GENERIC REPRESENTATIONS OF ABELIAN GROUPS AND EXTREME AMENABILITY

JULIEN MELLERAY AND TODOR TSANKOV

ABSTRACT. If G is a Polish group and Γ is a countable group, denote by $\mathrm{Hom}(\Gamma, G)$ the space of all homomorphisms $\Gamma \to G$. We study properties of the group $\overline{\pi(\Gamma)}$ for the generic $\pi \in \mathrm{Hom}(\Gamma, G)$, when Γ is abelian and G is one of the following three groups: the unitary group of an infinite-dimensional Hilbert space, the automorphism group of a standard probability space, and the isometry group of the Urysohn metric space. Under mild assumptions on Γ, we prove that in the first case, there is (up to isomorphism of topological groups) a unique generic $\overline{\pi(\Gamma)}$; in the other two, we show that the generic $\overline{\pi(\Gamma)}$ is extremely amenable. We also show that if Γ is torsion-free, the centralizer of the generic π is as small as possible, extending a result of King from ergodic theory.

1. INTRODUCTION

A *representation* of a countable group Γ into a Polish group G is a homomorphism $\Gamma \to G$. In the case when G is the group of symmetries of a certain object X, a representation can be thought of as an action $\Gamma \curvearrowright X$ that preserves the relevant structure; by varying X one obtains settings of rather different flavor. For example, representation theory studies linear representations, while understanding homomorphisms $\Gamma \to \mathrm{Aut}(\mu)$ (or, equivalently, measure-preserving actions of Γ on a standard probability space (X, μ)) is the subject of ergodic theory.

In this paper, we take a global view and, rather than study specific representations, we are interested in the space of all representations $\mathrm{Hom}(\Gamma, G)$. $\mathrm{Hom}(\Gamma, G)$ is a closed subspace of the Polish space G^Γ and is therefore also Polish. It is equipped with a natural action of G by conjugation

$$(g \cdot \pi)(\gamma) = g\pi(\gamma)g^{-1}, \quad \text{for } \pi \in \mathrm{Hom}(\Gamma, G), g \in G$$

and if two representations are conjugate, one usually considers them as isomorphic. We are interested in *generic* properties of representations (that is, properties that hold for a comeager set of $\pi \in \mathrm{Hom}(\Gamma, G)$) that are moreover invariant under the G-action. We note that when Γ is the free group $\mathbf{F}_n$, $\mathrm{Hom}(\Gamma, G)$ can be canonically identified with G^n (and in particular, as a G-space, $\mathrm{Hom}(\mathbf{Z}, G)$ is isomorphic to G equipped with the action by conjugation).

A very special situation is when the action $G \curvearrowright \mathrm{Hom}(\Gamma, G)$ has a comeager orbit; then, studying generic invariant properties of representations amounts to understanding the single generic point. Identifying when a comeager orbit exists has been the focus of active research during the last years, and looking for generic conjugacy classes is a well established topic in the study of automorphism

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groups of countable structures. Hodges–Hodkinson–Lascar–Shelah [HHLS] introduced the notion of ample generics (a Polish group G has *ample generics* if there is a generic element in $\mathrm{Hom}(\mathbf{F}_n, G)$ for every n) and gave the first examples, then Kechris–Rosendal [KR] undertook a systematic study of this notion; more recently, Rosendal [R2] proved that there is a generic point in $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{QU}))$ for any finitely generated abelian Γ ($\mathbf{QU}$ is the rational Urysohn space).

In many situations, however, it turns out that orbits are meager. Important examples of this phenomenon come from ergodic theory (that is, when $G = \mathrm{Aut}(\mu)$), and identifying generic properties in that setting has a long history, especially in the case $\Gamma = \mathbf{Z}$: for example, Halmos proved that the generic transformation in $\mathrm{Aut}(\mu)$ is weakly mixing, King showed that it is of rank 1, etc. We refer the reader to Kechris's book [K2] for a detailed study of the spaces $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ for various groups Γ.

In this paper, we study properties of the subgroup $\overline{\pi(\Gamma)} \leq G$ for a generic $\pi \in \mathrm{Hom}(\Gamma, G)$. Our first theorem is quite general and may be considered as a starting point for the rest of the results. Recall that a Polish group is called *extremely amenable* if every continuous action of the group on a compact space has a fixed point. The property of extreme amenability has received a lot of attention; we recommend Pestov's book [P2] as a general reference.

Theorem 1.1. *Let Γ be a countable group and G a Polish group. Then the set*

$$\{\pi : \overline{\pi(\Gamma)} \text{ is extremely amenable}\}$$

is G_δ in $\mathrm{Hom}(\Gamma, G)$.

Of course, it is possible that generically $\pi(\Gamma)$ is dense in G, for instance this often happens when Γ is free. The easiest way to assure that we are in a non-trivial situation is to assume that Γ satisfies some global law, for example that it is abelian. For that reason (among others), in the rest of the paper, we specialize to abelian groups Γ and three examples for G: the unitary group of a separable, infinite-dimensional Hilbert space $U(\mathcal{H})$, the group of measure-preserving automorphisms of a standard probability space $\mathrm{Aut}(\mu)$, and the group of isometries of the universal Urysohn metric space $\mathrm{Iso}(\mathbf{U})$ (see [LPR$^+$] for background on the Urysohn space). Conjugacy classes are meager in $\mathrm{Hom}(\Gamma, U(\mathcal{H}))$ for any countable infinite Γ [KLP]; it is an open question whether this is true for the other two groups. The orbits in $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ are known to be meager for any infinite amenable Γ [FW]. For $\mathrm{Iso}(\mathbf{U})$, the situation is less clear, though we prove here, using a technique due to Rosendal, that conjugacy classes in $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$ are meager for all torsion-free abelian Γ (see Theorem 8.12).

Of major importance in this paper is the group $L^0(\mathbf{T})$: the Polish group of (equivalence classes of) measurable maps $([0,1], \lambda) \to \mathbf{T}$, where λ denotes the Lebesgue measure, $\mathbf{T}$ is the torus, the group operation is pointwise multiplication, and the topology is given by convergence in measure (equivalently, $L^0(\mathbf{T})$ can be viewed as the unitary group of the abelian von Neumann algebra $L^\infty([0,1])$). Two important facts about this group, proved by Glasner [G1], are that it is *monothetic* (contains a dense cyclic subgroup) and that it is extremely amenable.

Recall that an abelian group is called *bounded* if there is an upper bound for the orders of its elements and *unbounded* otherwise. Even though our results hold in a slightly greater generality, in order to avoid introducing technical definitions, we state them here only for unbounded groups.

In the case of the unitary group, using spectral theory, we are able to identify exactly the group $\overline{\pi(\Gamma)}$ for a generic π.

Theorem 1.2. *Let Γ be a countable, unbounded, abelian group. Then the set*

$$\{\pi : \overline{\pi(\Gamma)} \cong L^0(\mathbf{T})\}$$

is comeager in $\mathrm{Hom}(\Gamma, U(\mathcal{H}))$.

For our other two examples, we are unable to decide whether generically one obtains a single group.

Question 1.3. For G either $\mathrm{Aut}(\mu)$ or $\mathrm{Iso}(\mathbf{U})$, does there exist a Polish group H such that for the generic $\pi \in \mathrm{Hom}(\mathbf{Z}, G)$, $\overline{\pi(\mathbf{Z})} \cong H$? In particular, is this true for $H = L^0(\mathbf{T})$?

This question was also raised (for $\mathrm{Aut}(\mu)$) by Solecki and by Pestov. A related problem was posed by Glasner and Weiss [GW]: is $\overline{\pi(\mathbf{Z})}$ a Lévy group for the generic $\pi \in \mathrm{Hom}(\mathbf{Z}, \mathrm{Aut}(\mu))$? (Recall that $L^0(\mathbf{T})$ is Lévy and that every Lévy group is extremely amenable (see [P2] for background on Lévy groups.)

Even though the questions above remain open, using Theorem 1.1, we can still show that extreme amenability is generic.

Theorem 1.4. *Let Γ be a countable, unbounded, abelian group and G be one of* $\mathrm{Aut}(\mu)$ *or* $\mathrm{Iso}(\mathbf{U})$. *Then the set*

$$\{\pi : \overline{\pi(\Gamma)} \cong L^0(\mathbf{T})\}$$

is dense in $\mathrm{Hom}(\Gamma, G)$ *and therefore, the generic $\overline{\pi(\Gamma)}$ is extremely amenable.*

In the special case $\Gamma = \mathbf{Z}$ and $G = \mathrm{Iso}(\mathbf{U})$, this answers a question of Glasner and Pestov ([P3, Question 8]).

A natural question that arises is how one can distinguish the generic groups $\overline{\pi(\Gamma)}$ for different abelian groups Γ. It turns out that at least for finitely generated free abelian groups, this is impossible.

Theorem 1.5. *Let G be either* $\mathrm{Aut}(\mu)$ *or* $\mathrm{Iso}(\mathbf{U})$, *let P be a property of abelian Polish groups such that the set $\{\pi \in \mathrm{Hom}(\mathbf{Z}, G) : \overline{\pi(\mathbf{Z})}$ has $P\}$ has the Baire property, and let d be a positive integer. Then the following are equivalent:*

(i) *for the generic $\pi \in \mathrm{Hom}(\mathbf{Z}, G)$, $\overline{\pi(\mathbf{Z})}$ has property P;*
(ii) *for the generic $\pi \in \mathrm{Hom}(\mathbf{Z}^d, G)$, $\overline{\pi(\mathbf{Z}^d)}$ has property P.*

In particular, the theorem implies that the generic $\overline{\pi(\mathbf{Z}^d)}$ is monothetic. It also implies that if the answer of Question 1.3 is positive for $\mathbf{Z}$, then it is positive for any $\mathbf{Z}^d$ and moreover, one obtains the same generic group. Finally, in the case $G = \mathrm{Aut}(\mu)$, $\mathbf{Z}^d$ can be replaced by any torsion-free abelian Γ (cf. Corollary 8.11).

Next we apply our techniques to studying stabilizers of representations. If $\pi \in \mathrm{Hom}(\Gamma, G)$, denote by $\mathcal{C}(\pi)$ the *centralizer* of π, i.e.

$$\mathcal{C}(\pi) = \{g \in G : g\pi(\gamma) = \pi(\gamma)g \text{ for all } \gamma \in \Gamma\}.$$

Then $\mathcal{C}(\pi)$ is a closed subgroup of G and $\overline{\pi(\Gamma)} \leq \mathcal{C}(\pi)$ if Γ is abelian. Our next result shows that, for a generic π, we often have equality.

Theorem 1.6. *Let G be either* $\mathrm{Aut}(\mu)$ *or* $\mathrm{Iso}(\mathbf{U})$ *and Γ be a countable, non-trivial, torsion-free, abelian group. Then for a generic $\pi \in \mathrm{Hom}(\Gamma, G)$, $\mathcal{C}(\pi) = \overline{\pi(\Gamma)}$.*

For $G = \mathrm{Aut}(\mu)$ and $\Gamma = \mathbf{Z}$, this result had been obtained by King [K3] in a very different manner.

We conclude with an amusing application of the methods developed in this paper. All three target groups that we consider ($U(\mathcal{H})$, $\mathrm{Aut}(\mu)$, $\mathrm{Iso}(\mathbf{U})$) are known to be Lévy and therefore extremely amenable (see [P2] for a discussion of Lévy groups). Using Theorem 1.1, we are able to give a "uniform" proof for their extreme amenability that does not pass through the fact that they are Lévy groups (but it does use the extreme amenability of $L^0(K)$ for compact groups K, and those are Lévy).

Theorem 1.7. *Let G be either of $U(\mathcal{H})$, $\mathrm{Aut}(\mu)$, $\mathrm{Iso}(\mathbf{U})$; denote by $U(n)$ the unitary group of dimension n and by $\mathbf{F}_\infty$ the free group on $\aleph_0$ generators. Then the set*

$$\{\pi \in \mathrm{Hom}(\mathbf{F}_\infty, G) : \exists n \ \overline{\pi(\mathbf{F}_\infty)} \cong L^0(U(n))\}$$

is dense in $\mathrm{Hom}(\mathbf{F}_\infty, G)$.

Corollary 1.8 (Gromov–Milman [GM]; Giordano–Pestov [GP]; Pestov [P1])**.** *The three groups $U(\mathcal{H})$, $\mathrm{Aut}(\mu)$, and $\mathrm{Iso}(\mathbf{U})$ are extremely amenable.*

Notation. All abelian groups are written multiplicatively and the neutral element is always denoted by 1. When G is a group and $A \subseteq G$, we denote by $\langle A \rangle$ the subgroup of G generated by A. (X, μ) is an atomless standard probability space; we write $\mathrm{Aut}(X, \mu)$ or simply $\mathrm{Aut}(\mu)$ to denote the group of measure-preserving bijections of X modulo a.e. equality (equipped with its usual Polish topology). $\mathcal{H}$ is a separable, infinite-dimensional, complex Hilbert space and $U(\mathcal{H})$ denotes its unitary group endowed with the strong operator topology. If G is a Polish group, $L^0(G)$ denotes the group of all (equivalence classes of) measurable maps from (X, μ) to G with pointwise multiplication and the topology of convergence in measure (see [K2, Section 19] for basic facts about those groups). Sometimes, when we need to explicitly mention the source measure space, we may write $L^0(X, \mu, G)$. $\mathbf{Z}$ denotes the infinite cyclic group; $\mathbf{Z}(n)$ is the finite cyclic group of order n; $\mathbf{T}$ is the multiplicative group of complex numbers of modulus 1; $\mathbf{F}_n$ is the free group with n generators ($n \leq \aleph_0$; if $n = \aleph_0$ we use the notation $\mathbf{F}_\infty$); $U(m)$ is the unitary group of an m-dimensional Hilbert space.

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2. Preliminaries on abelian groups

In this section, we prove several preliminary lemmas about abelian groups that will be useful in the sequel.

By a classical theorem of Prüfer (see [F2, Theorem 17.2]), every bounded countable abelian group is a direct sum of cyclic groups, moreover, the function

$$m_\Gamma \colon \{p^n : p \text{ prime}, n \in \mathbf{N}^+\} \to \mathbf{N} \cup \{\infty\}$$

indicating the number of summands isomorphic to $\mathbf{Z}(p^n)$ is a complete isomorphism invariant for bounded groups.

Consider the following property of a bounded abelian group:

$$(*) \qquad \forall p \forall n\ m_\Gamma(p^n) > 0 \implies \exists k \geq n\ m_\Gamma(p^k) = \infty.$$

The reason we are interested in it is that a bounded abelian group without $(*)$ cannot be densely embedded in an extremely amenable group. More precisely, we have the following.

Proposition 2.1. *Let Γ be a bounded abelian group that does not have property $(*)$. Then any Hausdorff topological group G in which Γ can be embedded as a dense subgroup has a non-trivial open subgroup of finite index.*

Proof. Let p and n be such that $0 < m_\Gamma(p^n) < \infty$ and $m_\Gamma(p^m) = 0$ for all $m > n$. Let k be the product of all orders of elements of G not divisible by p and consider the closed subgroup $H \leq G$ given by

$$H = \{g \in G : g^{kp^{n-1}} = 1\}.$$

Then $H \cap \Gamma$ has finite index in Γ (the quotient $\Gamma/(H \cap \Gamma)$ being isomorphic to $\mathbf{Z}(p)^{m_\Gamma(p^n)}$), showing that H has finite index in G and is therefore open (if $\Gamma = \bigcup_{i=1}^n \gamma_i(H \cap \Gamma)$, then $\bigcup_{i=1}^n \gamma_i H$ is closed and contains Γ). ☐

If Γ is bounded, denote by H_Γ the subgroup of $\mathbf{T}$ generated by

$$\{\exp(2\pi i/p^n) : m_\Gamma(p^n) = \infty\}$$

and note that H_Γ is finite, cyclic and that if Γ has property $(*)$, every character of Γ takes values in H_Γ.

For any countable abelian group Γ, denote by $\hat{\Gamma}$ its dual, i.e. the group of all homomorphisms $\Gamma \to \mathbf{T}$, and recall that as Γ is discrete, $\hat{\Gamma}$ is compact.

Lemma 2.2. *Let Γ be a countable abelian group and let $e\colon \Gamma \to \Gamma^n$ be the diagonal embedding.*

(i) *If Γ is unbounded, then the set*

$$\{\phi \in \hat{\Gamma}^n : \phi(e(\Gamma)) \text{ is dense in } \mathbf{T}^n\}$$

is dense G_δ in $\hat{\Gamma}^n$.

(ii) *If Γ is bounded and has property $(*)$, then the set*

$$\{\phi \in \hat{\Gamma}^n : \phi(e(\Gamma)) = H_\Gamma^n\}$$

is dense G_δ in $\hat{\Gamma}^n$.

Proof. In both cases it is clear that the set is G_δ, so we only have to check that it is dense. We will use the fact that every character of a subgroup $\Delta \leq \Gamma$ extends to a character of Γ (see [F1, Corollary 4.41]).

(i) **Case 1.** Γ contains an element of infinite order γ_0. Denote by Δ the cyclic subgroup generated by γ_0 and let $\theta\colon \hat{\Gamma} \to \hat{\Delta}$ be the dual of the inclusion map $\Delta \to \Gamma$. Let U be an open set in $\hat{\Gamma}^n$. Find $\phi \in U$ such that $\theta(\phi_1), \dots, \theta(\phi_n)$ are independent over $\mathbf{Q}$ (this is possible because θ is an open map and the set of rationally independent tuples is dense in $\hat{\Delta}^n$). Then by Kronecker's theorem, $\phi(e(\Delta))$ is dense in $\mathbf{T}^n$.

Case 2. Γ is torsion. By Baire's theorem, it suffices to show that for any open $V \subseteq \mathbf{T}^n$, the open set of $\phi \in \hat{\Gamma}^n$ such that there exists $\gamma \in \Gamma$ with $\phi(e(\gamma)) \in V$ is dense.

We let d denote the usual (shortest geodesic) metric on the torus. A basic open set U in $\hat{\Gamma}^n$ has the form

$$U = \{\phi \in \hat{\Gamma}^n : \forall f \in F \, \forall i \leq n \; \phi_i(f) = \theta_i(f)\},$$

where F is a finite subgroup of Γ and $\theta_i \in \hat{F}$. Let also V be given by

$$V = \{x \in \mathbf{T}^n : \max_i d(x_i, b_i) < \epsilon\}$$

for some $b \in \mathbf{T}^n$ and $\epsilon > 0$. Let γ_0 be an element of Γ of order greater than $|F|/\epsilon$. Let k be the least positive integer such that $\gamma_0^k \in F$ and note that $k > 1/\epsilon$. Let Δ be the subgroup of Γ generated by F and γ_0. For $i \leq n$, choose $c_i \in \mathbf{T}$ such that $d(c_i, b_i) < \epsilon$ and $c_i^k = \theta_i(\gamma_0^k)$ (this is possible because as $k > 1/\epsilon$, the preimage of any point in $\mathbf{T}$ under the map $x \mapsto x^k$ is ϵ-dense) and define $\psi_i \in \hat{\Delta}$ by

$$\psi_i(\gamma_0^j f) = c_i^j \theta_i(f), \quad \text{for } j \in \mathbf{Z}, f \in F.$$

(ψ_i is well-defined because of the choice of c_i.) Now extend each ψ_i to an element $\phi_i \in \hat{\Gamma}$. Then by definition, $\phi \in U$ and $\phi(e(\gamma_0)) \in V$.

(ii) Let $p_1, \ldots, p_k$ denote the prime numbers dividing the orders of elements of Γ, and $n_1, \ldots, n_k$ the integers such that $m_\Gamma(p_i^{n_i}) = \infty$ and $m_\Gamma(p_i^n) = 0$ for all $n > n_i$. Pick a nonempty open $U \subseteq \hat{\Gamma}^n$ and $x = (x_1, \ldots, x_n) \in H_\Gamma^n$; we need to show that there exists $\phi \in U$ such that $x \in \phi(\Gamma)$.

There exists $\psi = (\psi_1, \ldots, \psi_n) \in \hat{\Gamma}^n$ and a finite subgroup $\Delta \leq \Gamma$ such that any element of $\hat{\Gamma}^n$ coinciding with ψ on Δ belongs to U. From Prüfer's structure theorem and property $(*)$, we see that there exists $\gamma \in \Gamma$ of order $p_1^{n_1} \cdots p_k^{n_k}$ and such that $\langle \gamma \rangle \cap \Delta = \{1\}$. The order of x divides $p_1^{n_1} \cdots p_k^{n_k}$, so we may extend each ψ_i to a character ϕ_i of $\langle \Delta, \gamma \rangle \cong \Delta \times \langle \gamma \rangle$ by setting $\phi_i(\gamma) = x_i$. Further extending $(\phi_1, \ldots, \phi_n)$ to an element of $\hat{\Gamma}^n$, we are done. $\square$

Definition 2.3. Let $n \leq \aleph_0$. A topological group G is called *generically n-generated* if the set of n-tuples $(g_1, \ldots, g_n) \in G^n$ that generate a dense subgroup of G is dense G_δ in G^n. G is *generically monothetic* if it is generically 1-generated.

Note that in the definition above, the condition that the set be G_δ is automatic, so one only has to check that it is dense. Note also that if G is generically n-generated, then it is generically m-generated for every $m \geq n$ and that every Polish group is generically $\aleph_0$-generated.

The idea of the proof of the next proposition comes from Kechris [K2, p. 26], where by a similar argument he showed that $U(\mathcal{H})$ is generically 2-generated.

Proposition 2.4. *Let Γ be a countable abelian group.*

(i) *If Γ is unbounded, then*

$$\{\pi \in \mathrm{Hom}(\Gamma, L^0(\mathbf{T})) : \pi(\Gamma) \textit{ is dense in } L^0(\mathbf{T})\}$$

is dense G_δ in $\mathrm{Hom}(\Gamma, L^0(\mathbf{T}))$. In particular, $L^0(\mathbf{T})$ is generically monothetic.

(ii) *If Γ is bounded and has property $(*)$, then*

$$\{\pi \in \mathrm{Hom}(\Gamma, L^0(H_\Gamma)) : \pi(\Gamma) \textit{ is dense in } L^0(H_\Gamma)\}$$

is dense G_δ *in* $\mathrm{Hom}(\Gamma, L^0(H_\Gamma))$.

Proof. (i) Let K_n denote the subgroup of $L^0([0,1],\lambda,\mathbf{T})$ consisting of all functions that are constant on intervals of the type $[i2^{-n},(i+1)2^{-n}]$. Then K_n is isomorphic to $\mathbf{T}^n$, the sequence $\{K_n\}$ is increasing, and $\bigcup_n K_n$ is dense in $L^0(\mathbf{T})$. It is easy to check that whenever Γ is finitely generated, $\bigcup_n \mathrm{Hom}(\Gamma, K_n)$ is dense in $\mathrm{Hom}(\Gamma, L^0(\mathbf{T}))$ and by using again the result of extension of characters from subgroups, we obtain that this is true for all Γ.

Thanks to the Baire category theorem, we only need to show that for any non-empty open subset U of $L^0(\mathbf{T})$, the open set

$$\{\phi \in \mathrm{Hom}(\Gamma, L^0(\mathbf{T})) : \phi(\Gamma) \cap U \neq \emptyset\}$$

is dense in $\mathrm{Hom}(\Gamma, L^0(\mathbf{T}))$. So, pick a non-empty open subset V of $\mathrm{Hom}(\Gamma, L^0(\mathbf{T}))$ and let n be big enough that $K_n \cap U \neq \emptyset$ and $\mathrm{Hom}(\Gamma, K_n) \cap V \neq \emptyset$. Using Lemma 2.2 (i), we know that we can find some $\phi \in \mathrm{Hom}(\Gamma, K_n) \cap V$ such that $\phi(\Gamma) \cap U \neq \emptyset$, which concludes the proof.

(ii) The same proof, with obvious modifications, works. ☐

3. Extreme amenability is a G_δ property

The following result, while simple, is the key tool for most of our constructions.

Theorem 3.1. *Let G be a Polish group and Γ be a countable group. Then the set*

$$\{\pi \in \mathrm{Hom}(\Gamma, G) : \pi(\Gamma) \text{ is extremely amenable}\}$$

is G_δ *in* $\mathrm{Hom}(\Gamma, G)$.

Proof. We begin by fixing a left-invariant metric d inducing the topology of G. Recall from [P2, Theorem 2.1.11] that $\pi(\Gamma)$ is extremely amenable if and only if the left-translation action of $\pi(\Gamma)$ on $(\pi(\Gamma), d)$ is finitely oscillation stable. From [P2, Theorem 1.1.18], we know that this is equivalent to the following condition:

$$\text{(3.1)}\quad \forall \epsilon > 0\ \forall F \text{ finite} \subseteq \pi(\Gamma)\ \exists K \text{ finite} \subseteq \pi(\Gamma)\ \forall c\colon K \to \{0,1\}$$
$$\exists i \in \{0,1\}\ \exists g \in \pi(\Gamma)\ \forall f \in F\ \exists k \in c^{-1}(i)\ d(gf,k) < \epsilon.$$

In fact, the above property is implied by item (10) of Pestov's Theorem 1.1.18 and implies item (8) of that same theorem, which lists equivalent conditions for an action to be finitely oscillation stable.

We claim that (3.1) is equivalent to the following property:

$$\text{(3.2)}\quad \forall \epsilon > 0\ \forall A \text{ finite} \subseteq \Gamma\ \exists B \text{ finite} \subseteq \Gamma\ \forall c\colon B \to \{0,1\}$$
$$\exists i \in \{0,1\}\ \exists \gamma \in \Gamma\ \forall a \in A\ \exists \delta \in c^{-1}(i)\ d(\pi(\gamma a), \pi(\delta)) < \epsilon.$$

Indeed, since any coloring of $\pi(\Gamma)$ defines a coloring of Γ, it is clear that (3.2) is stronger than (3.1). To see the converse, assume that condition (3.1) is satisfied and fix $\epsilon > 0$ and a finite subset A of Γ. Then (3.1) applied to ϵ and $F = \pi(A)$ yields a finite subset K of $\pi(\Gamma)$ and we can pick a finite $B \subseteq \Gamma$ such that $\pi(B) = K$. We claim that this B witnesses that condition (3.2) is satisfied. To see this, fix an enumeration $\{\gamma_n\}_{n<\omega}$ of Γ. Given $c\colon B \to \{0,1\}$, we define $c_K\colon K \to \{0,1\}$ by setting

$$c_K(k) = c(\gamma_i) \text{ where } i = \min\{j\colon \gamma_j \in B \text{ and } \pi(\gamma_j) = k\}.$$

To conclude, it is enough to notice that the definition of c_K ensures that for any $i \in \{0,1\}$ and any $k \in c_K^{-1}(i)$, there is some $\delta \in B$ with $\pi(\delta) = k$ and $c(\delta) = i$.

It is easily checked that (3.2) is a G_δ condition on π, and we are done. □

Below, we will work with $\overline{\pi(\Gamma)}$ rather than $\pi(\Gamma)$. When it comes to extreme amenability the difference is immaterial since $\pi(\Gamma)$ is extremely amenable if and only if $\overline{\pi(\Gamma)}$ is extremely amenable.

4. The unitary group

In this section, we study generic unitary representations of abelian groups Γ, that is generic elements of $\mathrm{Hom}(\Gamma, U(\mathcal{H}))$, where $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space.

Lemma 4.1. *Let Γ be an abelian group. Then the set*

$$\{\pi \in \mathrm{Hom}(\Gamma, U(\mathcal{H})) : \pi \textit{ is a direct sum of finite-dimensional representations}\}$$

is dense in $\mathrm{Hom}(\Gamma, U(\mathcal{H}))$.

Proof. This is a direct consequence of [K2, Proposition H.2], [BdlHV, Proposition F.2.7], and the fact that irreducible representations of abelian groups are one-dimensional. □

Recall that a vector $\xi \in \mathcal{H}$ is *cyclic* for the representation π if the span of $\{\pi(\gamma)\xi : \gamma \in \Gamma\}$ is dense in $\mathcal{H}$.

Lemma 4.2. *Let Γ be an infinite abelian group and ξ be a unit vector in $\mathcal{H}$. Then the set*

$$\{\pi \in \mathrm{Hom}(\Gamma, U(\mathcal{H})) : \xi \textit{ is a cyclic vector for } \pi\}$$

is dense G_δ in $\mathrm{Hom}(\Gamma, U(\mathcal{H}))$.

Proof. We will first show that the set

$$\{(\pi, \xi) \in \mathrm{Hom}(\Gamma, U(\mathcal{H})) \times \mathcal{H} : \xi \text{ is a cyclic vector for } \pi\}$$

is comeager. By Baire's theorem, we only need to check that for a given non-empty open set $A \subseteq \mathcal{H}$, the open set

$$\{(\pi, \xi) : \exists \gamma_1, \dots, \gamma_n \in \Gamma\ \exists c_1, \dots, c_n \in \mathbf{C}\ \sum_i c_i \pi(\gamma)\xi \in A\}$$

is dense in $\mathrm{Hom}(\Gamma, U(\mathcal{H})) \times \mathcal{H}$. Fix A, an open set $V \subseteq \mathrm{Hom}(\Gamma, U(\mathcal{H}))$, and an open set $B \subseteq \mathcal{H}$. By Lemma 4.1, there is $\pi \in V$, a π-invariant finite-dimensional subspace $\mathcal{K} \subseteq \mathcal{H}$ and vectors $\xi \in B \cap \mathcal{K}$ and $\eta \in A \cap \mathcal{K}$. As every finite-dimensional representation of Γ is a direct sum of characters, there exists an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathcal{K}$ and $\phi \in \hat{\Gamma}^n$ such that $\pi(\gamma)e_i = \langle \gamma, \phi_i \rangle e_i$. Let $\xi = \sum_i a_i e_i$. By slightly modifying ξ if necessary, we can also assume that $a_i \neq 0$ for all i. As Γ is infinite, $\hat{\Gamma}$ is perfect, so without leaving V, we can assume that the characters $\phi_1, \dots, \phi_n$ are distinct. By a classical theorem of Artin (see for instance [L, Theorem 4.1]), they are then linearly independent, so there exist $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $\det(\langle \gamma_j, \phi_i \rangle)_{i,j=1}^n \neq 0$. We thus have

$$\det \left(a_i \langle \gamma_j, \phi_i \rangle\right)_{i,j=1}^n = \Big(\prod_i a_i\Big)(\det\langle \gamma_j, \phi_i \rangle) \neq 0,$$

so the vectors $\pi(\gamma_j)\xi = \sum_i a_i \langle \gamma_j, \phi_i \rangle e_i$ are linearly independent. As every tuple of n independent vectors spans $\mathcal{K}$, in particular, $\eta \in \operatorname{span}\{\pi(\gamma_j)\xi : j \leq n\}$.

Now by the Kuratowski–Ulam theorem [K1, 8.41], there exists $\xi \in \mathcal{H}$ for which the set

$$\{\pi \in \operatorname{Hom}(\Gamma, U(\mathcal{H})) : \xi \text{ is a cyclic vector for } \pi\}$$

is comeager. By homogeneity, this is in fact true for every $\xi \neq 0$. Finally, to see that the set is G_δ, it suffices to write down the definition. □

If $\pi \colon \Gamma \to U(\mathcal{H})$ is a representation of the countable abelian group Γ and $\xi \in \mathcal{H}$ is a unit vector, let $\phi_{\pi,\xi}$ be the positive-definite function corresponding to ξ, i.e.

$$\phi_{\pi,\xi}(\gamma) = \langle \pi(\gamma)\xi, \xi \rangle.$$

By Bochner's theorem [F1, Theorem 4.18], there exists a unique Borel probability measure $\mu_{\pi,\xi}$ on $\hat{\Gamma}$ such that

$$\int \langle \gamma, x \rangle \, \mathrm{d}\mu_{\pi,\xi}(x) = \phi_{\pi,\xi}(\gamma) \quad \text{for all } \gamma \in \Gamma. \tag{4.1}$$

Also, the map $(\pi, \xi) \mapsto \mu_{\pi,\xi}$ is continuous.

Then the cyclic subrepresentation of π generated by ξ is isomorphic to the representation of Γ on $L^2(\hat{\Gamma}, \mu_{\pi,\xi})$ given by

$$(\gamma \cdot f)(x) = \langle \gamma, x \rangle f(x) \tag{4.2}$$

with cyclic vector the constant 1. This representation naturally extends to a representation of the C^*-algebra $C(\hat{\Gamma})$ which we will again denote by π.

Lemma 4.3. *Let Γ be an infinite abelian group and ξ a unit vector in $\mathcal{H}$. Then the set*

$$\{\pi \in \operatorname{Hom}(\Gamma, U(\mathcal{H})) : \xi \text{ is a cyclic vector for } \pi \text{ and } \mu_{\pi,\xi} \text{ is non-atomic}\}$$

is dense G_δ in $\operatorname{Hom}(\Gamma, U(\mathcal{H}))$.

Proof. Note that $\mu_{\pi,\xi}$ is non-atomic iff the cyclic subrepresentation of π generated by ξ is *weakly mixing*, i.e. does not contain finite-dimensional subrepresentations. Apply Lemma 4.2, then use Proposition H.11 and Theorem H.12 from [K2]. □

We are now ready to prove the main theorem of this section.

Theorem 4.4. *Let Γ be a countable abelian group.*

(i) *If Γ is unbounded, then the set*

$$\{\pi \in \operatorname{Hom}(\Gamma, U(\mathcal{H})) : \overline{\pi(\Gamma)} \cong L^0(\mathbf{T})\}$$

is comeager in $\operatorname{Hom}(\Gamma, U(\mathcal{H}))$.

(ii) *If Γ is bounded and has property* $(*)$*, then the set*

$$\{\pi \in \operatorname{Hom}(\Gamma, U(\mathcal{H})) : \overline{\pi(\Gamma)} \cong L^0(H_\Gamma)\}$$

is comeager in $\operatorname{Hom}(\Gamma, U(\mathcal{H}))$.

Proof. Again, we only give the proof for (i), the proof of (ii) being similar.

Let D be a countable dense subset of $C(\hat{\Gamma}, \mathbf{T})$ (the space of all continuous functions $\hat{\Gamma} \to \mathbf{T}$ with the uniform convergence topology) and E a countable dense subset of the unit sphere of $\mathcal{H}$. Define the set $C_1 \subseteq \operatorname{Hom}(\Gamma, U(\mathcal{H}))$ by

$$\begin{aligned} \pi \in C_1 \iff & \forall \epsilon > 0 \, \forall f \in D \, \forall \eta_1, \ldots, \eta_k \in E \\ & \exists \gamma \in \Gamma \, \forall i \leq k \ \|\pi(f)\eta_i - \pi(\gamma)\eta_i\| < \epsilon. \end{aligned}$$

We will show that C_1 is dense G_δ. By Baire's theorem, it suffices to show that for fixed $\epsilon > 0$, $f \in D$, and $\eta_1, \dots, \eta_k \in E$, the open set of π satisfying the condition on the the second line of the definition of C_1 is dense. Fix an open set $V \subseteq \mathrm{Hom}(\Gamma, U(\mathcal{H}))$. By Lemma 4.1, there exist $\pi \in V$, a finite-dimensional π-invariant subspace $\mathcal{K} \subseteq \mathcal{H}$ and $\eta'_1, \dots, \eta'_k \in \mathcal{K}$ such that for all $i \leq k$, $\|\eta_i - \eta'_i\| < \epsilon/4$ and $\|\eta'_i\| \leq 1$. As before, there exists an orthonormal basis $e_1, \dots, e_n$ of $\mathcal{K}$ and $\phi \in \hat{\Gamma}^n$ such that $\pi(\gamma)(e_j) = \langle \gamma, \phi_j \rangle e_j$ for all $\gamma \in \Gamma$ and $j \leq n$. By Lemma 2.2, we can, without leaving V, further assume that $\phi(\Gamma)$ is dense in $\mathbf{T}^n$. Find now $\gamma \in \Gamma$ such that $|\langle \gamma, \phi_j \rangle - f(\phi_j)| < \epsilon/2$ for all $j \leq n$. Then for every $\eta = \sum_j a_j e_j \in \mathcal{K}$,

$$\begin{aligned} \|\pi(f)\eta - \pi(\gamma)\eta\|^2 &= \Big\| \sum_j a_j (f(\phi_j) - \langle \gamma, \phi_j \rangle) e_j \Big\|^2 \\ &\leq \sum_j |a_j|^2 (\epsilon/2)^2 = (\epsilon/2)^2 \|\eta\|^2, \end{aligned}$$

so using the choice of $\mathcal{K}$ and the fact that both $\pi(f)$ and $\pi(\gamma)$ are unitary, we obtain that $\|\pi(f)\eta_i - \pi(\gamma)\eta_i\| < \epsilon$ for all $i \leq k$ as required.

Fix a unit vector $\xi \in \mathcal{H}$ and define the set $C_2 \subseteq \mathrm{Hom}(\Gamma, U(\mathcal{H}))$ by

$$\pi \in C_2 \iff \pi \in C_1 \text{ and } \xi \text{ is cyclic for } \pi \text{ and } \mu_{\pi,\xi} \text{ is non-atomic.}$$

By Lemma 4.2, Lemma 4.3, and the argument above, C_2 is dense G_δ. We now show that for every $\pi \in C_2$, $\overline{\pi(\Gamma)} \cong L^0(\mathbf{T})$. Fix $\pi \in C_2$ and write μ for $\mu_{\pi,\xi}$. As ξ is cyclic, π is isomorphic to the representation on $L^2(\hat{\Gamma}, \mu)$ given by (4.2), so we can identify $\mathcal{H}$ with $L^2(\hat{\Gamma}, \mu)$. Note that $L^0(\hat{\Gamma}, \mu, \mathbf{T})$ embeds as a closed subgroup of $U(L^2(\hat{\Gamma}, \mu))$ (acting by pointwise multiplication) and that we can view $\pi(\Gamma)$ as a subgroup of $L^0(\hat{\Gamma}, \mu, \mathbf{T})$. As μ is non-atomic, it now suffices to check that $\pi(\Gamma)$ is dense in $L^0(\hat{\Gamma}, \mu, \mathbf{T})$. We first observe that the natural homomorphism $C(\hat{\Gamma}, \mathbf{T}) \to L^0(\hat{\Gamma}, \mu, \mathbf{T})$ has a dense image. Indeed, consider the commutative diagram

$$\begin{array}{ccc} C(\hat{\Gamma}, \mathbf{R}) & \xrightarrow{\theta_1} & L^1(\hat{\Gamma}, \mu, \mathbf{R}) \\ \big\downarrow e_1 & & \big\downarrow e_2 \\ C(\hat{\Gamma}, \mathbf{T}) & \xrightarrow{\theta_2} & L^0(\hat{\Gamma}, \mu, \mathbf{T}), \end{array}$$

where the horizontal arrows denote the quotient maps by μ-a.e. equivalence and the vertical ones are the exponential maps $f \mapsto e^{if}$. As the image of θ_1 is dense and e_2 is surjective, we obtain that the image of θ_2 is also dense. Now consider an open subset of $L^0(\hat{\Gamma}, \mu, \mathbf{T})$ of the form

$$V = \{g \in L^0(\hat{\Gamma}, \mu, \mathbf{T}) : \forall i \leq k \ \ \|f \cdot \eta_i - g \cdot \eta_i\| < \epsilon\}$$

for some $f \in D$, $\eta_1, \dots, \eta_k \in E$ and $\epsilon > 0$. As $L^0(\hat{\Gamma}, \mu, \mathbf{T})$ is a closed subgroup of $U(L^2(\hat{\Gamma}, \mu))$ and $\theta_2(D)$ is dense in $L^0(\hat{\Gamma}, \mu, \mathbf{T})$, sets of this form form a basis for the topology of L^0. As $\pi \in C_1$, each such open set meets $\pi(\Gamma)$, finishing the proof. □

5. Aut(μ)

Now, we turn to the study of generic measure-preserving actions of countable abelian groups Γ on a standard probability space, i.e generic elements of $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$. Specifically, we prove the following theorem.

Theorem 5.1. *Let* Γ *be a countable abelian group which is either unbounded or bounded with property* $(*)$. *Then*

$$\{\pi \in \mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu)) : \overline{\pi(\Gamma)} \textit{ is extremely amenable}\}$$

is dense G_δ *in* $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$.

We recall that a homomorphism $\pi \in \mathrm{Hom}(\Gamma, \mathrm{Aut}(X, \mu))$ can be viewed as a measure-preserving action of Γ on X. Such an action is called *(essentially) free* if for all $\gamma \in \Gamma \setminus \{1\}$, $\mu(\{x \in X : \pi(\gamma) \cdot x = x\}) = 0$. It is well-known that for abelian Γ, the conjugacy class of any free action of Γ is dense (see, e.g. [FW, Claim 19]). Since the isomorphism type of $\overline{\pi(\Gamma)}$ is invariant under conjugacy, it suffices, in view of Theorem 3.1, to prove that there exists a single free action π such that $\overline{\pi(\Gamma)}$ is extremely amenable. For this, we will use our results from the previous section and the Gaussian construction, which we quickly recall. As a reference for the facts stated below without proof, we refer the reader to [K2, Appendix E] and [KTD, Lecture 4]

Let ν denote the standard Gaussian measure on $\mathbf{R}$ and let $(X, \mu) = (\mathbf{R}^{\mathbf{N}}, \nu^{\mathbf{N}})$. Let $p_i \colon \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$, $i \in \mathbf{N}$ be the projection functions. Then p_i are independent Gaussian random variables, $p_i \in L^2(X, \mu, \mathbf{R})$, and $\langle p_i, p_j \rangle = \delta_{ij}$ for all i, j. Let H be the closed subspace of $L^2(X, \mu, \mathbf{R})$ spanned by the p_is. Then for every orthogonal operator $T \in O(H)$, we can define a measure-preserving transformation $\tilde{T} \colon X \to X$ by

$$\tilde{T}(x)(n) = (Tp_n)(x).$$

Then $T \mapsto \tilde{T}$ is a topological group isomorphism between the orthogonal group $O(H)$ and a closed subgroup of $\mathrm{Aut}(\mu)$. This defines (by composition) for every group Γ, a homeomorphic embedding $\mathrm{Hom}(\Gamma, O(H)) \to \mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$, $\pi \mapsto \tilde{\pi}$.

The construction described above applies to a real Hilbert space, yet, to use our results from the previous section, we need to embed the unitary group of a complex separable Hilbert space $\mathcal{H}$ into $\mathrm{Aut}(\mu)$. This is achieved by first taking the realification $\mathcal{H}_{\mathbf{R}}$ of $\mathcal{H}$, then embedding the unitary group of $\mathcal{H}$ into the orthogonal group of $\mathcal{H}_{\mathbf{R}}$ in the natural way, and then applying the Gaussian construction to $\mathcal{H}_{\mathbf{R}}$.

Note that it is clear from the construction that $\widetilde{\pi_1 \oplus \pi_2}$ is conjugate to $\tilde{\pi}_1 \times \tilde{\pi}_2$ for any $\pi_1, \pi_2 \in \mathrm{Hom}(\Gamma, U(\mathcal{H}))$. Finally, we will use the fact that the Gaussian action associated to the left-regular representation of Γ is isomorphic to the Bernoulli shift $\Gamma \curvearrowright [0,1]^\Gamma$.

We now have the following.

Proposition 5.2. *Let* Γ *be a countable, infinite group. Then the set*

$$F = \{\pi \in \mathrm{Hom}(\Gamma, U(\mathcal{H})) : \tilde{\pi} \textit{ is free}\}$$

is dense G_δ *in* $\mathrm{Hom}(\Gamma, U(\mathcal{H}))$.

Proof. As the set $\{\phi \in \mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu)) : \phi \text{ is free}\}$ is G_δ (see e.g. [K2, 10.8]), it suffices to check that F is dense. To that end, let $\pi \in \mathrm{Hom}(\Gamma, U(\mathcal{H}))$ be arbitrary and let V be an open neighborhood of π. Denote by λ_Γ the left regular representation of Γ. As any partial isometry (in particular, the embedding of π in $\pi \oplus \lambda_\Gamma$) can be approximated by unitary operators, π is in the closure of the conjugacy class of

$\pi \oplus \lambda_\Gamma$, and therefore, there exists $\sigma \in V$ isomorphic to $\pi \oplus \lambda_\Gamma$. Then $\tilde{\sigma} \cong \tilde{\pi} \times \tilde{\lambda}_\Gamma$ and it only remains to recall that $\tilde{\lambda}_\Gamma$ is free, and the product of any action with a free action is free. □

Proof of Theorem 5.1. Applying Theorem 4.4 and Proposition 5.2, we obtain that the set of $\pi \in \mathrm{Hom}(\Gamma, U(\mathcal{H}))$ such that $\tilde{\pi}$ is free, and $\overline{\pi(\Gamma)}$ is extremely amenable is comeager and in particular, non-empty. As $\overline{\pi(\Gamma)} \cong \overline{\tilde{\pi}(\Gamma)}$, there exists $\tilde{\pi} \in \mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ which is free and such that $\overline{\tilde{\pi}(\Gamma)}$ is extremely amenable. As the conjugacy class of $\tilde{\pi}$ is dense, there is a dense set of elements of $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ which generate an extremely amenable subgroup, and Theorem 3.1 gives the desired conclusion. □

6. The Urysohn space

We next consider generic representations of countable abelian groups by isometries of the Urysohn space, i.e. generic elements of $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$. We refer the reader to the volume [LPR+] for information about the Urysohn space $\mathbf{U}$ and its bounded variant, the Urysohn sphere $\mathbf{U}_1$; Chapter 5 of [P2] is also a good source of information. Let us note that all the results proved here are also true for $\mathrm{Iso}(\mathbf{U}_1)$, and the proofs are essentially the same as those given below. To simplify the exposition, we focus on $\mathrm{Iso}(\mathbf{U})$.

The next lemma is a metric variant of the induction construction of Mackey and will be used in many of our arguments about representations in the isometry group of the Urysohn space.

Lemma 6.1. *Let K be a Polish group admitting a compatible bi-invariant metric and $\Gamma \leq K$ be a discrete subgroup. Let (Z, d_Z) be a bounded Polish metric space on which Γ acts faithfully by isometries. Then there exists a bounded Polish metric space (Y, d_Y), a continuous, faithful, isometric action $K \curvearrowright Y$, and an isometric, Γ-equivariant embedding $e\colon Z \to Y$.*

Proof. Fix a compatible bounded bi-invariant metric δ on K (note that δ is necessarily complete) such that

$$\inf_{\gamma \in \Gamma} \delta(1, \gamma) > \operatorname{diam} Z. \tag{6.1}$$

Put $X = K \times Z$ and equip it with the metric

$$d_X((k_1, z_1), (k_2, z_2)) = \max(\delta(k_1, k_2), d_Z(z_1, z_2)).$$

The group Γ acts on X isometrically on the right by

$$(k, z) \cdot \gamma = (k\gamma, \gamma^{-1} \cdot z).$$

Set $Y = X/\Gamma$ and equip it with the metric

$$d_Y(x_1\Gamma, x_2\Gamma) = \inf_{\gamma_1, \gamma_2} d_X(x_1 \cdot \gamma_1, x_2 \cdot \gamma_2) = \inf_{\gamma} d_X(x_1, x_2 \cdot \gamma).$$

Note that d_Y is indeed a metric because the Γ-action is by isometries and each $x\Gamma$ is closed. It is also not difficult to see that d_Y is complete.

The group K acts on Y on the left by

$$k \cdot (k', z)\Gamma = (kk', z)\Gamma.$$

This action is by isometries, and is easily seen to be continuous. Define $e\colon Z \to Y$ by $e(z) = (1, z)\Gamma$. We check that e is Γ-equivariant:

$$e(\gamma \cdot z) = (1, \gamma \cdot z)\Gamma = (\gamma, z)\Gamma = \gamma \cdot (1, z)\Gamma = \gamma \cdot e(z).$$

We finally check that e is an isometric embedding. Using (6.1) and the fact that Γ acts on Z by isometries, we have:

$$\begin{aligned} d_Y(e(z_1), e(z_2)) &= d_Y((1, z_1)\Gamma, (1, z_2)\Gamma) \\ &= \inf_{\gamma_1, \gamma_2} \{d_X((\gamma_1, \gamma_1^{-1} \cdot z_1), (\gamma_2, \gamma_2^{-1} \cdot z_2))\} \\ &= \inf_{\gamma_1, \gamma_2} \{\max(\delta(\gamma_1, \gamma_2), d_Z(\gamma_1^{-1} \cdot z_1, \gamma_2^{-1} \cdot z_2))\} \\ &= d(z_1, z_2). \end{aligned}$$

To see that the action is faithful, note that $k \cdot (1, z)\Gamma = (1, z)\Gamma$ implies that $k \in \Gamma$ and $k \cdot z = z$. Since the action $\Gamma \curvearrowright Z$ is assumed to be faithful, we are done. □

We recall for future reference that compact Polish groups, as well as abelian Polish groups, admit a compatible bi-invariant metric.

For our construction below, we need to introduce some notation: whenever (Y, d_Y) is a bounded Polish metric space, we consider the space $L^1(Y)$ of all (equivalence classes of) measurable maps from a standard probability space (X, μ) to Y, endowed with the metric

$$d(f, g) = \int_X d_Y(f(x), g(x))\, \mathrm{d}\mu(x).$$

Then $L^1(Y)$ is a Polish metric space in which Y embeds isometrically (as constant functions). Also, if G is a Polish group acting continuously on Y by isometries, the action $G \curvearrowright Y$ extends to an isometric action $L^0(G) \curvearrowright L^1(Y)$ defined by

$$(g \cdot f)(x) = g(x) \cdot f(x).$$

This action is continuous, faithful if the original action $G \curvearrowright Y$ is faithful, and the image of $L^0(G)$ in the isometry group of $L^1(Y)$ is closed if G is a closed subgroup of $\mathrm{Iso}(Y)$.

Lemma 6.2. (i) *Let Γ be an unbounded abelian group. Then the set*

$$\{\pi \in \mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U})) : \overline{\pi(\Gamma)} \cong L^0(\mathbf{T})\}$$

is dense in $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$.

(ii) *Let Γ be a bounded abelian group with property* $(*)$. *Then the set*

$$\{\pi \in \mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U})) : \overline{\pi(\Gamma)} \cong L^0(H_\Gamma)\}$$

is dense in $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$.

Proof. (i) We denote $G = \mathrm{Iso}(\mathbf{U})$. Note that open sets in $\mathrm{Hom}(\Gamma, G)$ of the form

$$\{\pi : d(\pi(\gamma)a, \pi_0(\gamma)a) < \epsilon \text{ for all } \gamma \in F, a \in A\},$$

where $\pi_0 \in \mathrm{Hom}(\Gamma, G)$, $\epsilon > 0$, F is a finite subset of Γ, and A is a finite subset of $\mathbf{U}$, form a basis for the topology of $\mathrm{Hom}(\Gamma, G)$. We will show that every such open set contains a π such that $\overline{\pi(\Gamma)} \cong L^0(\mathbf{T})$.

To that end, fix π_0, ϵ, F, and A. Denote by Δ the subgroup of Γ generated by F. As Δ is residually finite, we may apply [PU, Theorem 1] to find an action $\pi_1 \colon \Delta \curvearrowright \mathbf{U}$ such that $d(\pi_1(\gamma)a, \pi_0(\gamma)a) < \epsilon$ for all $\gamma \in F$, $a \in A$ and such that π_1

has finite range in $\mathrm{Iso}(\mathbf{U})$. Let B denote the (finite) closure of A under π_1, and Δ' be the image of Δ in $\mathrm{Iso}(B)$. Find an integer n such that Δ' embeds as a subgroup of $\mathbf{T}^n$. By Lemma 6.1, there exists a bounded Polish metric space Y and a faithful action of $\mathbf{T}^n$ on Y which extends the action $\Delta' \curvearrowright B$. The natural homomorphism $\Delta \to \mathbf{T}^n$ extends to a homomorphism $\Gamma \to \mathbf{T}^n$ which we denote by ϕ. The action $\mathbf{T}^n \curvearrowright Y$ gives rise, as explained above, to an action $L^0(\mathbf{T}^n) \curvearrowright L^1(Y)$ which extends the action $\mathbf{T}^n \curvearrowright Y$, and we may identify $L^0(\mathbf{T}^n)$ with its image in $\mathrm{Iso}(L^1(Y))$. By the Katětov construction, the action $\mathrm{Iso}(L^1(Y)) \curvearrowright L^1(Y)$ extends to an action $\mathrm{Iso}(L^1(Y)) \curvearrowright \mathbf{U}$, and $\mathrm{Iso}(L^1(Y))$ embeds as a closed subgroup of G. Thus we have identified $L^0(\mathbf{T}^n)$ with a closed subgroup of G.

The composition of the homomorphisms

$$\Gamma \to \mathbf{T}^n \to L^0(\mathbf{T}^n) \to G$$

defines an element π of $\mathrm{Hom}(\Gamma, G)$ and we can assume, using the homogeneity of $\mathbf{U}$, that for all $\gamma \in \Delta$, one has $\pi(\gamma)|_A = \pi_1(\gamma)|_A$, hence $\pi \in U$. Now by perturbing π slightly, using Proposition 2.4, we can arrange so that π is still in U but $\pi(\Gamma)$ is dense in $L^0(\mathbf{T}^n)$. As $L^0(\mathbf{T}^n)$ is closed in G, we obtain that $\overline{\pi(\Gamma)} = L^0(\mathbf{T}^n) \cong L^0(\mathbf{T})$, finishing the proof.

(ii) Essentially the same proof works (replacing everywhere $\mathbf{T}$ by H_Γ). □

The next theorem follows immediately from Lemma 6.2 and Theorem 3.1.

Theorem 6.3. *Let Γ be a countable abelian group, and assume that Γ is either unbounded or is bounded and has property* $(*)$. *Then*

$$\{\pi \in \mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U})) : \overline{\pi(\Gamma)} \text{ is extremely amenable}\}$$

is dense G_δ in $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$.

Remark. We note that, in view of Proposition 2.1, Theorem 5.1 and Theorem 6.3 are best possible, in the sense that the condition on Γ cannot be replaced with a more general one.

7. The extreme amenability of $U(\mathcal{H})$, $\mathrm{Aut}(\mu)$ and $\mathrm{Iso}(\mathbf{U})$

In this section, we describe how to deduce the extreme amenability of $U(\mathcal{H})$, $\mathrm{Aut}(\mu)$, and $\mathrm{Iso}(\mathbf{U})$ from the extreme amenability of $L^0(K)$, where K is compact (see [G1] or [P2]), and Theorem 3.1. Even though the extreme amenability of those groups is known, we think that our "uniform" proof is of some independent interest.

We recall that any Polish group is generically $\aleph_0$-generated; it is then clear that the following theorem, along with Theorem 3.1 and the fact that $L^0(U(m))$ is extremely amenable for all m, implies the extreme amenability of our three groups.

Theorem 7.1. *Let G be either of $U(\mathcal{H})$, $\mathrm{Aut}(\mu)$, $\mathrm{Iso}(\mathbf{U})$. Then the set*

$$\{\pi \in \mathrm{Hom}(\mathbf{F}_\infty, G) : \exists m \ \overline{\pi(\mathbf{F}_\infty)} \cong L^0(U(m))\}$$

is dense in $\mathrm{Hom}(\mathbf{F}_\infty, G)$.

Proof. It suffices to show that for every n,

$$\{(g_1, \ldots, g_n) \in G^n : \exists m \ \langle g_1, \ldots, g_n \rangle \text{ is contained in a closed copy of } L^0(U(m))\}$$

is dense. Indeed, any open set in $\mathrm{Hom}(\mathbf{F}_\infty, G)$ is specified by imposing conditions on finitely many generators, say $a_1, \ldots, a_n$; if we can arrange that $\pi(a_1), \ldots, \pi(a_n)$ are contained in a closed copy of $L^0(U(m))$, we can use the other generators to ensure that $\overline{\pi(\mathbf{F}_\infty)} \cong L^0(U(m))$.

The unitary group. We fix an orthonormal basis $\{e_i\}$ of $\mathcal{H}$, and identify the finite-dimensional unitary group $U(m)$ with the subgroup of $U(\mathcal{H})$ that acts on the first m elements of the basis. Fix a non-empty open subset V of $U(\mathcal{H})^n$; then there exists m such that $U(m)^n \cap V \neq \emptyset$. Identifying the span of $\{e_1, \ldots, e_m\}$ with $\mathbf{C}^m$, we obtain a unitary action $U(m) \curvearrowright \mathbf{C}^m$, which then extends to an action of $L^0([0,1], \lambda, U(m)) \curvearrowright L^2([0,1], \lambda, \mathbf{C}^m) \cong \mathcal{H}$. This gives a homeomorphic embedding of $L^0(U(m))$ as a closed subgroup of the unitary group of $\mathcal{H}$. Using the obvious identifications, we just built a tuple $(g_1, \ldots, g_n) \in V$ which generates a group contained in a subgroup topologically isomorphic to $L^0(U(m))$, and we are done.

The isometry group of the Urysohn space. In that case, the proof is basically the same as that of Lemma 6.2. Pick a non-empty open subset V of $\mathrm{Iso}(\mathbf{U})^n$, and use the fact that $\mathbf{F}_n$ is residually finite and [PU, Theorem 1] to find $(g_1, \ldots, g_n)$ belonging to V and generating a finite subgroup Δ. Then pick a finite subset A of $\mathbf{U}$ such that $\Delta \curvearrowright A$ and any tuple coinciding with $(g_1, \ldots, g_n)$ on A belongs to V; let Δ' denote the image of Δ in $\mathrm{Iso}(A)$. Let $m = |\Delta'|$; then Δ' embeds in the unitary group $U(m)$ (by its left-regular representation). Using Lemma 6.1, the action $\Delta' \curvearrowright A$ extends to a faithful action $U(m) \curvearrowright Y$ for some bounded Polish metric space Y, which then extends to an action $L^0(U(m)) \curvearrowright L^1(Y)$, and $L^0(U(m))$ is isomorphic to the corresponding subgroup of $\mathrm{Iso}(L^1(Y))$. Applying the Katětov construction and using the homogeneity of $\mathbf{U}$, we obtain $(h_1, \ldots, h_n) \in V$ such that the subgroup generated by $(h_1, \ldots, h_n)$ is contained in an isomorphic copy of $L^0(U(m))$.

The automorphism group of (X, μ). Call an action $\mathbf{F}_n \curvearrowright (X, \mu)$ *finite* if it factors through the action of a finite group. It is an immediate consequence of the fact that $\mathrm{Aut}(\mu)$ has a locally finite dense subgroup that the finite actions of $\mathbf{F}_n$ are dense. Fix now an open neighborhood V of a finite action $\mathbf{F}_n \curvearrowright^{\phi_0} (X, \mu)$ that factors through the finite group Δ. As free actions of Δ are dense in $\mathrm{Hom}(\Delta, \mathrm{Aut}(\mu))$, we can assume that the action ψ_0 of Δ induced by ϕ_0 is free. Let m be the order of Δ and, as above, identify Δ with a subgroup of $U(m)$ via the left-regular representation. Let (Y, ν) be a standard probability space and consider the measure-preserving action $L^0(Y, \nu, U(m)) \curvearrowright^a Y \times U(m)$ defined by

$$f \cdot (y, u) = (y, f(y)u)$$

an observe that this action gives an embedding of $L^0(Y, \nu, U(m))$ as a closed subgroup of $\mathrm{Aut}(Y \times U(m), \nu \times \mathrm{Haar})$, which we identify with $\mathrm{Aut}(\mu)$. As $a|_\Delta$ is free and any two free actions of Δ are conjugate, we can assume that $\psi_0 = a|_\Delta$. Denoting by ϕ the action of $\mathbf{F}_n$ induced by $a|_\Delta$, we thus have that $\phi \in V$ and $\phi(\mathbf{F}_n)$ is contained in an isomorphic copy of $L^0(U(m))$. □

Remark. In fact, each of the three groups is generically 2-generated: for $U(\mathcal{H})$ this is proved on page 26 of [K2]; for $\mathrm{Aut}(\mu)$ it is a result of Prasad [P4]; and for $\mathrm{Iso}(\mathbf{U})$, this is a recent result of Slutsky [S, 4.19]. Hence, for the above proof, it would be enough to only consider pairs of elements of G (if one wants to follow that route, one also needs the fact that $L^0(U(m))$ is generically 2-generated; in fact, $L^0(K)$ is generically 2-generated whenever K is compact, connected and metrizable).

8. Generic properties of homomorphisms from torsion-free abelian groups into $\mathrm{Aut}(\mu)$ and $\mathrm{Iso}(\mathbf{U})$

We now turn to the question: how much do generic properties of $\overline{\pi(\Gamma)}$ depend on the (abelian) group Γ? Of course, if Γ is bounded, then the group $\overline{\pi(\Gamma)}$ is also bounded and in general, it will be different from a generic $\overline{\pi(\mathbf{Z})}$. However, as it turns out, if the target group is $\mathrm{Aut}(\mu)$ or $\mathrm{Iso}(\mathbf{U})$, and one restricts attention to free abelian groups, the generic properties of $\overline{\pi(\mathbf{Z}^d)}$ do not depend on d. As another application of our techniques (together with some Baire category tools developed in Appendix A), we also identify exactly the centralizers of generic $\pi(\Gamma)$ for torsion-free groups Γ.

8.1. **A criterion for monotheticity.** We now prove a simple criterion that will allow us to show that the generic $\overline{\pi(\Gamma)}$ is monothetic (of course, $\overline{\pi(\mathbf{Z})}$ is always monothetic but this is not a priori clear for other groups Γ).

Proposition 8.1. *Let G be a Polish group, Γ a countable group. Suppose that for every $\delta \in \Gamma$, $\gamma \in \Gamma \setminus \{1\}$, the set*

$$A_{\gamma,\delta} = \{\pi \in \mathrm{Hom}(\Gamma, G) : \pi(\delta) \in \langle \pi(\gamma) \rangle\}$$

is dense. Then both sets

$$B_1 = \{\pi \in \mathrm{Hom}(\Gamma, G) : \overline{\pi(\Gamma)} \text{ is generically monothetic}\}$$

and

$$B_2 = \{\pi \in \mathrm{Hom}(\Gamma, G) : \langle \pi(\gamma) \rangle \text{ is dense in } \overline{\pi(\Gamma)} \text{ for every } \gamma \in \Gamma \setminus \{1\}\}$$

are dense G_δ.

Proof. First we check that B_1 is G_δ. Let $\{U_n\}$ be a basis for the topology of G. We have

$$\pi \in B_1 \iff \Big(\forall m, n \ \big(\pi(\Gamma) \cap U_m \neq \emptyset \text{ and } \pi(\Gamma) \cap U_n \neq \emptyset\big) \implies \big(\exists \gamma \in \Gamma, k \in \mathbf{Z}\ \pi(\gamma) \in U_m \text{ and } \pi(\gamma^k) \in U_n\big)\Big)$$

Then, we show that $B_{\gamma,\delta} = \{\pi \in \mathrm{Hom}(\Gamma, G) : \pi(\delta) \in \overline{\langle \pi(\gamma) \rangle}\}$ is G_δ. As B_2 is the intersection of all $B_{\gamma,\delta}$ ($\gamma \in \Gamma \setminus \{1\}$, $\delta \in \Gamma$) this will show that B_2 is G_δ. We have

$$\pi \in B_{\gamma,\delta} \iff \Big(\forall n \big(\pi(\delta) \in U_n \implies \exists k \in \mathbf{Z}\ \pi(\gamma^k) \in U_n\big)\Big).$$

Finally, as $B_2 \subseteq B_1$, it suffices to check that B_2 is dense. This is a direct consequence of Baire's theorem and the fact that $A_{\gamma,\delta} \subseteq B_{\gamma,\delta}$. □

The following theorem is the main technical tool for our applications.

Theorem 8.2. *Let G be one of $\mathrm{Aut}(\mu)$ and $\mathrm{Iso}(\mathbf{U})$, and let Γ be a countable, torsion-free, abelian group. Then the sets*

$$\{\pi \in \mathrm{Hom}(\Gamma, G) \colon \overline{\pi(\Gamma)} \text{ is generically monothetic}\}$$

and

$$\{\pi \in \mathrm{Hom}(\Gamma, G) \colon \forall \gamma \in \Gamma \setminus \{1\}\ \overline{\langle \pi(\gamma) \rangle} = \overline{\pi(\Gamma)}\}$$

are dense G_δ *in* $\mathrm{Hom}(\Gamma, G)$.

Proof. We show that both $\mathrm{Aut}(\mu)$ and $\mathrm{Iso}(\mathbf{U})$ satisfy the hypothesis of Proposition 8.1. In what follows, denote by $e_1, \ldots, e_d$ be the standard generators of $\mathbf{Z}^d$.

We begin with the case $G = \mathrm{Aut}(\mu)$; the following is a consequence of the Rokhlin lemma for free abelian groups [C, Theorem 3.1].

Lemma 8.3. *Let* $\epsilon > 0$. *Then for every free* $\pi \in \mathrm{Hom}(\mathbf{Z}^d, \mathrm{Aut}(\mu))$ *and all* $n_1, \ldots, n_d \in \mathbf{N}$, *there exists* $\psi \in \mathrm{Hom}(\mathbf{Z}^d, \mathrm{Aut}(\mu))$ *such that for all* i,

- $\psi(e_i)^{n_i} = 1$;
- $\mu\big(\{x \colon \phi(e_i)(x) \neq \psi(e_i)(x)\}\big) \leq \epsilon + 1/n_i$.

For $\mathcal{A} = \{A_1, \ldots, A_n\}$ a finite measurable partition of X and $g, h \in G$, we use the notation $d(g\mathcal{A}, h\mathcal{A}) < \epsilon$ as shorthand for the more cumbersome $\forall i \leq n\ \mu(g(A_i) \triangle h(A_i)) < \epsilon$.

Fix $\gamma, \delta \in \Gamma$, $\gamma \neq 1$ and let U be a non-empty open set in $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$. We can assume that U is given by

$$\pi \in U \iff d(\pi(e_i)\mathcal{A}, \pi_0(e_i)\mathcal{A}) < \epsilon \text{ for } i = 1, \ldots, k,$$

where $\epsilon > 0$, $\mathcal{A}$ is a finite measurable partition of X, $e_1, \ldots, e_k$ are independent elements of Γ (i.e. $\langle e_1, \ldots, e_k \rangle \cong \mathbf{Z}^k$) and $\gamma, \delta \in \langle e_1, \ldots, e_k \rangle$. By changing the basis $e_1, \ldots, e_k$ if necessary, we can further assume that $\gamma = e_1^{s_1} \cdots e_k^{s_k}$ and $s_i \neq 0$ for all i. Put $\Delta = \langle e_1, \ldots, e_k \rangle$.

By Lemma 8.3, there exist distinct prime numbers $p_1, \ldots, p_k$ with $\min(p_i) > \max(s_i)$ such that if $\Delta' = \mathbf{Z}(p_1) \times \cdots \times \mathbf{Z}(p_k)$ and $q \colon \Delta \to \Delta'$ denotes the natural quotient map, then there exists an action $\Delta' \curvearrowright^{\psi} X$ such that for all i, $d((\psi \circ q)(e_i)\mathcal{A}, \pi_0(e_i)\mathcal{A}) < \epsilon$. Note also that Δ' is cyclic and $q(\gamma)$ is a generator. Consider now Δ' as a subgroup of $\mathbf{T}$, so that q becomes a character of Δ. Then q extends to a homomorphism $\Gamma \to \mathbf{T}$ (still denoted by q); denote its image by Γ'.

Let $\Gamma' \curvearrowright^{\psi'} X^{\Gamma'/\Delta'}$ be the action of Γ' co-induced from ψ (see e.g. [K2, 10 (G)] for details on co-induction) and let $\Theta \colon X^{\Gamma'/\Delta'} \to X$ be the factor map, $\Theta(\omega) = \omega(\Delta')$, which is Δ'-equivariant. Finally, let $T \colon X^{\Gamma'/\Delta'} \to X$ be any measure-preserving isomorphism such that $T^{-1}(\mathcal{A}) = \Theta^{-1}(\mathcal{A})$ and define $\pi \colon \Gamma \to \mathrm{Aut}(\mu)$ by $\pi = q \circ (T\psi' T^{-1})$. It is clear that $\pi \in U$ and $\pi(\delta) \in \langle \pi(\gamma) \rangle$ (as $\pi(\gamma)$ is a generator of $\pi(\Delta)$). This concludes the proof for $G = \mathrm{Aut}(\mu)$.

For $\mathrm{Iso}(\mathbf{U})$, we adapt an argument of Pestov and Uspenskij [PU].

Lemma 8.4. *Let* n *be a non-negative integer,* $\Lambda = \mathbf{Z}^d * \mathbf{F}_n$ *and* C *be a finite subset of* Λ. *Then there exists an integer* M *such that for any* $p_1, \ldots, p_d \geq M$ *there exists a finite group* L *and a morphism* $q \colon \Lambda \to L$ *such that* $q|_C$ *is injective and* $q(e_i)$ *has order* p_i *for all* $i \leq d$.

Proof. Let $a_1, \ldots, a_n$ be generators of $\mathbf{F}_n$. Consider each element of C as a reduced word on the letters $e_1, \ldots, e_d, a_1, \ldots, a_n$ and let N denote the greatest integer such that a subword of the form e_i^N appears in some element of C. Then, pick any p_i greater than $M = N + 1$. Let $q_0 \colon \Lambda \to (\prod_{i=1}^d \mathbf{Z}(p_i)) * \mathbf{F}_n$ be the homomorphism defined by extending the canonical projection $\mathbf{Z}^d \to \prod_{i=1}^d \mathbf{Z}(p_i)$. Note that the restriction $q_0|_C$ is one-to-one, and each $q_0(e_i)$ has order p_i. Now let

$C' = q_0(\mathbf{Z}^d) \cup q_0(C)$. Since free products of residually finite groups are residually finite (see [G2], or [P2, 5.3.5]) and C' is finite, there exists a finite group L and a morphism $q_1\colon (\prod_{i=1}^d \mathbf{Z}(p_i)) * \mathbf{F}_n \to L$ such that the restriction $q_1|_{C'}$ is injective. Setting $q = q_1 \circ q_0$, the lemma is proved. □

With this new ingredient, we obtain the following variant on Lemma 4 of [PU].

Lemma 8.5. (variation on Pestov–Uspenskij [PU]) *Let $\Lambda = \mathbf{Z}^d * \mathbf{F}_n$, and K be a finite subset of Λ. Fix a left-invariant pseudometric on Λ whose restriction to K is a metric. Then there exists $M \in \mathbf{N}$ such that for any $p_1, \dots, p_d \geq M$ there exists a finite group L, a morphism $q\colon \Lambda \to L$ and a left-invariant metric on L such that:*

- *the restriction of π to K is distance-preserving;*
- *$q(e_i)$ has order p_i.*

Proof. The proof is exactly the same as in [PU], using Lemma 8.4 above instead of simply using the fact that Λ is residually finite. □

Redoing the proof of Theorem 1 in [PU], with Lemma 8.5 replacing Lemma 4 of [PU], we obtain the following result.

Theorem 8.6 (variation on Pestov–Uspenskij [PU])**.** *Let $G = \mathrm{Iso}(\mathbf{U})$ and U be a nonempty open subset of $\mathrm{Hom}(\mathbf{Z}^d, G)$. Then there exists M such that for any $p_1, \dots, p_d \geq M$ there exists $\pi \in U$ such that $\pi(e_i)$ has order p_i for all i.*

We now proceed to show that the hypothesis of Proposition 8.1 is satisfied if Γ is a countable, torsion-free, abelian group and $G = \mathrm{Iso}(\mathbf{U})$. Let U be a non-empty open subset of $\mathrm{Hom}(\Gamma, G)$, let $\delta \in \Gamma$ and $\gamma \in \Gamma \setminus \{1\}$, and assume without loss of generality that there exist a finite $A \subseteq \mathbf{U}$, $\epsilon > 0$, $\pi_0 \in \mathrm{Hom}(\Gamma, G)$ and $e_1, \dots, e_k \in \Gamma$ such that

$$\pi \in U \iff \forall a \in A \; d\big(\pi(e_i)(a), \pi_0(e_i)(a)\big) < \epsilon.$$

As above, we also assume that $e_1, \dots, e_k$ are independent, that $\delta, \gamma \in \Delta = \langle e_1, \dots, e_k \rangle$ and that $\gamma = e_1^{s_1} \dots e_k^{s_k}$ with $s_i \neq 0$ for all i. Applying Theorem 8.6, we may find distinct prime numbers $p_1, \dots, p_k$ such that $\min p_i > \max s_i$ and, letting Δ' denote $\mathbf{Z}(p_1) \times \cdots \times \mathbf{Z}(p_k)$ and $q\colon \Delta \to \Delta'$ the natural quotient homomorphism, an action $\Delta' \curvearrowright^{\psi} \mathbf{U}$ such that

$$\forall a \in A \; \forall i \leq k \; d\big(\psi \circ q(e_i)(a), \pi_0(e_i)(a)\big) < \epsilon.$$

Note that, since $q(\gamma)$ is a generator of Δ', one has that $\psi \circ q(\delta) \in \langle \psi \circ q(\gamma) \rangle$. Now, $B = \psi(\Delta') \cdot A$ is finite; applying Lemma 6.1, we may extend $\psi \circ q\colon \Delta \curvearrowright B$ to $\tilde{\psi}\colon \Gamma \curvearrowright X$, where X is a countable metric space, and the construction ensures that we still have $\tilde{\psi}(\delta) \in \langle \tilde{\psi}(\gamma) \rangle$. Use the Katětov construction to extend $\tilde{\psi}$ to an isometric action, still denoted $\tilde{\psi}$, of Γ on $\mathbf{U}$. We still have $\tilde{\psi}(\delta) \in \langle \tilde{\psi}(\gamma) \rangle$ and, using the homogeneity of $\mathbf{U}$, we may assume that

$$\forall a \in A \; \forall i \leq k \; \tilde{\psi}(e_i)(a) = \psi \circ q(e_i)(a).$$

Hence $\tilde{\psi} \in U$, and we are done. □

8.2. **Applications.** In this subsection, we derive some consequences of Theorem 8.2. To that end, we need a generalization of the classical Kuratowski–Ulam theorem, which we discuss in Appendix A. For the moment, let us simply state the definition of a category-preserving map, which we use below: a continuous map between Polish spaces $f\colon X \to Y$ is *category-preserving* if $f^{-1}(A)$ is comeager whenever A is comeager in Y; equivalently, f is category-preserving if it is continuous and $f^{-1}(A)$ is dense whenever A is open and dense in Y. A coordinate projection from $X \times X$ to X is an example of a category-preserving map, as is any open map; the Kuratowski–Ulam theorem, usually stated for projections, extends to all category-preserving maps (see Theorem A.5).

The following lemma provides, in a rather special situation, a simple criterion for the restriction of homomorphisms to be a category-preserving map.

Lemma 8.7. *Let G be a Polish group and Γ a countable, abelian group. Assume that for a dense set of $\pi \in \mathrm{Hom}(\Gamma \times \mathbf{Z}, G)$, $\overline{\pi(\Gamma \times \mathbf{Z})} = \overline{\pi(\Gamma)}$. Then the restriction $\mathrm{Res}\colon \mathrm{Hom}(\Gamma \times \mathbf{Z}, G) \to \mathrm{Hom}(\Gamma, G)$ is category-preserving.*

Proof. Let A be a dense subset of $\mathrm{Hom}(\Gamma, G)$; we will show that $\mathrm{Res}^{-1}(A)$ is dense in $\mathrm{Hom}(\Gamma \times \mathbf{Z}, G)$. Let O be an open, non-empty subset of $\mathrm{Hom}(\Gamma \times \mathbf{Z}, G)$; we can assume that it is of the form

$$O = \{\pi \in \mathrm{Hom}(\Gamma \times \mathbf{Z}, G) : \mathrm{Res}(\pi) \in U \text{ and } \pi(g) \in V\},$$

where $g \in \Gamma \times \mathbf{Z}$ is a generator of $\mathbf{Z}$, U is an open subset of $\mathrm{Hom}(\Gamma, G)$, and V is an open subset of G. Let $\pi_0 \in O$ be such that $\overline{\pi_0(\Gamma \times \mathbf{Z})} = \overline{\pi_0(\Gamma)}$. Then there exists $\gamma_0 \in \Gamma$ with $\pi_0(\gamma_0) \in V$. By shrinking U if necessary, we can assume that $\pi \in U \implies \pi(\gamma_0) \in V$. Let now $\psi \in U \cap A$ be arbitrary and define $\pi\colon \Gamma \times \mathbf{Z} \to G$ by $\pi(\gamma, n) = (\psi(\gamma), \psi(\gamma_0)^n)$. Then clearly $\mathrm{Res}(\pi) \in A$ and $\pi \in O$. □

For the next results, we see $\mathbf{Z}$ embedded in $\mathbf{Z}^d$ as the subgroup generated by e_1.

Theorem 8.8. *Let G be either $\mathrm{Aut}(\mu)$ or $\mathrm{Iso}(\mathbf{U})$. Then the restriction map $\mathrm{Res}\colon \mathrm{Hom}(\mathbf{Z}^d, G) \to \mathrm{Hom}(\mathbf{Z}, G)$ is category-preserving.*

Proof. Since the composition of category-preserving maps is category-preserving, the theorem follows from applying Theorem 8.2 and Lemma 8.7 to the sequence

$$\mathrm{Hom}(\mathbf{Z}^d, G) \to \mathrm{Hom}(\mathbf{Z}^{d-1}, G) \to \cdots \to \mathrm{Hom}(\mathbf{Z}, G).$$

□

Using the following result of Ageev, it is possible to improve the above theorem in the case of $\mathrm{Aut}(\mu)$. Denote by $\mathrm{FREE}(\Gamma)$ the subset of $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ consisting of all free actions. It is well known that $\mathrm{FREE}(\Gamma)$ is dense G_δ in $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ (see e.g. [K2, Theorem 10.8]).

Theorem 8.9 (Ageev [A]). *Let $\Delta \leq \Gamma$ be countable abelian groups such that Δ is infinite cyclic. Let $\mathrm{Res}\colon \mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu)) \to \mathrm{Hom}(\Delta, \mathrm{Aut}(\mu))$ be the restriction; then $\mathrm{Res}(\mathrm{FREE}(\Gamma))$ is comeager in $\mathrm{Hom}(\Delta, \mathrm{Aut}(\mu))$.*

Corollary 8.10. *Let $\Delta \leq \Gamma$ be countable abelian groups such that Δ is infinite cyclic. Then $\mathrm{Res}\colon \mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu)) \to \mathrm{Hom}(\Delta, \mathrm{Aut}(\mu))$ is category-preserving.*

Proof. It is well known that the action $\mathrm{Aut}(\mu) \curvearrowright \mathrm{FREE}(\Gamma)$ is minimal (see e.g. [FW]); applying Proposition A.7, we obtain the desired conclusion. □

Corollary 8.11. *Let either G be* $\mathrm{Aut}(\mu)$ *and* Γ *be a non trivial torsion-free countable abelian group, or G be* $\mathrm{Iso}(\mathbf{U})$ *and* $\Gamma = \mathbf{Z}^d$. *Let P be a property of abelian Polish groups such that the set*

$$\{\pi \in \mathrm{Hom}(\mathbf{Z}, G) : \overline{\pi(\mathbf{Z})} \text{ has property } P\}$$

has the Baire property. Then the following are equivalent:

(i) *for the generic* $\pi \in \mathrm{Hom}(\mathbf{Z}, G)$, $\overline{\pi(\mathbf{Z})}$ *has property P;*
(ii) *for the generic* $\pi \in \mathrm{Hom}(\Gamma, G)$, $\overline{\pi(\Gamma)}$ *has property P.*

Proof. (i) $\Rightarrow$ (ii). Let A be the set of $\pi \in \mathrm{Hom}(\mathbf{Z}, G)$ such that $\overline{\pi(\mathbf{Z})}$ has P. In the case $G = \mathrm{Iso}(\mathbf{U})$ and $\Gamma = \mathbf{Z}^d$, let $g = e_1$ and in the other case, let g be any nontrivial element of Γ. Let $\mathrm{Res}\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\langle g \rangle, G)$ be the restriction map. We have by Theorem 8.8 or Corollary 8.10 that $\mathrm{Res}^{-1}(A)$ is comeager in $\mathrm{Hom}(\Gamma, G)$, and Theorem 8.2 shows that $B = \{\pi \in \mathrm{Hom}(\Gamma, G) : \overline{\pi(\Gamma)} = \overline{\pi(\langle g \rangle)}\}$ is comeager. Then for any $\pi \in \mathrm{Res}^{-1}(A) \cap B$, we have that $\overline{\pi(\Gamma)}$ has property P.

(ii) $\Rightarrow$ (i). Both $\mathrm{Aut}(\mu)$ and $\mathrm{Iso}(\mathbf{U})$ have a dense conjugacy class (for $\mathrm{Aut}(\mu)$, this is a classical result of Rokhlin and for $\mathrm{Iso}(\mathbf{U})$, it is a result of Kechris–Rosendal [KR]) and therefore if P is not generic in $\mathrm{Hom}(\mathbf{Z}, G)$, its negation is and we can apply the above argument to the negation of P. □

It is known that conjugacy classes are meager in $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ for any amenable group Γ (Foreman–Weiss [FW]). We state a result in that direction for $\mathrm{Iso}(\mathbf{U})$, generalizing the result of Kechris stating that conjugacy classes in $\mathrm{Iso}(\mathbf{U})$ are meager.

Theorem 8.12. *Let* Γ *be a non-trivial, torsion-free, countable, abelian group. Then conjugacy classes in* $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$ *are meager.*

Proof. We use Rosendal's technique of considering topological similarity classes, see [R1]. For the remainder of the proof, fix $\gamma \in \Gamma \setminus \{1\}$.

Let S be an infinite subset of $\mathbf{N}$. We first claim that

$$\{\pi \in \mathrm{Hom}(\Gamma, G) \colon \exists n \in S \ \pi(\gamma^n) = 1\}$$

is dense in $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$. To see this, pick a nonempty open subset U. There exists a finitely generated subgroup $\Delta \cong \mathbf{Z}^d$ containing γ and a nonempty open $V \subseteq \mathrm{Hom}(\Delta, \mathrm{Iso}(\mathbf{U}))$ such that, denoting by Res the restriction map from $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$ to $\mathrm{Hom}(\Delta, \mathrm{Iso}(\mathbf{U}))$, one has

$$\{\pi \in \mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U})) \colon \ \mathrm{Res}(\pi) \in V\} \subseteq U.$$

Let $\gamma_1, \ldots, \gamma_d$ be generators of Δ. Applying Theorem 8.6 to Δ, and using the fact that S is infinite, we can find $\pi \in \mathrm{Hom}(\Delta, \mathrm{Iso}(\mathbf{U}))$ and $n \in S$ such that $\pi(\gamma_i^n) = 1$ for all $i \in \{1, \ldots, d\}$ (hence also $\pi(\gamma^n) = 1$). Let $\Delta' = \prod_{i=1}^d \mathbf{Z}(n)$; embed Δ' in $\mathbf{T}^d$ and let $q\colon \Delta \to \mathbf{T}^d$ the natural morphism, which we may extend to a morphism (still denoted by q) from Γ to $\mathbf{T}^d$.

We can now find a finite $A \subseteq \mathbf{U}$ and an action $q(\Delta) \curvearrowright^{\pi} A$ such that, for any action $q(\Gamma) \curvearrowright^{\phi} \mathbf{U}$ extending $q(\Delta) \curvearrowright^{\pi} A$, one has $\phi \circ q \in U$. Applying Lemma 6.1 to $q(\Delta) \leq q(\Gamma)$, using Katětov's construction and the homogeneity of $\mathbf{U}$, we obtain such an action ϕ, and then $\phi \circ q \in U$ and $\phi \circ q(\gamma^n) = 1$. This proves the claim.

Now, let V be a nonempty open subset of $\mathbf{U}$ containing 1 and S an infinite subset of $\mathbf{N}$. It is clear that

$$\{\pi \in \mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U})) \colon \exists n \in S \ \pi(\gamma^n) \in V\}$$

is open, and the claim implies that it is dense. It follows that, for any infinite $S \subseteq \mathbf{N}$, the set

$$A_S = \{\pi \colon \exists (s_n) \in S^{\mathbf{N}} \colon \pi(\gamma^{s_n}) \to 1\}$$

is comeager. This set is conjugacy-invariant, hence if some π had a comeager conjugacy class, π would belong to A_S for all S, which would imply that $\pi(\gamma^n)$ converges to 1. This is only possible if $\pi(\gamma) = 1$; as it is not hard to see that the generic π is injective, we obtain a contradiction. □

We finally turn to studying centralizers of generic representations. Recall that $\mathcal{C}(\pi)$ denotes the centralizer of π.

Theorem 8.13. *Let G be either $\mathrm{Aut}(\mu)$ or $\mathrm{Iso}(\mathbf{U})$ and Γ be a countable, non-trivial, torsion-free, abelian group. Then for the generic $\pi \in \mathrm{Hom}(\Gamma, G)$, $\mathcal{C}(\pi) = \overline{\pi(\Gamma)}$.*

Proof. Denote by $\mathrm{Res} \colon \mathrm{Hom}(\Gamma \times \mathbf{Z}, G) \to \mathrm{Hom}(\Gamma, G)$ the restriction map and note that for any $\pi \in \mathrm{Hom}(\Gamma, G)$, we can naturally identify $\mathcal{C}(\pi)$ with $\mathrm{Res}^{-1}(\{\pi\})$. By Theorem 8.2,

$$\forall^* \psi \in \mathrm{Hom}(\Gamma \times \mathbf{Z}, G)\ \overline{\psi(\Gamma \times \mathbf{Z})} = \overline{\psi(\Gamma)}.$$

Applying Lemma 8.7 and Theorem A.5 to Res, we obtain

$$\forall^* \pi \in \mathrm{Hom}(\Gamma, G) \forall^* g \in \mathcal{C}(\pi)\ g \in \overline{\pi(\Gamma)}.$$

As $\overline{\pi(\Gamma)}$ is a closed subgroup of $\mathcal{C}(\pi)$, this implies the conclusion. □

Remark. We note that even though our hypothesis on Γ in the above theorem is probably not optimal, certain hypotheses are certainly necessary. For example, if $\Gamma = \bigoplus_\omega \mathbf{Z}(2)$ and $G = \mathrm{Aut}(\mu)$, then $\mathrm{Res} \colon \mathrm{Hom}(\Gamma \times \mathbf{Z}, G) \to \mathrm{Hom}(\Gamma, G)$ is still category-preserving. However, $F = \{\pi \in \mathrm{Hom}(\Gamma \times \mathbf{Z}, G) : \pi \text{ is free}\}$ is comeager, therefore

$$\mathrm{Res}(F) \supseteq \{\psi \in \mathrm{Hom}(\Gamma, G) : \mathcal{C}(\psi) \text{ contains an element of infinite order}\}$$

is non-meager, showing that for the generic ψ, $\overline{\psi(\Gamma)} \subsetneq \mathcal{C}(\psi)$.

We conclude with several open problems.

Questions:

(1) Are conjugacy classes in $\mathrm{Hom}(\Gamma, \mathrm{Aut}(\mu))$ and $\mathrm{Hom}(\Gamma, \mathrm{Iso}(\mathbf{U}))$ meager for every countable, infinite group Γ?
(2) For G either $\mathrm{Aut}(\mu)$ or $\mathrm{Iso}(\mathbf{U})$, does there exist a Polish group H such that for the generic $\pi \in \mathrm{Hom}(\mathbf{Z}, G)$, $\overline{\pi(\mathbf{Z})} \cong H$? In particular, is this true for $H = L^0(\mathbf{T})$?
(3) For which pairs of groups $\Delta \leq \Gamma$ is the restriction map $\mathrm{Res} \colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Delta, G)$ category-preserving (again for both $G = \mathrm{Aut}(\mu)$ and $G = \mathrm{Iso}(\mathbf{U})$)?

Appendix A. A generalization of the Kuratowski–Ulam theorem

In this appendix, we establish a generalization of the Kuratowski–Ulam theorem (Theorem A.5 below), similar to the disintegration theorem in measure theory. We believe this theorem to be of some independent interest.

Let us begin by pointing out that from the proof of the classical Kuratowski–Ulam theorem (see for example, [K1, 8K]), one can obtain the following result.

Theorem A.1. *Let X, Y be Polish spaces and $f\colon X \to Y$ be a continuous, open map. Let A be a Baire measurable subset of X. Then the following are equivalent:*

(i) *A is comeager in X;*
(ii) *$\forall^* y \in Y\ A \cap f^{-1}(\{y\})$ is comeager in $f^{-1}(\{y\})$.*

Proof. We begin by proving that (i) implies (ii) in the particular case when A is a dense open subset of X. It is enough to prove that

$$\forall^* y \in Y\ A \cap f^{-1}(\{y\}) \text{ is dense in } f^{-1}(\{y\}).$$

Fix a countable basis $\{U_n\}$ for the topology of X; using Baire's theorem, the above statement is equivalent to saying that

$$\forall n\ \forall^* y \in Y\ \Big(f^{-1}(\{y\}) \cap U_n \neq \emptyset \Rightarrow A \cap f^{-1}(\{y\}) \cap U_n \neq \emptyset\Big)\ .$$

Fix $n < \omega$. The fact that f is open implies that

$$C_n = \{y \in Y\colon f^{-1}(\{y\}) \cap U_n \neq \emptyset \Rightarrow A \cap f^{-1}(\{y\}) \cap U_n \neq \emptyset\}$$

is G_δ (it is the union of a closed set and an open set), and to show that it is comeager, we simply need to show that it is dense in Y. To that end, pick a non-empty open subset V of Y. We may restrict our attention to the case when $V \subseteq f(U_n)$. Then $f^{-1}(V) \cap U_n \cap A$ is non-empty since A is dense, and for any $x \in f^{-1}(V) \cap A \cap U_n$ we have, letting $y = f(x)$, that $y \in V$ and $f^{-1}(\{y\}) \cap A \cap U_n \neq \emptyset$. This proves that (i) implies (ii) when A is a dense open subset of X. The general case follows easily from Baire's theorem.

Now we turn to the proof that (ii) implies (i). We proceed by contradiction and assume that A is a Baire measurable subset of X that satisfies condition (ii) but is not comeager. Using the fact that A is Baire measurable, we know that there exists some non-empty open subset O of X such that $A \cap O$ is meager, whence (using the fact that (i) implies (ii) applied to $A \cap O$)

$$\forall^* y \in Y\ (A \cap O) \cap f^{-1}(\{y\}) \text{ is meager in } f^{-1}(\{y\}).$$

Since $f(O)$ is non-empty and open in Y, we may apply the above condition and (ii) to pick $y \in f(O)$ such that $(A \cap O) \cap f^{-1}(\{y\})$ is meager in $f^{-1}(\{y\})$, and $A \cap f^{-1}(\{y\})$ is comeager in $f^{-1}(\{y\})$. This contradicts Baire's theorem in the closed non-empty set $f^{-1}(\{y\})$, and we are done. □

Definition A.2. Let X, Y be Polish spaces and $f\colon X \to Y$ a continuous map. We say that f is *category-preserving* if $f^{-1}(A)$ is comeager in X whenever A is comeager in Y.

The definition above is equivalent to saying that $f^{-1}(A)$ is meager whenever A is meager, and also to the fact that $f^{-1}(O)$ is dense whenever O is open and dense in Y. Also, note that the composition of two category-preserving maps is still category-preserving.

Let us point out the following easy fact.

Proposition A.3. *Let X, Y be Polish spaces, and $f\colon X \to Y$ a continuous map. Then the following are equivalent:*

(i) *f is category-preserving;*
(ii) *for any non-empty open subset U of X, $f(U)$ is not meager;*
(iii) *for any non-empty open subset U of X, $f(U)$ is somewhere dense.*

Proof. Suppose that (i) holds but (ii) does not, and let U be a non-empty open subset of X such that $f(U)$ is meager. Then $f^{-1}(f(U))$ contains U and does not intersect the comeager set $f^{-1}(Y \setminus f(U))$, a contradiction.

It is immediate that (ii) implies (iii). To see that (iii)⇔(i), observe that (iii) is a rephrasing of the condition that $f^{-1}(O)$ is dense whenever O is open and dense in Y. □

In particular, any open map between Polish spaces is category-preserving.

Our aim is to obtain an extension of Theorem A.1 to category-preserving maps. The following easy lemma is a crucial ingredient.

Lemma A.4. *Let X, Y be Polish metric spaces, and $f\colon X \to Y$ be a continuous map. Then there exists a dense G_δ subset $A \subseteq Y$ such that $f\colon f^{-1}(A) \to A$ is open. (We allow here $f^{-1}(A) = \emptyset$.)*

Proof. Fix a countable basis $\{U_n\}_{n<\omega}$ of the topology of X. Since each $f(U_n)$ is analytic, there exists for all n an open $O_n \subseteq Y$ and a meager $M_n \subseteq Y$ such that $f(U_n) \triangle O_n = M_n$. Then $B = Y \setminus (\bigcup_{n<\omega} M_n)$ is comeager, and we can find a dense G_δ subset A of Y which is contained in B. We claim that this A satisfies the conclusion of the lemma. To see this, note first that for any n, we have that $f(U_n) \cap A$ is open in A. From this, it follows that $f(U) \cap A$ is open in A for any open subset U of X and we are done. □

We are now ready to prove the following strengthening of Theorem A.1.

Theorem A.5. *Let X, Y be Polish spaces and $f\colon X \to Y$ be a continuous, category-preserving map. Let Ω be a Baire measurable subset of X. Then the following are equivalent:*

(i) *Ω is comeager in X;*
(ii) *$\forall^* y \in Y\ \ \Omega \cap f^{-1}(\{y\})$ is comeager in $f^{-1}(\{y\})$.*

Proof. From Lemma A.4 we obtain a dense G_δ subset A of Y such that $f\colon f^{-1}(A) \to A$ is open. Since A is G_δ and f is continuous, $B = f^{-1}(A)$ is G_δ in X, and it is dense since f is category-preserving.

Assume that (i) holds. Since Ω is comeager in X and B is dense G_δ, it follows from Baire's theorem that $\Omega \cap B$ is comeager in B. Then we may apply Theorem A.1 to the open map $f\colon B \to A$ to obtain that

$$\forall^* y \in Y\ (\Omega \cap B) \cap f^{-1}(\{y\}) \text{ is comeager in } f^{-1}(\{y\}),$$

which implies the desired conclusion.

Now assume that (ii) holds. As A is comeager in Y, we have

$$\forall^* a \in A\ \ \Omega \cap f^{-1}(\{a\}) \text{ is comeager in } f^{-1}(\{a\}).$$

Then Theorem A.1 yields that $\Omega \cap B$ is comeager in B, and this implies that Ω is comeager in X. □

It may be worth pointing out that Theorem A.5 is best possible in the sense that, if $f\colon X \to Y$ is a continuous map between two Polish spaces such that (i) and (ii) are equivalent for any Baire measurable subset Ω of X, then f must be category-preserving.

It is particularly interesting, given a Polish group G and two Polish G-spaces X and Y, to obtain conditions that imply that a G-map $\pi\colon X \to Y$ is category-preserving. An easy example of such a condition is the following.

Proposition A.6. *Let G be a Polish group and X, Y be two Polish G-spaces. Let*

$$A = \{y \in Y \colon G \cdot y \text{ is not meager}\}$$

and $\pi\colon X \to Y$ be a continuous G-map such that $\pi^{-1}(A)$ is dense in X. Then π is category-preserving.

Proof. Let U be a non-empty open subset of X. By assumption, we may pick $x \in U$ such that $G \cdot \pi(x)$ is not meager. Find an open subset V of G such that $V \cdot x \subseteq U$. Then $\pi(V \cdot x) = V \cdot \pi(x)$ is not meager, since a countable union of translates of $V \cdot \pi(x)$ covers $G \cdot \pi(x)$. Hence $\pi(U)$ is not meager and we are done. □

This result applies in particular when there exists $y \in Y$ such that $G \cdot y$ is comeager and $\pi^{-1}(G \cdot y)$ is dense in X. This may for instance be used to give a considerably simplified proof of [R2, Theorem 17].

Another situation in which we obtain the same conclusion is the following.

Proposition A.7. *Let G be a Polish group and X and Y Polish G-spaces such that X is minimal (i.e. every orbit is dense). If $\pi\colon X \to Y$ is a G-map and $\pi(X)$ is comeager, then π is category-preserving.*

Proof. Let G_0 be a dense countable subgroup of G and $U \subseteq X$ be open non-empty. Since the action $G \curvearrowright X$ is minimal, we have $G_0 \cdot U = G \cdot U = X$ and hence $\pi(X) = \pi(G_0 \cdot U) = G_0 \cdot \pi(U)$ is comeager in Y, hence $\pi(U)$ is non-meager and π is therefore category-preserving. □

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Université de Lyon, CNRS UMR 5208, Université Lyon 1, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne Cedex, France
E-mail address: melleray@math.univ-lyon1.fr

Université Paris 7, UFR de Mathématiques, case 7012, 75205 Paris CEDEX 13
E-mail address: todor@math.jussieu.fr