) - f(y)\|_X^p \omega(x,y),$$

where Qf is the mean value of f. When $X = L_2$, the constant $\kappa_2(S, L_2) = \kappa_2(S,\mathbb{R})$ can be expressed in terms of of the first eigenvalue of the discrete Laplacian.

Our main result shows that sufficiently small constants in Poincaré inequalities for the graph $\mathcal{L}(S)$ imply the required cohomological vanishing. Given a number $1 < p < \infty$ we denote by p^* its adjoint index satisfying $\frac{1}{p} + \frac{1}{p^*} = 1$.

Theorem 1. *Let X be a reflexive Banach space and let G be a group generated by a finite, symmetric set S, not containing the identity element. If the link graph $\mathcal{L}(S)$ is connected and for some $1 < p < \infty$ the associated Poincaré constants satisfy*

$$\max\left\{2^{-\frac{1}{p}}\kappa_p(S,X), 2^{-\frac{1}{p^*}}\kappa_{p^*}(S,X^*)\right\} < 1,$$

then

$$H^1(G,\pi) = 0$$

for every isometric representation π of G on X.

Clearly, by reflexivity, the same conclusion holds for actions on X^*. Interestingly, the roles of the the two constants in the proof of the above theorem are not symmetric.

We apply Theorem 1 to L_p spaces. The interesting case is $p > 2$. Indeed, when $1 < p \leq 2$, affine isometric actions exhibit the same behavior as for the Hilbert space. G has property (T) if and only if any affine action on an L_p space for $1 < p \leq 2$ has a fixed point [2]. Also, G admits a metrically proper affine isometric action on the Hilbert space (i.e., is a-T-menable) if and only if it admits such an action on any $L_p[0,1]$ for $1 < p \leq 2$ [26] (see corrected version [27]). This last property is a strong negation of the existence of a fixed point.

Fixed point properties for groups acting on L_p spaces for $p > 2$ are difficult to prove and only a handful of results are known:

(1) certain higher rank algebraic groups and their lattices have fixed points for every affine isometric action on L_p-spaces for all $p > 1$ [2];
(2) in [22] it was proved that $\mathrm{SL}_n(\mathbb{Z}[x_1, \ldots, x_k])$ has fixed points for every affine isometric on L_p for every $p > 1$ and $n \geq 4$;
(3) Naor and Silberman [23] showed that Gromov's random groups, containing (in a certain weak sense) expanders in their Cayley graphs, have a fixed point for affine isometric actions on any L_p for $p > 1$;
(4) a general argument due to Fisher and Margulis (see the proof in [2]) shows, that for every property (T) group G there exists a constant $\varepsilon = \varepsilon(G) > 0$ such that any affine isometric action on L_p for $p \in [2, 2 + \varepsilon)$, has a fixed point. However, their argument does not give any control over ε.

On the other hand there are also groups which have property (T) but act without fixed points on certain L_p spaces. One example is furnished by the group $\mathrm{Sp}(n, 1)$, which has property (T) but has non-vanishing L_p-cohomology for $p > 4n + 2$, by a result of Pansu [29]. It also known that there exist hyperbolic groups which have property (T). Nevertheless, Bourdon and Pajot [5] showed that for every hyperbolic group G and sufficiently large $p > 2$ there is an affine isometric action on $\ell_p(G)$, whose linear part is the regular representation and which does not have a fixed point. Moreover, Yu [34] showed that every hyperbolic group admits a proper, affine isometric action on $\ell_p(G \times G)$ for all sufficiently large $p > 2$.

The techniques we use to establish fixed point properties are different that the ones used previously for general Banach spaces. In particular, the expected outcome is also slightly different, as our methods are not expected to give fixed points on L_p for all $p > 1$. One reason is that the p-Poincaré constants usually increase above $2^{1/p}$ as p grows to infinity. The second reason is that the main result applies to certain hyperbolic groups, which, as remarked earlier, do not act without fixed points on some L_p-spaces. Using our approach we obtain the appropriate vanishing of cohomology for an interval $[2, 2 + c)$, where the value of c depends on the group and can be estimated explicitly.

To apply Theorem 1 we need to estimate p-Poincaré constants for $p > 2$. Even in classical settings, such as convex domains in $\mathbb{R}^n$, estimates exist but exact values of p-Poincaré constants are not known, except a few special cases. The situation is even worse for finite graphs, where very few estimates are known for cases other than $p = 1, 2$. Here we consider the family of $\widetilde{A}_2$-groups, indexed by powers of primes. These groups were introduced and studied in [6, 7]. For every q the group G_q has a generating sets whose link graph is the incidence graph of the finite projective plane over the field $\mathbb{F}_q$. Spectra of such graphs were computed in [11] and give, in particular, the exact value of the Poincaré constant $\kappa_2(S, \mathbb{R})$. We use this fact to estimate $\kappa_p(S, L_p)$ for these graphs, which allows to obtain for each q a number c_q such that any affine isometric action of G on any L_p has a fixed point for $p \in [2, 2 + c_q)$. The explicit estimates of c_q are given in Theorem 17.

As mentioned earlier, our results also apply to some hyperbolic groups, more precisely, to random groups in the Gromov density model with densities $1/3 < d < 1/2$. These groups are hyperbolic and have Kazhdan's property (T) with probability

1 [19, 36]. We give lower bounds on p for which fixed points exists for all isometric actions on any L_p-space. An interesting connection with the conformal dimension arises due to the work of Bourdon and Pajot [5] and allows us to give a lower bound on the conformal dimension of a boundary of a hyperbolic group, using a certain associated link graph. Details are discussed in Section 6.

Our methods also apply to certain affine actions, whose linear part is a uniformly bounded representation on a Hilbert space. More precisely, we show that $H^1(G,\pi) = 0$, whenever π is a uniformly bounded representation with norms of all operators bounded by a certain constant, which depends on the group but is close to $\sqrt{2}$ in many cases, see Theorems 21 and 24.We remark that Y. Shalom proved that there exists a uniformly bounded representation π of $\mathrm{Sp}(n,1)$ on a Hilbert space, such that $H^1(\mathrm{Sp}(n,1),\pi) \neq 0$.

Finally, we present other applications. We improve the differentiability class of diffeomorphic actions on the circle in the rigidity theorem in [24, 25] and estimate eigenvalues of the discrete p-Laplacian on finite quotients of groups using Kazhdan-type constants.

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Contents

2. Actions on Banach spaces

2.1. **Generating sets and link graphs.** Let G denote a discrete group generated by a finite symmetric set $S = S^{-1}$. Let $\mathcal{L}(S)$ denote the following graph, called the link graph of S. The vertices are given by $\mathcal{V} = S$. Two vertices $s, t \in S$ are connected by an edge, denoted $s \sim t$, if and only if $s^{-1}t \in S$ and $t^{-1}s \in S$.

The set E is defined as follows:

$$E = \left\{(s,t) \in S \times S \ : \ s^{-1}t \in S\right\}.$$

Note that E can be viewed as the set of oriented edges and in E every edge is counted twice.

A weight on $\mathcal{L}(S)$ consists of a function $\omega : E \to (0,\infty)$ such that $\omega(s,t) = \omega(t,s)$, for every $s,t \in S$. Given a weight on the link graph the associated degree of a vertex $r \in S$ is defined to be

$$\deg_\omega(s) = \sum_{t,\ t\sim s} \omega(t,s).$$

A weight ω on a link graph $\mathcal{L}(S)$ is admissible if it satisfies

(1) $\deg_\omega(s) = \deg_\omega(s^{-1})$, and
(2) $\deg_\omega(r) = \sum_{(s,t):\ s^{-1}t=r} \omega(s,t)$,

for every $r, s, t \in S$.

Note that

$$\sum_{s\in S} \deg_\omega(s) = \omega(E).$$

Throughout the article we consider only admissible weights on link graphs of generating sets.

2.2. **Isometric representations and associated Banach spaces.** Let X be a Banach space equipped with a norm $\|\cdot\|_X$. We assume throughout that X is reflexive and that $\pi : G \to B(X)$ is a representation of G into the bounded operators on X. Let X^* denote the continuous dual of X with its standard norm. X^* is naturally equipped with the adjoint representation of G, $\overline{\pi} : G \to B(X^*)$ given by

$$\overline{\pi}_g = \pi^*_{g^{-1}}.$$

Throughout we fix $1 < p < \infty$. This p is later chosen depending on the context. We denote by p^* the adjoint index satisfying $\frac{1}{p} + \frac{1}{p^*} = 1$ and by L_p the space $L_p(\mu)$ for any measure μ (our results will apply with no assumptions on the measure). We also use $\simeq$ to denote an isomorphism and $\cong$ to denote an isometric isomorphism of Banach spaces.

Define the Banach space $C^{(0,p)}(G,X)$ to be the linear space X with the norm

$$\|v\|_{(0,p)} = \omega(E)^{\frac{1}{p}} \|v\|_X.$$

Let $\langle\cdot,\cdot\rangle_X$ denote the natural pairing between X and X^*. The pairing between $C^{(0,p)}(G,X)$ and $C^{(0,p^*)}(G,X^*)$ is given by

$$\langle v, w\rangle_0 = \omega(E)\,\langle v,w\rangle_X.$$

Then $C^{(0,p^*)}(G,X^*)$ is the dual space of $C^{(0,p)}(G,X)$.

We define $C^{(1,p)}(G,X)$ to be the finite direct sum $\bigoplus_{s\in S} X$ with a norm given by

$$\|f\|_{(1,p)} = \left(\sum_{s\in S} \|f(s)\|_X^p \deg_\omega(s)\right)^{\frac{1}{p}}.$$

The dual of $C^{(1,p)}(G,X)$ is $C^{(1,p^*)}(G,X^*)$, via the pairing

$$\langle f,\phi\rangle_1 = \sum_{s\in S} \langle f(s),\phi(s)\rangle_X \deg_\omega(s),$$

for $f\in C^{(1,p)}(G,X)$ and $\phi\in C^{(1,p)}(G,X)^*$.

Consider the following subspaces of $C^{(1,p)}(G,X)$:

$$C_+^{(1,p)}(G,X) = \left\{ f\in C^{(1,p)}(G,X) :\ \ f(s^{-1}) = \pi_{s^{-1}} f(s)\right\},$$

and

$$C_-^{(1,p)}(G,X) = \left\{ f\in C^{(1,p)}(G,X) :\ \ f(s^{-1}) = -\pi_{s^{-1}} f(s)\right\}.$$

Lemma 2. *For any* $1 < p < \infty$ *we have* $C^{(1,p)}(G,X) = C_+^{(1,p)}(G,X)\oplus C_-^{(1,p)}(G,X)$.

Proof. We define two operators: $P_+ : C^{(1,p)}(G,X)\to C_+^{(1,p)}(G,X)$,

$$P_+ f(s) = \frac{f(s)+\pi_s f(s^{-1})}{2},$$

and $P_- : C^{(1,p)}(G,X)\to C_-^{(1,p)}(G,X)$,

$$P_- f(s) = \frac{f(s)-\pi_s f(s^{-1})}{2}.$$

Clearly $P_+ + P_- = \mathrm{Id}$. Additionally, $C_+^{(1,p)}(G,X) = \ker P_- = \operatorname{im} P_+$ and $C_-^{(1,p)}(G,X) = \ker P_+ = \operatorname{im} P_-$. Indeed, we have

$$\pi_{s^{-1}}(P_+ f(s)) = \frac{\pi_{s^{-1}} f(s) + f(s^{-1})}{2} = P_+ f(s^{-1}).$$

Also, P_+ restricted to $C_+^{(1,p)}(G,X)$ is the identity operator. We also estimate

$$\begin{aligned}
\|P_+ f\|_{(1,p)}^p &= \frac{1}{2}\sum_{s\in S} \|f(s)+\pi_s f(s^{-1})\|_X^p \deg(s)\\
&\le \frac{2^{p-1}}{2}\sum_{s\in S}\left(\|f(s)\|_X^p + \|f(s^{-1})\|_X^p\right)\deg(s)\\
&\le 2^{p-1}\sum_{s\in S}\|f(s)\|_X^p \deg(s)\\
&\le 2^{p-1}\|f\|_{(1,p)}^p
\end{aligned}$$

A similar argument for P_- shows that both P_- and P_+ are bounded projections, which gives the claimed decomposition. □

We now need to analyze the structure of $C^{(1,p)}(G,X)$ in relation to the one of $C^{(1,p^*)}(G,X^*)$.

Convention. We will denote by a bar versions of the same constants and operators for the dual X^*. All the following facts hold in both cases unless noted otherwise.

2.3. **Duality for $C_-^{(1,p)}(G,X)$.** The dual of $C^{(1,p)}(G,X)$ is $C^{(1,p^*)}(G,X^*)$. Let $\overline{P}_+ : C^{(1,p^*)}(G,X^*) \to C_+^{(1,p^*)}(G,X^*)$, $\overline{P}_- : C^{(1,p^*)}(G,X^*) \to C_-^{(1,p^*)}(G,X^*)$ denote the projections as above.

Lemma 3. *We have $\overline{P}_+ = P_+^*$ and $\overline{P}_- = P_-^*$.*

Proof. Let $f \in C^{(1,p)}(G,X)$ and $\phi \in C^{(1,p^*)}(G,X^*)$. Then

$$\begin{aligned}
\langle P_- f, \phi\rangle_1 &= \frac{1}{2}\sum_{s\in S}\langle f(s) - \pi_s f(s^{-1}), \phi(s)\rangle_X \deg_\omega(s)\\
&= \frac{1}{2}\sum_{s\in S}\Big(\langle f(s),\phi(s)\rangle_X - \langle f(s^{-1}), \overline{\pi}_{s^{-1}}\phi(s)\rangle_X\Big)\deg_\omega(s)\\
&= \frac{1}{2}\sum_{s\in S}\langle f(s), \phi(s) - \overline{\pi}_s\phi(s^{-1})\rangle_X \deg_\omega(s)\\
&= \langle f, \overline{P}_-\phi\rangle_1.
\end{aligned}$$

The proof is similar for P_+. □

Lemma 4. *We have the following isomorphisms: $C_-^{(1,p)}(G,X)^* \simeq C_-^{(1,p^*)}(G,X^*)$ and $C_+^{(1,p)}(G,X)^* \simeq C_+^{(1,p^*)}(G,X^*)$.*

Proof. Consider $f \in C_-^{(1,p)}(G,X)$ and let $\phi \in C^{(1,p^*)}(G,X^*)$. Then

$$\begin{aligned}
\langle f,\phi\rangle_1 &= \sum_{s\in S}\langle f(s),\phi(s)\rangle_X \deg_\omega(s)\\
&= \sum_{s\in S}\langle -\pi_s f(s^{-1}),\phi(s)\rangle_X \deg_\omega(s)\\
&= \sum_{s\in S}\langle f(s^{-1}), -\overline{\pi}_{s^{-1}}\phi(s)\rangle_X \deg_\omega(s)\\
&= \sum_{s\in S}\langle f(s), -\overline{\pi}_s\phi(s^{-1})\rangle_X \deg_\omega(s).
\end{aligned}$$

Therefore,

$$2\langle f, \overline{P}_+\phi\rangle_1 = \sum_{s\in S}\langle f(s), \phi(s) + \overline{\pi}_s\phi(s^{-1})\rangle_X \deg_\omega(s) = 0,$$

which shows that $C_+^{(1,p^*)}(G,X^*)$ annihilates $C_-^{(1,p)}(G,X)$.

Conversely, if $\phi \in C^{(1,p^*)}(G,X^*)$ annihilates $C_-^{(1,p)}(G,X)$ then

$$\langle P_- f,\phi\rangle_1 = \langle f, \overline{P}_-\phi\rangle = 0$$

for every $f \in C^{(1,p)}(G,X)$. Consequently, $\overline{P}_{-}\phi = 0$ and $\phi = \overline{P}_{+}\phi$, which means it belongs to $C_{+}^{(1,p^*)}(G,X^*)$. Thus,

$$C_{-}^{(1,p)}(G,X)^* \cong C^{(1,p)}(G,X)\Big/C_{+}^{(1,p^*)}(G,X^*) \simeq C_{-}^{(1,p^*)}(G,X^*).$$

Other cases are proved similarly. □

In order to identify the dual of $C_{-}^{(1,p)}(G,X)$ an isomorphism is not sufficient, we need an isometric isomorphism instead. For a representation π, $C^{(1,p)}(G,X)^*$ is in general not isometrically isomorphic to $C^{(1,p^*)}(G,X^*)$. However, it turns out that this additional property holds when the representation π is isometric.

Theorem 5. *Assume that π_s is an isometry for every $s \in S$. Then we have the following isometric isomorphisms: $C_{-}^{(1,p)}(G,X)^* \cong C_{+}^{(1,p^*)}(G,X^*)$ and $C_{+}^{(1,p)}(G,X)^* \cong C_{+}^{(1,p^*)}(G,X^*)$.*

Proof. Assume that $\phi \in C_{-}^{(1,p^*)}(G,X^*)$ and consider $C^{(1,p^*)}(G,X)\Big/C_{+}^{(1,p^*)}(G,X^*)$, which consists of cosets $[\phi] = C_{+}^{(1,p^*)}(G,X^*) + \phi$, for $\phi \in C^{(1,p)}(G,X)$. We need to show that for each such coset N, $\inf\{\|\phi\| : N = [\phi]\}$ is attained when $\phi \in C_{-}^{(1,p^*)}(G,X^*)$.

For $\phi \in C_{-}^{(1,p^*)}(G,X^*)$ and $\psi \in C_{+}^{(1,p^*)}(G,X^*)$ we have

$$\begin{aligned}
\|\phi+\psi\|_{(1,p^*)}^{p^*} &= \sum_{s\in S} \|\phi(s)+\psi(s)\|_X^{p^*} \deg_\omega(s)\\
&= \sum_{s\in S} \|-\pi_s\phi(s^{-1})+\pi_s\psi(s^{-1})\|_X^{p^*} \deg_\omega(s)\\
&= \sum_{s\in S} \|\phi(s)-\psi(s)\|_X^{p^*} \deg_\omega(s)\\
&= \|\phi-\psi\|_{(1,p^*)}^{p^*}.
\end{aligned}$$

Now consider the coset $[\phi]$ for $\phi \in C_{-}^{(1,p^*)}(G,X^*)$ and consider another element, $\zeta \in C^{(1,p^*)}(G,X^*)$ such that $\zeta - \phi \in C_{+}^{(1,p^*)}(G,X^*)$, so that $\zeta = \phi + \psi$, for some $\psi \in C_{+}^{(1,p^*)}(G,X^*)$. This implies

$$\|\phi\|_{(1,p^*)} \leq \frac{\|\phi-\psi\|_{(1,p^*)} + \|\phi+\psi\|_{(1,p^*)}}{2} = \|\zeta\|_{(1,p^*)},$$

which proves the claim. □

This last statement allows us to identify $C_{-}^{(1,p)}(G,X)^*$ with $C_{-}^{(1,p^*)}(G,X^*)$ for isometric representations and is crucial in the proof.

2.4. **The operator δ.** We define the operator $\delta : C^{(0,p)}(G,X) \to C_{-}^{(1,p)}(G,X)$ by the formula

$$\delta v(s) = v - \pi_s v.$$

Theorem 5 allows to express the adjoint of δ in a way which is convenient for calculations. We have the following explicit formula for δ^*.

Lemma 6. *The operator* $\delta^* : C_-^{(1,p^*)}(G, X^*) \to C^{(0,p^*)}(G, X^*)$ *is given by*

$$\delta^*\phi = 2\sum_{s\in S} \phi(s)\frac{\deg_\omega(s)}{\omega(E)}. \tag{1}$$

Proof.

$$\begin{aligned}
\langle \delta v, \phi\rangle_1 &= \sum_{s\in S} \langle v - \pi_s v, \phi(s)\rangle_X \deg_\omega(s) \\
&= \sum_{s\in S} \left(\langle v, \phi(s)\rangle_X - \langle v, \overline{\pi}_{s^{-1}}\phi(s)\rangle_X\right) \deg_\omega(s) \\
&= \sum_{s\in S} \left(\langle v, \phi(s), \rangle_X + \langle v, \phi(s^{-1})\rangle_X\right) \deg_\omega(s) \\
&= \left\langle v, 2\sum_{s\in S} \phi(s)\frac{\deg_\omega(s)}{\omega(E)}\right\rangle_0 .
\end{aligned}$$

□

It is now clear that δ^* admits a continuous extension to the space $C^{(1,p^*)}(G, X^*)$, defined by the right hand side of the formula (1).

2.5. **The operators D, L, and d.** We define the Banach space,

$$C^{(2,p)}(G, X) = \left\{ \eta \in \bigoplus_{(s,t)\in E} X \,:\, \eta(s,t) = -\eta(t,s) \right\},$$

equipped with the norm

$$\|\eta\|_{(2,p)} = \left(\sum_{(s,t)\in E} \|\eta(s,t)\|_X^p \omega(s,t) \right)^{\frac{1}{p}}.$$

We also define operators $D, L : C_-^{(1,p)}(G, X) \to C^{(2,p)}(G, X)$ by the formulas

$$Df(s,t) = f(t) - f(s),$$

$$Lf(s,t) = \pi_s f(s^{-1}t).$$

Then the operator d is defined by

$$d = L - D.$$

Similarly, define $\overline{D}$, $\overline{L}$ and $\overline{d}$ for the dual space.

Lemma 7. *The operator L is an isometry onto its image. Consequently, D is an isometry as well when restricted to* $\ker d$. *(The same claim holds for $\overline{L}$ and $\overline{D}$ restricted to* $\ker \overline{d}$*).*

Proof. By direct calculation,

$$\begin{aligned}\|Lf\|_{(2,p)}^p &= \sum_{(s,t)\in E} \|\pi_s f(s^{-1}t)\|_X^p \omega(s,t)\\ &= \sum_{s\in S} \|f(s)\|_X^p \deg_\omega(s)\\ &= \|f\|_{(1,p)}^p.\end{aligned}$$

□

The kernel of the operator $\overline{D}$ consists of the constant functions on S, which is a complemented subspace of $C^{(1,p)}(G,X)$. The projection onto this subspace is given by

$$\overline{M}\phi(s) = \sum_{s\in S} \phi(s) \frac{\deg_\omega(s)}{\omega(E)}.$$

Note that for $\phi \in C_-^{(1,p^*)}(G,X^*)$ we have

$$\overline{M}\phi(s) = \frac{1}{2}\delta^*\phi,$$

for every $s \in S$.

Lemma 8. *Let* $\phi \in C_-^{(1,p^*)}(G,X^*)$. *Then* $\|\overline{M}\phi\|_{(1,p^*)} = \dfrac{1}{2}\|\delta^*\phi\|_{(0,p^*)}$.

Proof. We have the following equalities:

$$\begin{aligned}\|\overline{M}\phi\|_{(1,p^*)}^{p^*} &= \sum_{s\in S} \left\|\frac{\delta^*\phi}{2}\right\|_X^{p^*} \deg_\omega(s)\\ &= \frac{1}{2^{p^*}}\|\delta^*\phi\|_X^{p^*}\left(\sum_{s\in S}\deg_\omega(s)\right)\\ &= \frac{1}{2^{p^*}}\|\delta^*\phi\|_0^{p^*}.\end{aligned}$$

□

2.6. **Sufficients conditions for vanishing of cohomology.** Given a group G, the 1-cocycles associated to π are functions $b : G \to X$ satisfying the cocycle condition,

$$b(gh) = \pi_g b(h) + b(g),$$

for every $g, h \in G$. The coboundaries are those cocycles which are of the form

$$b(g) = \pi_g v - v$$

for some $v \in X$ and all $g \in G$. The first cohomology of G with coefficients in π is defined to be $H^1(G,\pi) = \text{cocycles}/\text{coboundaries}$.

An affine action of G on X is defined as

$$A_g v = \pi_g v + b_g,$$

where π is called the linear part of the action and b is a cocycle. Vanishing of cohomology $H^1(G,\pi)$ is equivalent to existence of a fixed point for any affine action with linear part π. We refer to [4] for background on cohomology and affine actions.

The reader can easily verify the following lemma.

Lemma 9. $\operatorname{image}(\delta) \subseteq \ker d$.

This fact allows to formulate the following sufficient condition for the fixed point property for affine actions on X.

Proposition 10. *If the image of δ is equal to $\ker d$ then any affine isometric action with linear part π has a fixed point,*

Proof. Let $b : G \to X$ be a 1-cocycle for π and let b' denote the restriction of b to the generating set S. The cocycle condition implies that $b' \in C_-^{(1,p)}(G,X)$ and, furthermore, that $b' \in \ker d$. If δ is onto $\ker d$ then $b' = \delta v$ for some $v \in X$. Since b is trivial on the generators, we conclude that b is trivial. □

It is important to remark that the technical details here are slightly different than the one in [36], where the original condition in terms of almost invariant vectors is deduced, and one needs to use the Delorme-Guichardet theorem to obtain cohomological vanishing. The above argument allows to bypass the use of the Delorme-Guichardet theorem and obtain vanishing of cohomology directly.

Note that the image of δ is always properly contained in $C_-^{(1,p)}(G,X)$. By the open mapping theorem we also have the following

Corollary 11. *Assume π does not have invariant vectors. If δ is onto $\ker d$ then there is a constant K such that*

$$\sup_{s\in S} \|v - \pi_s v\|_X \geq K\|v\|_X$$

for every $v \in X$.

The constant K in the above statement can be viewed as a version of Kazhdan constant for isometric representations of G on X.

3. Poincaré inequalities associated to norms

Consider a weighted, finite graph $\Gamma = (\mathcal{V},\mathcal{E})$, a number $p \geq 1$ and a Banach space X. The p-Poincaré inequality for Γ and the norm of X is the inequality

$$\left(\sum_{x\in\mathcal{V}} \|f(x) - Qf\|_X^p \deg_\omega(x)\right)^{\frac{1}{p}} \leq \kappa_p \left(\sum_{x\sim y} \|f(x) - f(y)\|_X^p \omega(x,y)\right)^{\frac{1}{p}} \tag{2}$$

for all functions $f : \mathcal{V} \to X$, where $Qf = \frac{1}{2\omega(\mathcal{E})} \sum_{x\in\mathcal{V}} f(x) \deg_\omega(x)$. The operator defined by $\nabla f(x,y) = f(x) - f(y)$ whenever $x \sim y$. On a finite graph, the inequality (2) is always satisfied for some $\kappa_p > 0$.

Definition 12. *Let $\mathcal{L}(S)$ be a link graph of a generating set S, with weight ω. For a Banach space X and a number $1 < p < \infty$ we define the constant $\kappa_p(S, X)$ of $\mathcal{L}(S)$ by setting*

$$\kappa_p(S, X) = \inf \kappa_p,$$

where the infimum is taken over all κ_p, for which inequality (2) holds.

We will omit the reference to ω in the notation for κ.

3.0.1. *Hilbert spaces.* When $X = L_2$ is the Hilbert space this constant is related to the smallest positive eigenvalue λ_1 of the Laplacian on the graph as follows:

$$\kappa_2(S, L_2) = \sqrt{\lambda_1^{-1}},$$

since the latter can be defined via the variational expression and the Rayleigh quotient.

3.0.2. *L_p-spaces,* $1 \le p < \infty$. Let (Y, μ) be any measure space and for $X = \mathbb{R}$ consider a p-Poincaré inequality

$$\sum_{x \in \mathcal{V}} |f(x) - Qf|^p \deg_\omega(s) \le \kappa_p^p \sum_{x \sim y} |f(x) - f(y)|^p \omega(x, y) \tag{3}$$

on a finite graph. By integrating over Y with respect to μ we obtain

$$\sum_{x \in \mathcal{V}} \|f(x) - Qf\|_{L_p}^p \deg_\omega(x) \le \kappa_p^p \sum_{x \sim y} \|f(x) - f(y)\|_{L_p}^p \omega(x, y),$$

for any $f : \mathcal{V} \to L_p$. This gives

$$\|f - Qf\|_{(1,p)} \le \kappa_p \|\nabla f\|_{(2,p)},$$

so that $\kappa_p(S, L_p)$ is equal to $\kappa_p(S, \mathbb{R})$ in the inequality (3).

3.0.3. *Direct sums.* More generally, consider an ℓ_p-direct sum $X = \left(\bigoplus_{s \in S} X_i\right)_p$ of Banach spaces $\{X_i\}_{i \in I}$. Then a similar argument as above shows that $\kappa_p(S, X) \le \sup_{i \in I} \kappa_p(S, X_i)$.

3.0.4. *The case $p = \infty$.* Consider $s, s^{-1} \in S$ and choose $x \in X$ such that $\|x\|_X = 1$. Let d_S denote the path metric on $\mathcal{L}(S)$. Define $f : \Gamma \to \mathbb{R}$ by the formula

$$f(t) = \begin{cases} \left(1 - \dfrac{1}{d_S(s, t)}\right) x & \text{if } d_S(s, t) \le \dfrac{d_S(s, s^{-1})}{2} \\ \left(-1 + \dfrac{1}{d_S(s^{-1}, t)}\right) x & \text{if } d_S(s^{-1}, t) \le \dfrac{d_S(s, s^{-1})}{2} \\ 0 & \text{if } d_S(s, t) > \dfrac{d_S(s, s^{-1})}{2} \text{ and } d_S(s^{-1}, t) > \dfrac{d_S(s, s^{-1})}{2} \end{cases}$$

For such f we have $\|f\|_{(X,\infty)} = 1$ and $Qf = 0$, however $\|Df\|_{(X,\infty)} = \dfrac{1}{d_S(s, s^{-1})}$. Thus we have

$$\kappa_\infty(G, X) \ge \max_{s \in S} d_S(s, s^{-1})$$

and for sufficiently large S, the above Poincaré constant is at least 1. Additionally, for any $\varepsilon > 0$ there exists a sufficiently large $p < \infty$, such that the norms $\|f\|_{(1,p)}$ and $\|f\|_{(1,\infty)}$ are ε-close. For a sufficiently small $\varepsilon > 0$ and the corresponding p as above, we also have $2^{-\frac{1}{p}}\kappa_p(S,X) \geq 1$.

3.0.5. *Behavior under isomorphisms.* Let $T : X \to Y$ be an isomorphism of Banach spaces X, Y, satisfying $\|x\|_X \leq \|Tx\|_y \leq L\|x\|_X$ for every $x \in X$. Then $\kappa_p(G,X) \leq L\kappa_p(G,Y)$.

4. Vanishing of cohomology

4.1. **An inequality for κ_p and δ^*.** Note that since in E each edge of $\mathcal{L}(S)$ is counted twice we have $\|Df\|_{(2,p)} = 2^{\frac{1}{p}}\|\nabla f\|_{\ell_p(S,X)}$ and $Mf = Qf$. The following result describes the relation between Poincaré constants and the operator δ^*.

Theorem 13. *The inequality*

$$2\left(1 - 2^{-\frac{1}{p^*}}\kappa_{p^*}(S,X)\right)\|\phi\|_{(1,p^*)} \leq \|\delta^*\phi\|_{(0,p^*)} \tag{4}$$

holds for every $\phi \in \ker \overline{d}$.

Proof. Let $\phi : S \to X^*$. Then

$$\begin{aligned}
\kappa_{p^*}(S,X^*)\,\|\overline{D}\phi\|_{(2,p^*)} &= \kappa_{p^*}(S,X^*)\,2^{\frac{1}{p^*}}\|\overline{\nabla}\phi\|_{\ell_{p^*}(\mathcal{E},X)}\\
&\geq 2^{\frac{1}{p^*}}\left\|\phi - \overline{M}\phi\right\|_{(1,p^*)}\\
&\geq 2^{\frac{1}{p^*}}\left(\|\phi\|_{(1,p^*)} - \left\|\overline{M}\phi\right\|_{(1,p^*)}\right)
\end{aligned}$$

Since $\overline{D}$ is an isometry on $\ker \overline{d}$,

$$2^{\frac{1}{p^*}}\|\phi\|_{(1,p^*)} - \kappa_{p^*}(S,X^*)\|\phi\|_{(1,p^*)} \leq 2^{\frac{1}{p^*}}\|\overline{M}\phi\|_{(1,p^*)},$$

which, by lemma 8, becomes

$$\left(1 - 2^{-\frac{1}{p^*}}\kappa_{p^*}(S,X^*)\right)\|\phi\|_{(1,p^*)} \leq \frac{1}{2}\|\delta^*\phi\|_{(0,p^*)}.$$

□

Remark 14. The above inequality does not reduce to the one in [36] in the case $X = L_2$ and $p = 2$, even though in both cases the constant is non-zero if $\kappa_2(S,\mathbb{R}) < \sqrt{2}$. For $X = L_2$ and $p = 2$, Theorem 13 gives a strictly smaller lower estimate for the norm of the operator δ^*. Indeed, in that case the estimate obtained using spectral methods is

$$\sqrt{2\left(2 - \kappa_2(S,L_2)^2\right)}\,\|\phi\|_{(1,2)} \leq \|\delta^*\phi\|_{(2,2)}.$$

This difference is a consequence of the fact that in the case $p = 2$ and $X = L_2$ we can apply the Pythagorean rule instead of the triangle inequality in the first sequence of inequalities.

A similar inequality as in Theorem 13 holds for $\kappa_p(S,X)$ and the norm of $\overline{\delta}^*$. The above inequality can be now used to show that sufficiently small constants in Poincaré inequalities on the link graph imply fixed point properties.

4.2. **Proof of the main theorem.**

Theorem 1. *Let X be a reflexive Banach space and let G be a group generated by a finite, symmetric set S, not containing the identity element. If the link graph $\mathcal{L}(S)$ is connected and for some $1 < p < \infty$ the associated Poincaré constants satisfy*

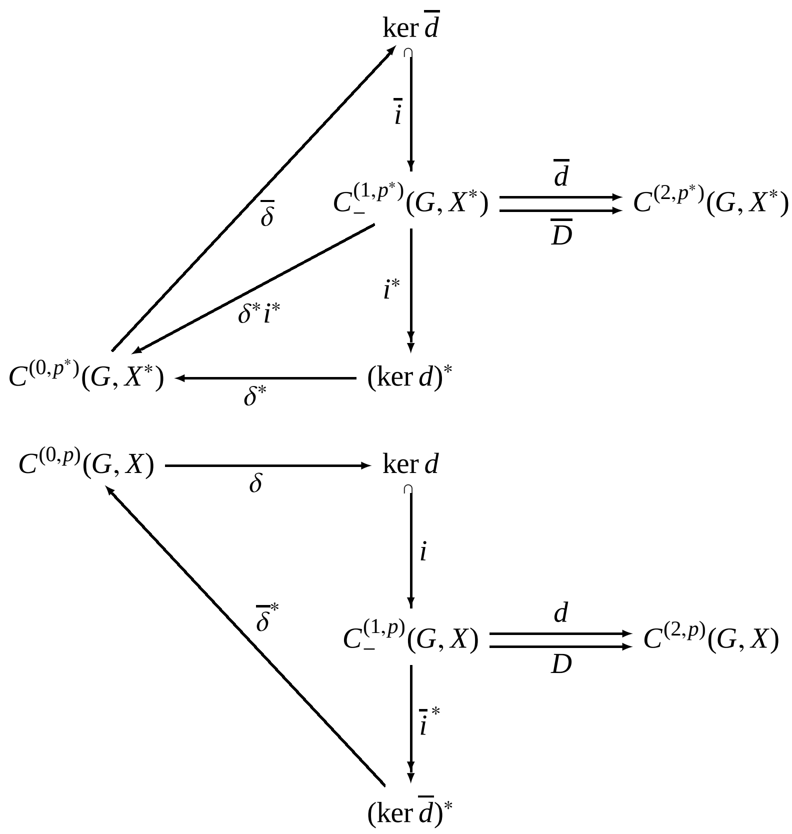
$$\max\left\{2^{-\frac{1}{p}}\kappa_p(S,X), 2^{-\frac{1}{p^*}}\kappa_{p^*}(S,X^*)\right\} < 1,$$

then

$$H^1(G,\pi) = 0,$$

for every isometric representation π of G on X.

Proof. We have the following dual diagrams:



By Theorem 13, if $2^{-\frac{1}{p}}\kappa_p(S,X) < 1$ we conclude that $\overline{\delta}^*$ has closed range when restricted to $\ker d$. In fact, this means that the composition $\overline{\delta}^* \circ \overline{i}^* \circ i$ has closed range. This implies that in particular $\overline{i}^* \circ i$ has closed range, and thus its dual, $i^* \circ \overline{i} : \ker \overline{d} \to (\ker d)^*$ is surjective.

A similar argument applied to $2^{-\frac{1}{p^*}}\kappa_{p^*}(S,X^*) < 1$ yields $\delta^*\circ\bar{i}^*\circ i$ also has closed range, which implies that δ^* has closed range on the image of $\bar{i}^*\circ i$. Since the latter is surjective, δ^* has closed range on $(\ker d)^*$. This on the other hand implies that δ is onto, which proves the theorem by Proposition 10. □

Remark 15. Note that under the assumptions of Theorem 1 $(\ker d)^*$ and $\ker\bar{d}$ are isomorphic (a similar fact holds for $\ker d$ and $(\ker\bar{d})^*$). These spaces are closely related to the spaces of cocycles for the given representation.

It is an interesting question, for which isometric representations is $i\circ\bar{i}^*$ automatically an isomorphism, or at least is surjective. This property would eliminate for such representations the need to use the inequality $2^{-\frac{1}{p}}\kappa_p < 1$, which is necessary in the above proof. A special case is discussed and applied in Section 7.2.

Remark 16. Note that it is not clear whether the above method can be extended to subspaces of $Y \subseteq X$. This would require estimating $\kappa_p(S,X)$ for some p, together with $\kappa_{p^*}(S,X^*)$ where Y^* is a quotient of X^*.

5. $\widetilde{A}_2$ GROUPS

In this section we apply Theorem 1 to specific groups and Banach spaces. In [7] the authors studied a family of groups $\{G_q\}$ called the $\widetilde{A}_2$-groups. These groups were introduced in [6], see also [4] for a detailed discussion. The group G_q has a presentation, whose associated link graph $\mathcal{L}(S)$ is the incidence graph of a finite projective plane $\mathbf{P}^2(\mathbb{F}_q)$ (here, q is a power of a prime number). Spectra of such graphs, with weight $\omega \equiv 1$, were computed by Feit and Higman [11], see also [4, 36]. It follows that

$$\kappa_2(S,\mathbb{R}) = \left(1 - \frac{\sqrt{q}}{q+1}\right)^{-\frac{1}{2}}.$$

In general, any estimates of p-Poincaré constants are difficult to obtain. In our case, the link graphs are finite graphs and we can use norm inequalities and a version of 3.0.5 to give the necessary estimates.

Theorem 17. *For each $q = k^n$ for some $n \in \mathbb{N}$ and prime number k we have*

$$H^1(G_q,\pi) = 0$$

for all

$$2 \le p < \frac{\ln(q^2+q+1) + \ln(q+1)}{\frac{1}{2}\ln(2(q^2+q+1)(q+1)) - \ln(2) - \ln\left(\sqrt{1 - \frac{\sqrt{q}}{q+1}}\right)}$$

and for any isometric representation π of G on L_p, where L_p denotes $L_p(Y,\mu)$ on any measure space.

Proof. We proceed by estimating $\kappa_p(S, L_p)$ and applying Theorem 1. Recall that given the space $\ell_p(\Omega)$ for $2 \leq p$, where the set Ω is finite, the following norm inequalities hold,

$$\|f\|_{\ell_p(\Omega)} \leq \|f\|_{\ell_2(\Omega)} \leq (\#\Omega)^{\frac{1}{2}-\frac{1}{p}} \|f\|_{\ell_p(\Omega)},$$

where $\|f\|_{\ell_p(\Omega)} = \left(\sum_{x\in\Omega} |f(x)|^p\right)^{\frac{1}{p}}$. Since the degree of the incidence graphs of finite projective planes is constant and equal to $q+1$ we obtain for $f : S \to \mathbb{R}$ satisfying $Mf = 0$,

$$\begin{aligned} \|f\|_{(1,p)} &= (q+1)^{\frac{1}{p}} \|f\|_{\ell_p(S,X)} \\ &\leq (q+1)^{\frac{1}{p}-\frac{1}{2}} \|f\|_{(1,2)} \\ &\leq (q+1)^{\frac{1}{p}-\frac{1}{2}} \kappa_2(S, L_2) \|\nabla f\|_{\ell_2(\mathcal{E},X)} \\ &\leq (q+1)^{\frac{1}{p}-\frac{1}{2}} \kappa_2(S, L_2) \left(\frac{\omega(E)}{2}\right)^{\frac{1}{2}-\frac{1}{p}} \|\nabla f\|_{\ell_p(\mathcal{E},X)}. \end{aligned}$$

For each q we have $\omega(E) = 2(q^2+q+1)(q+1)$, which gives the inequality

$$\begin{aligned} 2^{-\frac{1}{p}} \kappa_2(S, L_p) &\leq 2^{-\frac{1}{p}} (q+1)^{\frac{1}{p}-\frac{1}{2}} \kappa_2(S, L_2) \left(\frac{\omega(E)}{2}\right)^{\frac{1}{2}-\frac{1}{p}} \\ &= 2^{-\frac{1}{p}} \left(\sqrt{1 - \frac{\sqrt{q}}{q+1}}\right)^{-1} \left(q^2+q+1\right)^{\frac{1}{2}-\frac{1}{p}}. \end{aligned}$$

Bounding the above quantity by 1 from above gives

$$p < \frac{2\ln(2(q^2+q+1))}{\ln(2(q^2+q+1)) - \ln\left(2\left(1 - \dfrac{\sqrt{q}}{q+1}\right)\right)}.$$

A similar norm estimate for $p^* \leq 2$, by virtue of the inequality

$$\|f\|_{\ell_2(\Omega)} \leq \|f\|_{\ell_{p^*}(\Omega)} \leq (\#\Omega)^{\frac{1}{p}-\frac{1}{2}} \|f\|_{\ell_2(\Omega)},$$

yields

$$p^* > \frac{2\ln((q+1)(q^2+q+1))}{\ln\left(2(q^2+q+1)(q+1)\right) + \ln\left(2\left(1 - \frac{\sqrt{q}}{q+1}\right)\right)}$$

Simplifying and comparing p and $\dfrac{p^*}{p^*-1}$ we obtain the claim. □

Remark 18. The same argument gives a similar conclusion for the Banach space $X = \left(\bigoplus X_i\right)_p$, the ℓ_p-sum in which X_i is finite-dimensional with a norm sufficiently, and uniformly in i, close to the Euclidean norm. We leave the details to the reader.

Remark 19. The largest value of p in Theorem 17 is approximately 2.106 attained for $q = 13$. As q increases to infinity the values of p, for which cohomology vanishes, converge to 2 from above.

Remark 20. Although our estimate of the constant in the p-Poincaré inequality is not expected to be optimal, other interpolation methods do not seem to yield better constants. For instance, Matousek's interpolation method for p-Poincaré inequalities [21] gives a constant strictly less that 1 for any $p \geq 2$, since it emphasizes independence of dimension and is much better suited to deal with sequences of graphs (e.g. expanders).

Recall that the Banach-Mazur distance $d_{BM}(x, y)$ between two Banach spaces is the infimum of the set of numbers L, for which there exists an isomorphism $T : X \to Y$ satifying $\|x\|_x \leq \|T_x\|_Y \leq L\|x\|_X$. Another consequence of Theorem 1 is that we obtain vanishing of cohomology for representations on Banach spaces, whose Banach-Mazur distance from the Hilbert space is controlled. We phrase this property in terms of uniformly bounded representations.

Theorem 21. *Let G_q be an $\widetilde{A}_2$-group and π be a uniformly bounded representation of G_q on a Hilbert space H, satsfying*

$$\sup_{g\in G} \|\pi_g\| < \sqrt{2\left(1 - \frac{\sqrt{q}}{q+1}\right)}.$$

Then $H^1(G, \pi) = 0$.

Proof. Let $\|v\|' = \sup_{g\in G} \|\pi_g v\|$. Then $\|\cdot\|'$ is a norm and π is an isometric representation on $X = (H, \|\cdot\|')$. The identity is an isomorphism $\mathcal{I} : X \to H$ with $L = \sqrt{2\left(1 - \frac{\sqrt{q}}{q+1}\right)}$, and $L\mathcal{I} : X^* \to H$ is an isomorphism with the same property. The estimate now follows by letting $p = 2$ and using the relation between $\kappa_2(S, X)$, $\kappa_2(S, X^*)$ and $\kappa_2(S, H)$ described in 3.0.5. □

A similar fact (with appropriate constants) holds for L_p spaces, for the range of p as in the previous theorem.

6. Hyperbolic groups

In [36] Żuk used spectral methods to show that certain random groups have property (T) with overwhelming probability. A detailed account was recently provided in [19]. We sketch the strategy of the proof and generalize it to L_p-spaces.

In [13] it was shown that for a certain random graphs on n vertices of degree deg there exists a constant such that for any $\varepsilon > 1$ we have

$$\lim_{n\to\infty} \mathbb{P}\left(\kappa_2(S, \mathbb{R}) \leq \left(1 - \left(\frac{\sqrt{2\deg}(\deg - 1)^{\frac{1}{4}}}{\deg} + \frac{\varepsilon}{\deg}\right)\right)\right) \to 1. \tag{5}$$

In [36] a modified link graph, denoted $L'(S)$, with multiple edges was considered. $L'(S)$ decomposes into random graphs as above and it is shown, using the above estimate, that it has a spectral gap strictly large than $1/2$ with probability 1. In our setting, the modified link graph $L'(S)$ can be viewed as a link graph with an admissible weight $\omega(s, t)$, which is defined to be the number of edges connecting s and t. Thus we can apply Theorem 1. Recall that in the Gromov model $\mathcal{G}(n, l, d)$

for random groups one chooses a density $0 < d < 1$ and considers a group given by a generating set S of cardinality n and $(2n-1)^{ld}$ relations of length l, chosen at random, letting l increase to infinity.

Theorem 22 ([36]; see also [19] for a detailed proof)**.** *Let G be a random group in the density model, where $1/3 < d < 1/2$. Then, with probability 1, G is hyperbolic and there exists a group Γ and a homomorphism $\phi : \Gamma \to G$ with the following properties:*

(1) *Γ has a generating set S, whose link graph satisfies $2^{-1/2}\kappa_2(S, L_2) < 1$,*
(2) *$\phi(\Gamma)$ is of finite index in G.*

Given the above, we apply similar norm inequalities as in the case of $\widetilde{A}_2$-groups to the link graph of Γ and, as before, obtain fixed point properties for affine isometric actions of the group Γ on L_p for certain $p > 2$. For any given $p > 1$, the property of having vanishing cohomology $H^1(G,\pi)$ for *all* isometric representations π on L_p-spaces passes to quotients and from finite index subgroups to the ambient group. We thus have

Theorem 23. *With the assumptions of the previous theorem, with probability 1, Theorem 1 applies to hyperbolic groups. More precisely, let G, Γ and ϕ be as above, and let $\mathcal{L}(S) = (\mathcal{V}, \mathcal{E})$ denote the link graph of Γ. Then $H^1(G,\pi) = 0$ for every isometric representation π of G on L_p for*

$$p < \min\{p_0, \overline{p}_0{}^*\},$$

where

$$p_0 = \frac{\ln \deg_\omega - \ln(2\#\mathcal{E})}{\frac{1}{2}\ln\left(\frac{\deg_\omega}{\#\mathcal{E}}\right) - \ln\kappa_2(S,\mathbb{R})} \quad \textit{and} \quad \overline{p}_0 = \frac{\ln(\#\mathcal{V}\deg_\omega) - \ln 2}{\frac{1}{2}\ln(\#\mathcal{V}\deg_\omega) - \ln\kappa_2(S,\mathbb{R})}.$$

One also has a counterpart of Theorem 21 for hyperbolic groups.

Theorem 24. *Let G be a hyperbolic group in the Gromov model as above and π be a uniformly bounded representation of G on a Hilbert space H, satisfying*

$$\sup_{g\in G}\|\pi_g\| < \frac{\sqrt{2}}{\kappa_2(S,\mathbb{R})},$$

Then $H^1(G,\pi) = 0$.

In other words, the $H^1(G,\pi)$ vanishes with probability 1 for representations bounded by $\sqrt{2}$.

We remark, that in (5), $\kappa_2(S,\mathbb{R})$ tends to 1 as $\deg \to \infty$. Thus the above upper bound on the norm of the representation is $\sqrt{2}$ with probability 1. On the other hand, Shalom showed that the $\mathrm{Sp}(n,1)$ has non-trivial cohomology with respect to some uniformly bounded representations (unpublished). It is not known whether a similar fact holds for hyperbolic groups. M. Cowling proposed to define a numerical invariant of a hyperbolic group by setting $\inf\{\sup_{g\in G}\|\pi_g\| : H^1(G,\pi) = 0\}$. Theorem 24 gives a uniform lower bound of $\sqrt{2}$ on such an invariant, with probability 1, for hyperbolic groups in the Gromov model.

Theorem 23 brings us to another interesting application. Pansu [28] defined a quasi-isometry invariant of a hyperbolic group, called the conformal dimension, to be the number

$$\operatorname{confdim}(\partial G) = \inf \left\{ \dim_{\mathrm{Haus}}(\partial G, d) : \ d \text{ quasi-conformally equivalent to } d_v \right\},$$

where $\dim_{\mathrm{Haus}}$ denotes the Hausdorff dimension, ∂G denotes the boundary of the hyperbolic group and d_v denotes any visual metric on ∂G. We refer to [17] for a brief overview of conformal dimension of boundaries of hyperbolic groups. Bourdon and Pajot [5] showed that a hyperbolic group acts by affine isometries without fixed points on $L_p(G)$ for p greater than the conformal dimension of ∂G. Combining this with vanishing of cohomology as studied here we see, that if $H^1(G,\pi)$ vanishes for all isometric representations on L_p then

$$p \leq \operatorname{confdim}(\partial G).$$

As a consequence we have the following statement,

Corollary 25. *With the assumptions and notation of Theorem 23,*

$$\operatorname{confdim}(\partial G) \geq \min \left\{ p_0, \overline{p}_0^{\,*} \right\}.$$

Finally, as mentioned in the introduction, the above facts show that the method of Poincaré inequalities cannot in general give vanishing of cohomology as studied in this paper, for all $2 < p < \infty$. In addition, we have the following quantitative statement about Poincaré constants.

Corollary 26. *For any hyperbolic group G and any generating set S not containing the identity element, the Poincaré constants on the link graph associated to S satisfy*

$$\kappa_p(S, L_p) \geq 2^{\frac{1}{p}} \quad \text{or} \quad \kappa_{p^*}(S, L_{p^*}) \geq 2^{\frac{1}{p^*}},$$

for $p > \operatorname{confdim}(\partial G)$.

7. Other Applications

7.1. **Actions on the circle.** Fixed point properties for the spaces L_p, $p > 2$, can be applied to studying actions on the circle, by applying the vanishing of cohomology to the L_p-Liouville cocycle. In [25] the following theorem was proved.

Theorem 27. *Let G be a discrete group, such that $H^1(G,\pi) = 0$ for every isometric representation of G on L_p for some $p > 2$. Then for every $\alpha > \dfrac{1}{p}$ every homomorphism $h : G \to \mathrm{Diff}_+^{1+\alpha}(S^1)$ has finite image.*

Combining this result with, for instance, Theorem 17 we obtain

Corollary 28. *Let q be a power of a prime number and G_q be be the corresponding $\widetilde{A}_2$ group. Then every homomorphism $h : G \to \mathrm{Diff}_+^{1+\alpha}(S^1)$ has finite image for*

$$\alpha > \frac{\frac{1}{2}\ln(2(q^2+q+1)(q+1)) - \ln(2) - \ln\left(\sqrt{1 - \frac{\sqrt{q}}{q+1}}\right)}{\ln(q^2+q+1) + \ln(q+1)}.$$

7.2. **The finite–dimensional case and the p-Laplacian.** Let $1 < p < \infty$. The p-Laplacian Δ_p is an operator $\Delta_p : \ell_p(V) \to \ell_p(V)$, defined by the formula

$$\Delta_p f(x) = \sum_{x\sim y} (f(x) - f(y))^{[p]} \omega(x, y),$$

for $f : V \to \mathbb{R}$, where $a^{[p]} = |a|^{p-1} \operatorname{sign}(a)$. The p-Laplacian reduces to the standard discrete Laplacian for $p = 2$, and is non-linear when $p \neq 2$. The p-Laplacian is of great importance in the study of partial differential equations. Its discrete version was studied e.g. in [1, 31]

A real number λ is an eigenvalue of the p-Laplacian Δ_p if there exists a function $f : V \to \mathbb{R}$ such that

$$\Delta_p f = \lambda f^{[p]}.$$

The eigenvalues of the p-Laplacian are difficult to compute in the case $p \neq 2$, due to non-linearity of Δ_p, see [15] for explicit estimates. Define

$$\lambda_1^{(p)}(\Gamma) = \inf \left\{ \frac{\sum_{x\in V} \sum_{y\sim x} |f(x) - f(y)|^p \omega(x, y)}{\inf_{\alpha\in\mathbb{R}} \sum_{x\in V} |f(x) - \alpha|^p \deg_\omega(x)} \right\} \tag{6}$$

with the infimum taken over all $f : V \to \mathbb{R}$ such that f is not constant. $\lambda_1^{(p)}$ is the smallest positive eigenvalue of the discrete p-Laplacian Δ_p or the *p-spectral gap*.

We now apply an estimate similar to the one in Corollary 11, to finite quotients of groups. Let G be a finitely generated group and consider a homomorphism $h : G \to H$, where H is a finite group. Let $p > 1$ and let $\ell_p^0(H)$ denote the subspace of $\ell_p(H)$ consisting of those functions, which sum to 0.

We can identify the dual $\ell_p^0(H)^*$ with the space $\ell_{p^*}^0(H)$, with the norm

$$\|f\| = \inf_{\alpha\in\mathbb{R}} \|f - \alpha\|_{p^*}.$$

We will use our results to estimate the p^*-spectral gap for this norm on the Cayley graph of H.

Let $X = \ell_p^0(H)^*$ be equipped with the adjoint of the left regular representation λ on $\ell_p(H)$, restricted to $X^* = \ell_p^0(H)$. We have

$$\kappa_p(S, X^*) \leq \kappa_p(S, \ell_p(H)) = \kappa_p(S, \mathbb{R}).$$

Computing the Poincaré constant of the link graph for the norm of X is not straightforward. However, following the strategy outlined in Remark 15, we will show that we can bypass this condition. In order to do this we need to show that $i^* \circ \bar{i}$ is onto. In fact, a stronger statement is true.

Lemma 29. *Under the above assumptions, the map $i^* \circ \bar{i} : \ker d \to (\ker \bar{d})$, is an isomorphism.*

Proof. We can view X and X^* as having the same underlying vector space (real-valued functions $f : H \to X$ with mean value 0), equipped with two different norms. Similarly, $C_-^{(1,p^*)}(G, X)$ and $C_-^{(1,p)}(G, X^*)$ also have the same underlying vector space, equipped with two different norms. The adjoint $\bar{\lambda}$ of the left regular representation, coincides with λ. For that reason $\ker d$ and $\ker \bar{d}$ describe the same vector subspace. The claim follows from the fact that all the spaces involved are finite-dimensional and complemented. □

Now, since the representation of G on X does not have invariant vectors and δ is onto $\ker d$, we can conclude, by the Open Mapping Theorem, that δ in fact induces an isomorphism between $C^{(0,p^*)}(G,X)$ and $\ker d$. It follows from Theorem 13, that

$$2\left(1-2^{-\frac{1}{p}}\kappa_p(S,\mathbb{R})\right)\|f\|_{(0,p^*)} \leq \|\delta f\|_{1,p^*}.$$

Since $f \in \ell_p^0(H)$ this gives

$$\left(2\left(1-2^{-\frac{1}{p}}\kappa_p(S,\mathbb{R})\right)\right)^{p^*}\|f\|_X^{p^*} \leq \sum_{s\in S}\|f-\lambda_s f\|_X^{p^*}\frac{\deg_\omega(s)}{\omega(E)}$$

Since $\|v\|_X \leq \|v\|_{\ell_p(H)}$, this yields

$$\left(2\left(1-2^{-\frac{1}{p}}\kappa_p(S,\mathbb{R})\right)\right)^{p^*}\inf_{c\in\mathbb{R}}\sum_{h\in H}|f(h)-c|^{p^*}\deg_\omega(h)$$
$$\leq \sum_{h\in H}\sum_{g\sim h}|f(h)-f(g)|^{p^*}\frac{\deg_\omega(g^{-1}h)}{\omega(E)}.$$

(Note that $\deg_\omega(g^{-1}h)$ refer to $\mathcal{L}(S)$, not the Cayley graph of H.)

Corollary 30. *Let G be a group generated by a finite, symmetric set S not containing the identity element. If the link graph $\mathcal{L}(S)$ is connected and for some $1 < p < \infty$ the Poincaré constant satisfies*

$$2^{-\frac{1}{p}}\kappa_p(S,\mathbb{R}) < 1,$$

then

$$\lambda_1^{(p)} \geq 2\left(1-2^{-\frac{1}{p}}\kappa_p(S,\mathbb{R})\right)$$

on the Cayley graph of any finite quotient of G, for any weight $\omega(g,h) \geq \dfrac{\deg_\omega(g^{-1}h)}{\omega(E)}$.

Remark 31. Note that $\lambda_1^{(p)}$ is in general smaller than the p-Poincaré constant on any graph. They are related by the isomorphism constant for the two norms on the functions $f : V \to \mathbb{R}$ with mean value zero: the p-norm and the quotient norm $\inf_{c\in\mathbb{R}}\|f-c\|_p$.

Remark 32. A similar claim as in lemma 29 holds for any orthogonal representation which is also isometric on $\ell_p(H)$.

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